PhD Thesis

The Interplay of Unitary and Permutation Symmetries in Composite Quantum Systems

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1 Introduction

In the thesis, I undertake three separate projects, all having as a central theme an interplay between permutation and SU(d) symmetries. In the first two topics I investigate the ground state properties of highly symmetric magnetic systems, and in the third one, I solve the quantum shareability problem for two different classes of bipartite quantum states.

My approach to magnetic systems is from a theorist's point of view, as I look for setups simple enough that it is possible to derive the ground states exactly, yet still general enough that the results may prove useful. This simplicity is bestowed by various degrees of permutation symmetry, which is a way of applying mean field approximation to lattice models. The connection between the general notion of a mean field approximation, and the permutation symmetric definition used by the mathematical physics community, is formalized by the quantum de Finetti theorem [5–8]. Permutation symmetry, along with the innate SU(d) symmetries of the interactions that I study, makes it possible to diagonalize the Hamiltonians exactly through the application of the representation theory of Lie groups.

When can quantum correlations in overlapping subsystems of a larger, composite system be compatible with each other? This fundamental question in quantum physics is the quantum marginal problem; of which the final topic of the thesis, quantum shareability, is a permutation symmetric subcase. The shareability problem, posed for classes of SU(d) symmetric quantum states, is closely related to my study of magnetic systems in a mathematical sense through its symmetries. One could even argue that it is a generalized version of one of the ground state problems I study restated from a different point of view. Accordingly, the solution uses the extension of the same representation theoretic tools as the ground state problems.

2 The bilinear biquadratic model on the complete graph

In this project, I investigate the ground state phases of the most general three-level SU(2) symmetric two-particle interaction in a completely permutation symmetric setting. This is a three-level generalization of the Heisenberg model. The two-particle Hilbert space decomposes into 3 irreducible subspaces under global SU(2) transformations, labeled by spin 0, 1 and 2. Thus, the desired interaction Hamiltonian can be constructed as the linear combination of the identity matrix, and two other linearly independent, SU(2) invariant two-particle operators. We choose these to be the two-particle representations of the quadratic Casimir operators of SU(2) and SU(3), and get a normalized interaction Hamiltonian with a single parameter θ,

$$H_{ij} = \sin(\theta) C_{ij}^{\mathrm{SU}(3)} + \cos(\theta) C_{ij}^{\mathrm{SU}(2)}. \tag{1}$$

The same two particle interaction has been studied for a long time by solid state physicists; although, with different choice of the two SU(2) symmetric operators. It is called the bilinear-biquadratic (BLBQ) interaction, and its Hamiltonian is traditionally expressed as:

$$H_{ij} = \cos(\gamma)\mathbf{S}_i\mathbf{S}_j + \sin(\gamma)(\mathbf{S}_i\mathbf{S}_j)^2. \tag{2}$$

The BLBQ model was at the center of attention in the mid '80s, after Haldane's discovery that spin-1 Heisenberg chains can have a gapped excitation spectrum in contrary to spin-1/2 systems, where the spectrum is always gapless [9, 10]. This remarkable difference initiated an intensive study of the BLBQ model, particularly the chain, and its phase diagram.

Complete permutation symmetry in a spin model with two-particle interaction implies that instead of a regular lattice, the interconnectedness of the spins is described by a complete graph. Classical spin models on complete graphs, such as the Curie-Weiss [11] or Sherrington-Kirkpatrick [12] models, play an important role in statistical mechanics. The reason being that these can be treated relatively easily, yet still describe general features of the corresponding model on high-dimensional lattices. Besides their usefulness as mean field approximations, with the advent of ultracold atom experiments, complete graph models have a possible realization, and also a possible application in metrology, in such a fashion as was proposed for the SU(d) model in [13].

On a complete graph of N sites, the Hamiltonian of our general SU(2) symmetric interaction reduces to a linear combination of the N-particle representations of the two quadratic Casimir operators from Eq (1).

$$H_{\mathrm{BLBQ}} = \sin(\theta) C_N^{\mathrm{SU}(3)} + \cos(\theta) C_N^{\mathrm{SU}(2)}. \tag{3}$$

The ground state properties of this Hamiltonian are determined by a competition between the SU(2) and SU(3) symmetric terms, and its eigenspace decomposition can be obtained entirely through representation theoretic considerations. As the Casimir operators in Eq (3) commute, and their eigenspaces are the SU(2) and SU(3) irreducible subspaces of the N-particle Hilbert space, the eigenspaces of H_{BLBQ} correspond to a pair of SU(2) and SU(3) irreducible representations (irreps). The heart of solving the eigenproblem of H_{BLBQ} then lies in determining which of these irrep pairs are compatible. The image of the spin-1 representation of SU(2) can be considered a subgroup of SU(3); accordingly, each irrep of SU(3) decomposes into a direct sum of SU(2) irreps when we restrict the representation to the SU(2) subgroup. Working out the exact details of this decomposition lets us determine the compatible Young diagram spin pairs.

3 Collective SU(3) spin system with bipartite symmetry

In this project, I expand upon the theme started in the previous one, and investigate a different three-level generalization of the Heisenberg model in a highly permutation symmetric setting. Here, the spins interact with an SU(3) symmetric exchange interaction, and instead of breaking the SU(3) symmetry, the complete permutation invariance is broken in a minimal fashion, by the division of the system into two equal sized, permutation invariant subsystems. In this way, the behavior of the system is determined by the competition of the interactions within, and between the two subsystems. Introducing this bipartite structure into a mean field type model makes the phase structure more interesting, as it relaxes the extreme frustration of the complete graph, and opens the possibility for bipartite symmetry breaking. As it turns out, the breaking of bipartite symmetry does happen in one of the ground state phases. It is interesting that already such a simple bipartite long-range model provides a phase that is absent in the literature on short-ranged bipartite models.

In a similar fashion to the bilinear-biquadratic model on the complete graph, due to the permutation symmetry, it is possible to express the collective bipartite exchange (CBE) Hamiltonian governing the entire system with Casimir operators. The normalized Hamiltonian has a single parameter θ, and it contains the natural representations of the quadratic SU(3) Casimir operator on the two subsystems, $C_{\mathrm{A}}^{\mathrm{SU(3)}}, C_{B}^{\mathrm{SU(3)}}$, and on the entire system $C_{\mathrm{AB}}^{\mathrm{SU(3)}}$.

$$H_{\mathrm{CBE}} = \sin(\theta) C_{\mathrm{AB}}^{\mathrm{SU(3)}} + \cos(\theta)\left(C_{\mathrm{A}}^{\mathrm{SU(3)}} + C_{\mathrm{B}}^{\mathrm{SU(3)}}\right). \tag{4}$$

The physical intuition one can gain from this form is that the right parameters that characterize the system are actually not the strengths of the interaction within and between the subsystems, but the strength of the “baseline” uniform exchange interaction on the entire system represented by C_{AB}, and the strength of the additional uniform interaction on the subsystems superposed with the former, represented by C_{A} and C_{B}.

The Casimir operators in Eq. (4) once again make it possible to solve the ground state problem exactly by the use of representation theory. The eigenvalues of H_{CBE} are labeled by triples of SU(3) irreps, two for the subsystems, and the third for the entire system. Whether three particular irreps are compatible with each other, depends on the fusion rules of SU(3), which can be calculated using the Littlewood-Richardson rule. This is a combinatorial algorithm involving Young diagrams. In the thesis, I solve a combinatorial problem associated with the Littlewood-Richardson rule by relying on the closed form expression of the Littlewood-Richardson diagrams made by Schlosser [14]. This allows me to reduce the number of variables to two SU(3) irreps, then solve the remaining equations required to obtain the ground state as a function of θ.

4 The shareability of Werner and Isotropic states

The essence of the shareability problem is the following: Alice and Bob are each given a composite quantum system made out of n_{A} and n_{B} copies of the same system respectively. Is it possible, that each bipartite subsystem in which one part is owned by Alice, and the other by Bob, is simultaneously in the state ρ? If it is possible, then ρ is called n_{A}-n_{B} shareable. On one hand, pure entangled states are clearly not, or in other words, only 1-1 shareable, and on the other hand, separable states are arbitrarily, i.e. $\infty-\infty$ shareable. In general, the maximal values of n_{A} and n_{B} for which a given bipartite state ρ is shareable, serves as a measure of the entanglement of ρ.

In an attempt to make the problem approachable, I restrict the candidates for ρ to two classes of $\mathrm{U}(d)$ symmetric bipartite states. The Werner states, invariant to global unitary transformations, and the isotropic states, invariant to transformations of the form $U \otimes U^*$, where $U \in \mathrm{U}(d)$ and $*$ denotes complex conjugation. Both classes play an important role in our understanding of entanglement, Werner states in particular were originally defined in the same paper as entanglement itself [15]. The unitary symmetry of these states, coupled with the bipartite permutation symmetry inherent to the shareability scenario itself, makes it possible to solve the problem by using representation theory. This same approach was previously taken by Johnson and Viola [16], who derived necessary and sufficient conditions of the 1-n shareability of both Werner and isotropic states. In my work, I extend this result to arbitrary values of n_{A} and n_{B}.

In the thesis, I show that determining the full sets of n_{A}-n_{B} shareable Werner and isotropic states is equivalent to finding the extremal eigenvalues of certain linear operators, not unlike the Hamiltonian in Eq. (4) of the bipartite spin model I study. In the same fashion as in the ground state problem of the CBE Hamiltonian, the eigenvalues of these linear operators are labeled by triples of $\mathrm{SU}(d)$ irreps. The set of "valid" irrep-triples, that correspond to eigenvalues of these operators are determined through the Littlewood-Richardson rule that governs the irrep decomposition of products of $\mathrm{SU}(d)$ irreps. In order to find the extremal eigenvalues, I generalize the procedure I used to solve the ground state problem of the previous, bipartite spin model to arbitrary dimensions. In particular, I rely on the works of Lam [17] and Azenhas [18] that describes the partial order of the Young diagrams that appear in the Littlewood-Richardson algorithm.

5 Thesis points

1. Through representation theoretic considerations, I exactly diagonalized the Hamiltonian of the spin-1 bilinear-biquadratic model on the complete graph (Eq. (3)), and analyzed its ground state as a function of the external control parameter θ. I have found that the model has four distinct ground state phases belonging to different symmetry sectors. There is a ferromagnetic phase, a gapless partially magnetized phase in which the quantum numbers describing the ground state change gradually with θ, a completely permutation symmetric SU(2) singlet phase, and a phase in which the ground state is both an SU(2) and SU(3) singlet. I have a published paper in Journal of Physics A about this topic [1].

2. I exactly diagonalized the collective bipartite exchange Hamiltonian (Eq. (4)) in the thermodynamic limit, and studied its ground state as a function of the external control parameter θ. The model has five different ground state phases: A ferromagnetic phase, a Néel-type antiferromagnetic phase with ferromagnetically aligned bipartite subsystems, an SU(3) singlet phase, a gapless partially magnetized phase in which the ground state changes gradually with the control parameter, and a bipartite symmetry breaking

phase in which two subsystems are characterized by different SU(3) representations. I have published a paper about this topic in Physical Review B [2].

3. I have determined necessary and sufficient conditions for the n_A-n_B shareability of SU(d) Werner and isotropic states, for arbitrary values of $n_\mathrm{A}, n_\mathrm{B}$, and d. As of the writing of this thesis, I have a preprint available about this topic [3].

The author's publications

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