Mathematica Pannonica
2/1 (1991), 3 – 11

NOTE ON MODULES OVER PRÜFER DOMAINS*

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Received February 1990

AMS Subject Classification: 13 F 05, 13 C 05

Keywords: Prüfer domain, almost maximal, h-local domain, absolutely pure, finitely embedded module

Abstract: We give various characterizations – in terms of module properties – for Prüfer domains in general, and for (locally) almost maximal Prüfer domains, in particular. A domain R is a Prüfer domain if and only if pure-injective divisible R-modules are injective. A Prüfer domain R is locally almost maximal exactly if finitely embedded R-modules are pure-injective. An h-local domain R is almost maximal Prüfer if and only if finitely embedded R-modules are direct sums of cocyclic R-modules.

All rings will be commutative with 1. A ring R is *maximal* if it is linearly compact in the discrete topology (this is the topology in which

* The authors gratefully acknowledge the hospitality and support of the Department of Mathematics, University of Orange Free State, Bloemfontein, where this research was carried out.

linear compactness will be considered here); R is *almost maximal* if the ring R/I is maximal for every non-zero ideal I of R. A domain R (a ring without divisors of zero) is a *valuation domain* if its ideals form a chain under inclusion; it is a *Prüfer domain* if its finitely generated ideals are projective, or, equivalently, its localization R_M is a valuation domain for every maximal ideal M of R. If all of these localizations R_M are almost maximal, R is said to be *locally almost maximal* (Brandal [2, p.20]). A domain R is called *h-local* if (i) every non-zero prime ideal of R is contained in exactly one maximal ideal, and (ii) every non-zero element is contained in but a finite number of maximal ideals. By Brandal [2, Th. 2.9], a domain is almost maximal if and only if it is an h-local, locally almost maximal domain.

A module D over a domain R is called *divisible* if $rD = \{rd | d \in D\}$ is equal to D for all $0 \neq r \in R$, and *h-divisible* if it is an epic image of a direct sum of copies of the field Q of quotients of R (as an R-module). D is *absulutely pure* if it is pure in every R-module in which it is contained. Megibben [8] has shown (in a more general form) that a domain R is Prüfer if and only if every divisible module is absolutely pure. Naudé-Naudé-Pretorius [9] proved that a domain R is Prüfer exactly if all pure-injective modules are RD-injective (*RD-injectivity* is defined as the injective property relative to inclusions $A \longrightarrow B$ where $rA = A \cap rB$ for all $r \in R$; see [5, p.210]). This result will be sharpened: it is enough to require that the pure-injective divisible modules be RD-injective. See Theorem 5.

An R-module C is said to be *cocycylic* if it is an essential extension of a simple R-module S, i.e. it is contained in the injective hull $E(S)$ of S. An R-module F is *finitely embedded* if it is an essential extension of a finite direct sum of simple R-modules. In investigating classical rings R (i.e. $E(S)$ is linearly compact for every simple R-module S), Vámos [10] identified the classical Prüfer domains as those classical domains over which the finitely embedded modules are direct sums of cocyclic modules. We prove a similar result (Theorem 8) characterizing the h-local domains over which such decompositions hold: these are exactly the almost maximal Prüfer domains. This is the dual of a result by Matlis [7, Th. 5.7] which deals with the decompositions of finitely generated modules into direct sums of cyclics. Those Prüfer domains will also be described over which the finitely embedded modules are linearly compact (or pure-injective); see Theorem 6. A similar problem was investigated by Facchini [4]: he characterized the rings over which finitely embededded modules have injective dimension ≤ 1. (We wish to thank Willy Brandal for calling our attention to this paper.)

For unexplained terminology we refer to standard texts or to Fuchs-Salce [5].

1. Preliminaries

We start our discussion with lemmas on modules over arbitrary domains R. For an R-module D and $r \in R$, we set $D[r] = \{d \in D | rd = = 0\}$.

Lemma 1. *The R-module* $\mathrm{Hom}_R(D, *)$ *is torsion-free whenever D is a divisible R-module.*

Proof. From the exact sequence $0 \to D[r] \to D \xrightarrow{r} D \to 0$ we infer that the sequence $0 \to \mathrm{Hom}(D, *) \xrightarrow{r} \mathrm{Hom}(D, *) \to \mathrm{Hom}(D[r], *)$ is exact. $\diamondsuit$

Lemma 2. *If A is a torsion-free and E is an injective R-module, then* $\mathrm{Hom}_R(A, E)$ *is divisible and pure-injective.*

Proof. The pure-injectivity of $\mathrm{Hom}_R(*, E)$ for E pure-injective is well known (see e.g. [5, p.217]). The exact sequence $0 \to E[r] \to E \xrightarrow{r} E \to \to 0$ implies the exactness of

$$0 \to \mathrm{Hom}(A, E[r]) \to \mathrm{Hom}(A, E) \xrightarrow{r} \mathrm{Hom}(A, E) \to \mathrm{Ext}^1(A, E[r]).$$

As $E[r]$ is RD-injective (see [5,p.210]) and A is torsion-free, the last term vanishes. Hence $\mathrm{Hom}(A, E)$ is divisible. $\diamondsuit$

Lemma 3. *The pure-injective hull of a divisible module is divisible.*

Proof. If E is an injective cogenerator of the category of R-modules, then for every R-module M, there is a pure embedding

$$M \to \mathrm{Hom}_R(\mathrm{Hom}_R(M, E), E) = H$$

and the pure-injective hull $PE(M)$ of M is a summand of the pure-injective module H (see [5,p.217]). It is therefore enough to show that if M is divisible, then so is H. By Lemma 1, if M is divisible, then $\mathrm{Hom}(M, E)$ is torsion-free. Hence Lemma 2 implies H is divisible. $\diamondsuit$

Lemma 4. *An RD-injective divisible module is injective.*

Proof. By [5, p.213], an RD-injective module M decomposes as $M = = E \oplus N$ where E is injective and $N^1 = \bigcap_{0 \neq r \in R} rN = 0$. If M is divisible, then necessarily $N = 0$, and $M = E$ is injective. $\Diamond$

Remark. Actually, the following converse of Lemma 2 holds: *if E is an injective cogenerator of the category of R-modules, then* $\mathrm{Hom}_R(A, E)$ *is (pure-injective) divisible if and only if A is torsion-free.* To see this, consider the isomorphism [3, p.120]

$$\mathrm{Ext}^1_R(B, \mathrm{Hom}_R(A, E)) \cong \mathrm{Hom}_R(\mathrm{Tor}^R_1(B, A), E)$$

which holds for all R-modules A, B and injective E. Recall that an R-module D is divisible exactly if $\mathrm{Ext}^1_R(R/Rr, D) = 0$ for all $r \in R$ [5, p.36]. In view of the above isomorphism, $D = \mathrm{Hom}_R(A, E)$ is divisible if and only if, for all $r \in R$, $\mathrm{Hom}_R(\mathrm{Tor}^R_1(R/Rr, A)E) = 0$. This amounts to $\mathrm{Tor}^R_1(R/Rr, A) = 0$ whenever E is an injective cogenerator. The exact sequence $0 \to R \xrightarrow{r} R \to R/Rr \to 0$ induces the exact sequence $0 \to \mathrm{Tor}_1(R/Rr, A) \to R \otimes A \cong A \xrightarrow{r} R \otimes A \cong A$. This shows that $\mathrm{Tor}_1(R/Rr, A) = 0$ for all $r \in R$ is equivalent to the torsion-freeness of A.

The reader is advised to compare our remark with the well-known fact that if E is an injective cogenerator, then the injectivity of $\mathrm{Hom}_R(A, E)$ is equivalent to the flatness of A. (Hence the equivalence of (i) and (ii) in Theorem 5 can easily be derived: just recall flatness and torsion-freeness are equivalent exactly for Prüfer domains.)

2. Characterizations of Prüfer domains

The next result gives various equivalent properties which characterize Prüfer domains among the domains. The equivalence of (i) and (iv) is due to Megibben [8], while the equivalence of (i) and (ii) improves on a result by Naudé-Naudé-Pretorious [9].

Theorem 5. *For a domain R, the following are equvalent:*

(i) *R is a Prüfer domain;*

(ii) *pure-injective divisible R-modules are injective;*

(iii) *pure-injective hulls of divisible R-modules are injective;*

(iv) *divisible R-modules are absolutely pure;*

(v) *h-divisible R-modules are absolutely pure.*

Proof. (i) $\Rightarrow$ (ii): For Prüfer domains, purity and RD-property are equivalent (see [5, p.47]). Hence Lemma 4 shows that (ii) holds for Prüfer domains.

(ii) $\Rightarrow$ (iii) is obvious in view of Lemma 3.

(iii) $\Rightarrow$ (iv): Let D be a divisible module in the exact sequence $0 \to D \to A \to B \to 0$. Using the canonical embedding $\delta : D \to \to PE(D)$, form the pushout diagram

$$\begin{array}{ccccccccc} 0 & \to & D & \to & A & \to & B & \to & 0 \\ & & \downarrow \delta & & \downarrow \alpha & & \| & & \\ 0 & \to & PE(D) & \xrightarrow{\gamma} & C & \to & B & \to & 0 \end{array}$$

where α is monic. By (iii), $PE(D)$ is injective, and therefore $\operatorname{Im}\gamma$ is a summand of C. It follows that $\gamma\delta D$ is pure in C, and so D is pure in A. Thus D is absolutely pure.

(iv) $\Rightarrow$ (v) is trivial.

(v) $\Rightarrow$ (i): Let L be a finitely generated ideal of R, and D an h-divisible R-module. By (v), D is absolutely pure, thus every extension of D by a finitely presented R-module is splitting. In particular, $\operatorname{Ext}^1(R/L, D) = 0$. Given an R-module M and its injective hull E, the module D in the exact sequence $0 \to M \to E \to D \to 0$ is h-divisible. Form the commutative diagram

$$\begin{array}{ccccccc} \operatorname{Hom}(R,E) & \to & \operatorname{Hom}(R,D) & \to & 0 & & \\ \downarrow & & \downarrow & & & & \\ \operatorname{Hom}(L,E) & \to & \operatorname{Hom}(L,D) & \to & \operatorname{Ext}^1(L,M) & \to & 0 \\ \downarrow & & \downarrow & & & & \\ \operatorname{Ext}^1(R/L,E) = 0 & & \operatorname{Ext}^1(R/L,D) = 0 & & & & \end{array}$$

with exact rows and columns. The composite map $\operatorname{Hom}(R,E) \to \to \operatorname{Hom}(L,D)$ being surjective, $\operatorname{Ext}^1(L,M) = 0$ follows. This holds for every M, so L is projective and R is Prüfer. $\Diamond$

3. Locally almost maximal Prüfer domains

Among the valuation domains, the almost maximal ones are distinguished by a number of attractive properties. Some of these properties carry over to almost maximal Prüfer domains. We are particularly interested in those which relate to the finitely embedded modules.

Theorem 6. *For a Prüfer domain R, the following are equivalent:*

(i) *R is locally almost maximal;*
(ii) *finitely embedded R-modules are linearly compact;*
(iii) *finitely embedded R-modules are pure-injective;*
(iv) *cocyclic R-modules are pure-injective.*

Proof. (i) $\Rightarrow$ (ii): If R_M is an almost maximal valuation domain for every maximal ideal M, then $Q/R_M M$ is linearly compact for every M, both as an R_M- and as an R-module. A finitely embedded R-module is a submodule of a finite direct sum of linearly compact R-modules of the form $Q/R_M M$, hence itself linearly compact.

(ii) $\Rightarrow$ (iii) is clear, since linear compactness over a commutative ring always implies pure-injectivity.

(iii) $\Rightarrow$ (iv) is trivial.

(iv) $\Rightarrow$ (i): The R-module $Q/R_M M$ is cocyclic, and therefore pure-injective. It is moreover, injective, since over Prüfer domains divisible pure-injective modules are injective (cf. Theorem 5). The exact sequence $0 \to R_M/R_M M \to Q/R_M M \to Q/R_M \to 0$ implies the exactness of

$$0 = \mathrm{Ext}^1_R(R/I, Q/R_M M) \to \mathrm{Ext}^1_R(R/I, Q/R_M) \to \\ \to \mathrm{Ext}^2_R(R/I, R_M/R_M M)$$

for every ideal I for R. Simple modules are always RD-injective, and hence they have injective dimension 1 [5, p.243]. If the last Ext vanishes, then so does the middle one. This implies that Q/R_M is (an injective R-module and so) an injective R_M-module, proving the almost maximality of R_M. $\Diamond$

4. h-local almost maximal Prüfer domains

Our final goal is to find all h-local domains over which the finitely embedded modules are direct sums of cocyclics.

Recall that a torsion module T over an h-local domain R is the direct sum of its localizations: $T_M = R_M \otimes_R T$. Here T_M is an R_M-module whose R- and R_M-module structures coincide (see Brandal [1, Lemma 2.7]).

We start with a lemma; this is the dual of a result by Matlis [7] and Gill [6].

Lemma 7. *Let R be a local domain. If every finitely embedded R-*

module is a direct sum of cocyclic R-modules, then R is a valuation domain.

Proof. Suppose R has the stated property, but is not a valuation domain. Choose $a, b \in R$ such that $b \notin Ra$ and $a \notin Rb$. There is an ideal A of R which is maximal with respect to the properties $a \in A$ and $b \notin A$. Similarly, there is an ideal B of R maximal with respect to $b \in B$, $a \notin B$. Consider the R-module $F = R/(A \cap B)$ which is evidently a submodule of $R/A \oplus R/B$. Here R/A is subdirectly irreducible with $b + A$ generating its socle; thus R/A is cocyclic. The same holds for R/B. We conclude that $R/A \oplus R/B$ and hence F is finitely embedded. Neither a nor b is a unit of R, thus both A and B are contained in the maximal ideal M of R. Consequently, F is indecomposable, and hence – by hypothesis – cocyclic. But F has a non-simple socle $R(b + A) \oplus R(a + B)$, a contradiction. $\diamondsuit$

Observe that the last lemma holds for all commutative local rings.

We are now able to prove the dual of a theorem of Matlis [7, Th. 5.7]. (Since the ring is not assumed to be classical, duality arguments can not be applied.)

Theorem 8. *Let R be an h-local domain. The following are equivalent:*

(a) *R is an almost maximal Prüfer domain;*

(b) *every finitely embedded R-module is a direct sum of cocyclic R-modules.*

Proof. (a) $\Rightarrow$ (b): Since R is h-local, every finitely embedded R-module F is a finite direct sum $F = \oplus F_M$ where F_M is a finitely embedded R_M-module. The R- and R_M-module structures of F_M are identical, thus it suffices to verify the implication for an almost maximal valuation domain R (with maximal ideal M).

In this case, the injective hull E of a finitely embedded R-module F is the direct sum of a finite number of copies of Q/M. Hence we conclude that F is a submodule of a finite direct sum of uniserial R-modules, and so it is polyserial in the sense of [5,p.190]. Polyserial torsion modules over an almost maximal valuation domain are direct sums of uniserials, hence (b) holds.

(b) $\Rightarrow$ (a): We argue as before that it is enough to prove that a local domain R with property (b) has to be an almost maximal valuation domain.

That R is a valuation domain has been proved in Lemma 7. By way of contradicition, suppose R is not almost maximal. Then there is a unit u in a maximal immediate extension $\widehat{R}$ of R which is not in R and whose breadth ideal

$$B = B(u) = \{r \in R | u \notin R + r\widehat{R}\} \neq 0.$$

For every $x \in R \setminus B$ there is a (unit) $u_x \in R$ such that $u - u_x \in Rx$. Visibly, the family of units $\{u_x \in R | x \in R \setminus B\}$ satisfies
(i) $u_x - u_y \in Rx$ if $y \in Rx$,
(ii) there is no $v \in R$ such that $v - u_x \in Rx$ for all $x \in R \setminus B$.
Define the fractional ideal $C = B^{-1} = \{q \in Q | qB \leq R\}$; thus for $r \in R$, $r^{-1} \in C$ exactly if $B \leq rR$, i.e.

$$C = \bigcup_{r \in R \setminus B} Rr^{-1}.$$

Multiplication by u_x induces an automorphism α_x of C/R. If $y \in Rx$, then $u_x x^{-1} - u_y x^{-1} \in R$ shows that $\alpha_x w = \alpha_y w$ for all $w \in Rx^{-1}$. The automorphism α of C/R defined by $\alpha w = \alpha_x w$ for $w \in Rx^{-1}$ is not induced by any element of R.

For some non-unit t of R, consider the cocyclic uniserial R-module $V = C/Mt$. There is no automorphism θ of U which would induce α on C/R, because of the choice of C. Using the submodule $V = R/Mt$ and the canonical map $\pi : U \to U/V$, form the following commutative diagram with exact rows:

$$\begin{tikzcd}
 & & V \arrow[r, equal] \arrow[d, tail] & V \arrow[d, tail] & \\
0 \arrow[r] & V \arrow[r] \arrow[d, equal] & X \arrow[r] \arrow[d] & U \arrow[r] \arrow[d, "\alpha\pi"] & 0 \\
0 \arrow[r] & V \arrow[r] & U \arrow[r, "\pi"] \arrow[ur, dashed, <->] & U/V \arrow[r] & 0
\end{tikzcd}$$

Since there is no automorphism $\theta : U \to U$ making the arising lower triangle commute in either direction, neither the middle row nor the middle column splits. Manifestly, $V \oplus V \leq X \leq U \oplus U$, so X is finitely embedded, and as such it is a direct sum $X = X_1 \oplus X_2$ where X_i are cocyclic. The proof of [5, p.190] shows that the intersection of X with one of the U's is pure in X. This amounts to the purity of one of the V's in X. By [5, p.192], pure submodules of a finite direct sum of uniserials are summands; consequently, either the middle row or the middle column splits. This contradiction shows that no $u \in \widehat{R}$ can exist with $B(u) \neq 0$, i.e. R is almost maximal. $\diamondsuit$

The characterization of rings R for which part (b) of Theorem 8 holds is an open question. The condition of R being h-local can be weakened by demanding only that every prime $\neq 0$ in R be contained in exactly one maximal ideal of R.

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Mathematica Pannonica
2/1 (1991), 13 – 18

ON THE NUMBER OF PERMUTATIONS ARISING FROM A PROBLEM IN CELL-BIOLOGY

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Received March 1990

AMS Subject Classification: 05 A 05, 92 A 09, 20 B 30

Keywords: Disposition of chromosomes, admissible permutations, Hamiltonian circles and lines, numbers of admissible permutations, upper and lower bounds, limits

Abstract: Arrangements of chromosomes according to Bennett's model can be characterized by a certain type of permutations of n objects. The number of these permutations is determined for arbitrary even n and upper and lower bounds are given for any odd n. As a consequence it is proved that in both cases the relative frequency of the considered permutations converges to zero with n increasing to infinity (which is of interest especially from the biological point of view).

In cytogenetics the question is important whether there exists an ordered arrangement of the n chromosomes of a haploid genome during metaphase (a certain stage of cell division). The best known theory in favour of an ordered disposition of the chromosomes is Bennett's model (cf [1], [4], [5]). In terms of permutations the problem of the existence of an ordered arrangement according to Bennett's model can be formulated as follows (cf [2]):

Given $\pi \in S_n$, does there always exist $h, \rho \in S_n$ such that

$$(1) \qquad |h(2k) - h(2k+1)| = 1 \text{ for } k = 1, 2, \ldots, [\tfrac{n}{2}]$$

$$(2) \qquad |\rho(2k-1) - \rho(2k)| = 1 \text{ for } k = 1, 2, \ldots, [\tfrac{n}{2}]$$

and $\pi = h\rho^{-1}$?

Here S_n denotes the symmetric group on n letters, [r] indicates the greatest integer that does not exede $r, h(n+1)$ stands for $h(1)$ and $h\rho^{-1}$ is ment to indicate that first the inverse of ρ has to be performed and than h. If $\pi \in S_n$ admits a representation in the form $\pi = h\rho^{-1}$ with h and ρ satisfying (1) and (2), π is called *admissible.*

The essential question now is: How many admissible permutations exist in S_n?

Let AP_n be the set of admissible permutations of S_n and $A(n) = |AP_n|$. For n odd an $n \leq 11$ the numbers $A(n)$ were computed in [3]; in particular, for $n \leq 5$ $AP_n = S_n$ and for $5 < n \leq 11$ the number $A(n)$ does not much deviate from $|S_n|$. The question arises whether this is also true for an arbitrary odd n. (Some theoretical background on finding $A(n)$ for odd n can be found in [2].) On the other hand for even n computations show that $A(n)$ deviates very fast from $n!$ with increasing n.

In the following we will determine the exact value of $A(n)$ for all even n as well as lower and upper bounds for any odd n. As a consequence, we prove that in both cases the relative frequency of admissible permutations converges to zero.

As it was pointed out in [2] (for odd n but is analogously true for even n), an ordered arrangement of chromosomes according to Bennett's model can also be considered as an unorientated graph $G = \langle V, E \rangle$ with vertex set $V = \{1, 2, \ldots, n\}$ and edge set E which is the product of two 1-factors F_1 and F_2, i.e. $G = F_1 \times F_2$. We assign two colours to the edges of G, namely colour 1 to the edges of F_1 and colour 2 to the edges of F_2. Now, if π is an admissible permutation to which the ordered arrangement represented by G belongs, and $\pi = h\rho^{-1}$ with

h, ρ satisfying (1) and (2), then according to [2]

$$(3) \qquad \begin{aligned} F_1 &= < V, \{[1,2],[3,4],\ldots,[n-1,n]\} > \\ F_2 &= < V, \{[\pi 1, \pi 2],[\pi 3, \pi 4],\ldots,[\pi(n-1),\pi n] > \end{aligned}$$

for even n and

$$(4) \quad \begin{aligned} F_1 &= < V, \{[1,2],\ldots,[p-2,p-1],[p+1,p+2],\ldots,[n-1,n]\} > \\ F_2 &= < V, \{[\pi 1,\pi 2],\ldots,[\pi(q-2),\pi(q-1)], \\ &\qquad [\pi(q+1),\pi(q+2),\ldots,[\pi(n-1),\pi n]\} > \end{aligned}$$

for odd n with $p, q \in \{1,3,5,\ldots n\}$.

Further $G = F_1 \times F_2$ is a Hamiltonian circle in the even case and a Hamiltonian path in the odd case, with edges of alternating colours in both cases (shortly: alternating H-circle and H-path resp.). Moreover the sequence of vertices within the alternating H-circle and H-path resp. is given by $h1, h2, \ldots, hn$, and the notation is chosen in such a way that the first edge $[h1, h2]$ always belongs to F_2. Hence, if n is odd, $h1 = p$ and $hn = \pi q$. The same graph G may be induced by different admissible permutations π. On the other hand, G is determined uniquely by π if n is even, but not for odd n.

The graph $G = F_1 \times F_2$ can be defined by (3) or (4) for an arbitrary $\pi \in S_n$. But then in general G consists of several alternating H-circles in the even case, of an H-path and one or more circles in the odd case. Actually, for n even, π is admissible if and only if G is an alternating H-circle. For an odd n, this is the case if p, q can be chosen appropriately so that an alternating H-path results.

Now, for any n, let $\alpha_n = \frac{A(n)}{n!}$. This is the relative frequency of admissible permutations. For odd n we define further:

$$\begin{aligned} K_n &= 2^{n-1}\left(\tfrac{n-1}{2}!\right)^2 = (n-1)^2(n-3)^2\ldots 2^2, \\ k_n &= \frac{K_n}{n!} = \frac{(n-1)(n-3)\ldots\cdot 2}{n(n-2)\ldots\cdot 3\cdot 1} \end{aligned}$$

Remark 1. $\lim\limits_{n\to\infty} k_n = 0$, since

$$\begin{aligned} \frac{1}{k_n} &= \left(1+\frac{1}{n-1}\right)\left(1+\frac{1}{n-3}\right)\ldots\left(1+\frac{1}{2}\right) = \\ &= 1 + \left(\frac{1}{2}+\frac{1}{4}+\ldots+\frac{1}{n-1}\right)+\ldots \to \infty. \end{aligned}$$

Theorem 1. *For any even n, $\alpha_n = k_{n-1}$, hence $\lim\limits_{n\to\infty} \alpha_n = 0$.*

Proof. The number of possible alternating H-circles is $2^{\frac{n}{2}-1}(\frac{n}{2}-1)!$. To see this, we assume one edge of colour 1, say [1,2], in a fixed position, provide the edge with an orientation, which we keep also fixed, then we count the number of permutations of the remaining edges of the same colour and consider that each edge may be orientated in two ways. Thus there are $2^{\frac{n}{2}} \cdot \frac{n}{2}!$ permutations π which induce the same set of edges $\{[\pi 1, \pi 2], \ldots, [\pi(n-1), \pi n]\}$, i.e. the same alternating H-circle. This gives $A(n) = 2^{n-1}(\frac{n}{2}-1)!\frac{n}{2}! = n(n-2)^2(n-4)^2 \ldots \cdot 2^2 = nK_{n-1}$, therefore $\alpha_n = \frac{nK_{n-1}}{n!} = k_{n-1}$. $\diamondsuit$

In the odd case, the difficulty arises that an adimissible permutation π may induce different alternating H-pathes trough different choices of p and q in (4). We recall that the corresponding H-path is given by the sequence $h1, h2, \ldots, hn$, where $h = h\rho$, h, ρ satisfy (1) and (2), $h1 = p$, and $\rho n = q$. First we compute the number of admissible permutations for fixed p, q, then for a fixed p with arbitrary q.

Lemma 1. *Let n be an odd number and $A_n(p,q)$ be the set of permutations $\pi = h\rho^{-1} \in AP_n$ with $h1 = p$ and $\rho n = q$. Then $|A_n(p,q)| = K_n$ for any $p, q \in (1, 3, \ldots, n\}$.*

Proof. There are $\frac{n-1}{2}!$ possiblities to choose the order of the edges of F_1 (or F_2) in the alternating H-path and for each edge there are two possible orientations to fit them in. Then, each H-path is induced by $2^{\frac{n-1}{2}} \cdot \frac{n-1}{2}!$ permutations in $A_n(p,q)$ which gives the total number of $2^{n-1}(\frac{n-1}{2}!)^2 = K_n$. $\diamondsuit$

Lemma 2. *Let n be odd and $\overline{A_n}(p)$ the set of all $\pi = h\rho^{-1} \in AP_n$ with $h1 = p$. Then $|\overline{A_n}(p)| = 2(1-(\frac{1}{2})^{\frac{n+1}{2}})K_n$ for any $p = \{1, 3, \ldots, n\}$.*

Proof. We now write $F(p)$, $F_\pi(q)$ instead of F_1, F_2 in (4), in order to express theire dependence on p, q and π. Let $\pi = h\rho^{-1} \in A_n(p,q)$ with $q \geq 3$, thus $F(p) \times F_\pi(q)$ is an alternating H-path. Then π belongs to $A_n(p, q-2)$ if and only if $F(p) \times F_\pi(q-2)$ is an alternating H-path, too, i.e. if and only if the substitution of the edge $[\pi(q-2), \pi(q-1)]$ by $[\pi(q-1), \pi q]$ produces another Hamiltonian path. This is the case if $\pi(q-1) = hi$, $\pi(q-2) = h(i+1)$ for some i (then $i = \rho^{-1}(q-1)$ is odd and $i+1 = \rho^{-1}(q-2)$ is even). If, on the contrary, $\rho^{-1}(q-1)$ is even, i.e. $\pi(q-2) = hi$, $\pi(q-1) = h(i+1)$ for some i, then deleting the edge $[\pi(q-2), \pi(q-1)]$ and linking $\pi(q-1)$ to πq splits the path into a path (possibly a single vertex) and a circle.

For any $\pi \in A_n(p,q)$ let $\pi' = \pi\tau_{q-1,q-2}$ where $\tau_{q-1,q-2}$ is the transposition interchanging $q-1$ with $q-2$. Then π' also is in $A_n(p,q)$, but $\pi' \in A_n(p, q-2)$ if and only if $\pi \notin A_n(p, q-2)$. Thus, to any $\pi \in A_n(p,q) \cap A_n(p, q-2)$ there corresponds a $\pi' \in A_n(p,q) - A_n(p, q-2)$ and vice-versa. Since $|A_n(p,q)| = |A_n(p,q-2)|$ by Lemma 1, exactly half of the elements of one set belong to the intersection of the two. Taking into account that π and π' as above either both belong to $A_n(p,r)$ for some $r > q$ or neither of them does, we infer that an analogous argument is valid for the sets $A_n(p,q) - \bigcup_{r>q} A_n(p,r)$ and $A_n(p,q-2) - \bigcup_{r>q} (p,r)$.

Now let $B_q = A_n(p,q) - \bigcup_{r>q} A_n(p,r)$ for any odd $q \leq n$, then $B_n = A_n(p,n)$, $B_{n-2} = A_n(p,n-2) - A_n(p,n)$, etc. Further $\overline{A_n}(p) = B_n \cup B_{n-2} \cup \ldots \cup B_1$, where these sets are pairwise disjoint, and $|B_{q-2}| = |(A_n(p,q-2) - \bigcup_{r>q} A_n(p,r)) - (A_n(p,q) - \bigcup_{r>q} A_n(p,r))| = \frac{1}{2}|A_n(p,q) - \bigcup_{r>q} A_n(p,r)| = \frac{1}{2}|B_q|$. Consequently, $|\overline{A_n}(p)| = |B_n| + |B_{n-2}| + \ldots + |B_1| = |A_n(p,n)(1 + \frac{1}{2} + \ldots + \frac{1}{2^{\frac{n-1}{2}}}) = K_n(2 - (\frac{1}{2})^{\frac{n-1}{2}}) = 2(1 - (\frac{1}{2})^{\frac{n+1}{2}})K_n$. $\diamondsuit$

As an immediate consequence we get the following lower bound for the relative frequency of admissible permutations.

Theorem 2. *For any odd $n, \alpha_n \geq 2(1 - (\frac{1}{2})^{\frac{n+1}{2}})k_n$.* $\diamondsuit$

Finally we deduce an upper bound of α_n by a similar method.

Theorem 3. *For any odd n, $\alpha_n \leq 2(1 - (\frac{1}{2})^{\frac{n+1}{2}})^2 k_n$. Therefore, $\lim_{n\to\infty} \alpha_n = 0$.*

Proof. For $\pi \in \overline{A_n}(p)$, $\pi' = \tau_{p-1,p-2}\pi$ also belongs to $\overline{A_n}(p)$. It suffices to interchange the vertices $p-1$ and $p-2$ in a corresponding H-path. Now let $\pi \in \overline{A_n}(n)$, so $\pi \in A_n(n,q)$ for some $q \in 1,2,\ldots,n)$. As in the proof of Lemma 2, $\pi \in A_n(n-2,q)$ holds if and only if $[n-2,n-1]$ can be substituted by $[n-1,n]$ so that another alternating H-path results. This is the case if and only if $h^{-1}(p-2)$ is even and $h^{-1}(p-1)$ is odd (then $p-2 = hi$, $p-1 = h(i+1)$ for some i). Otherwise, $\pi \notin A_n(n-2,q)$ but still $\pi \in A_n(n-2,q')$ is possible for same $q' \neq q$ (it is easy to find examples). Since $\pi \notin A_n(n-2,q)$ implies $\pi' \in A_n(n-2,q) \subseteq \overline{A_n}(n-2)$, at least half of the elements of $\overline{A_n}(n)$ belong also to $\overline{A_n}(n-2)$. Therefore, since $|\overline{A_n}(n) = |\overline{A_n}(n-2)|$ by Lemma 2, $\overline{A_n}(n-2) - \overline{A_n}(n)| \leq \frac{1}{2}|\overline{A_n}(n)|$. In the same way, with $C_p = \overline{A_n}(p) - \bigcup_{s>p} \overline{A_n}(s)$, we infer $|C_{p-2}| \leq \frac{1}{2}|C_p|$. Since $AP_n = C_n \cup C_{n-2} \cup \ldots \cup C_1$ with $C_n = \overline{A_n}(n)$ we obtain $A(n) = |AP_n| \leq |A_n\overline{(n)}|(1 + \frac{1}{2} + \ldots + \frac{1}{2^{\frac{n-1}{2}}}) = 4(1 - (\frac{1}{2})^{\frac{n+1}{2}})^2 K_n$. To complete the proof, we need only divide by $n!$. $\diamondsuit$

Remark 2. The upper bound of Theorem 3 is not strict (for small n, we obtain values greater than 1), but it is sufficient to prove that the relative frequency of admissible permutations converges to zero. Compare the table below where we indicate the upper and lower bounds of α_n according to Theorem 2 and 3 for some biologically relevant values of n.

n	lower bound	upper bound
5	0.9333334	1.633334
7	0.8571429	1.607143
9	0.7873017	1.525397
11	0.7272728	1.431818
13	0.6766568	1.342741
15	0.6340328	1.263112
17	0.5979067	1.193478
21	0.5402465	1.079985
25	0.4962781	0.992435
29	0.4614604	0.9228927
33	0.433052	0.8660973
39	0.3988169	0.7976331
45	0.3715961	0.743192

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Mathematica Pannonica
2/1 (1991), 19 – 35

SIMULTANEOUS EXTENSIONS OF PROXIMITIES, SEMI-UNIFORMITIES, CONTIGUITIES AND MEROTOPIES II*

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Received March 1990

AMS Subject Classification: 54 E 15, 54 E 17

Keywords: Symmetric closure, weakly separated closure, S_1-topology, (Riesz/ /Lodato) semi-uniformity, (Riesz/Lodato) contiguity, (Riesz/Lodato) merotopy, Cauchy filter, extension.

Abstract: Given compatible semi-uniformities (or contiguities, or merotopies) on some subspaces of a closure space, we are looking for a common extension of these structures.

Notations. In addition to the notations and conventions introduced in §0 (see in [1]), let $A^2 = A \times A$, $A^r = X \setminus A$ (for $A \subset X$); if $\mathcal{U}$

* Research supported by Hungarian National Foundation for Scientific Research, grant no. 1807.

is a semi-uniformity then $s\mathcal{U}$ denotes the collection of the symmetric entourages contained by $\mathcal{U}$.

2. Extending a family of semi-uniformities in a closure space

A. WITHOUT SEPARATION AXIOMS

2.1 If a family of semi-uniformities can be extended in a closure space then the closure is necessarily symmetric; this condition will turn out to be sufficient, too. We are going to construct the finest and the coarsest extension.

Definitions. For a family of semi-uniformities in a closure space,

a) Let $\mathcal{U}^0$ be these semi-uniformity on $\dot{X}$ for which the following entourages form a subbase $\mathcal{B}$:

(1) $$U_i^0 = U_i \cup (X^2 \setminus X_i^2) \qquad (i \in I,\ U_i s \in \mathcal{U}_i);$$
(2) $$U_{x,B} = \{x\}^{r2} \cup B^{r2} \qquad (x \in X,\ B \subset X,\ x \notin c(B)).$$

b) Let $\mathcal{U}^1$ consist of the entourges U on X that satisfy the following conditions:

(3) $$Ux \in \mathsf{v}(x) \qquad (x \in X);$$
(4) $$U|X_i \in \mathcal{U}_i \qquad (i \in I). \ \diamondsuit$$

$\mathcal{B}$ is a collection of symmetric entourages, and, assuming $X \neq \emptyset$, $\mathcal{B}$ is non-empty (take $B = \emptyset$ in (2)), so it is indeed a subbase for a semi-uniformity. $\mathcal{U}^0$ would not change if $s\mathcal{U}_i$ were replaced by $\mathcal{U}_i$ in (1). It is straightforward to check that $\mathcal{U}^1$ is a semi-uniformity, too.

Similarly to the convention introduced in §1 for proximities, we shall write, if necessary, $\mathcal{U}^k(c,\mathcal{U}_i)$, or even $\mathcal{U}^k(c,\{\mathcal{U}_i : i \in I\})$. In particular, $\mathcal{U}^k(c) = L$, $\mathcal{U}^k(c,\emptyset)$. $(k = 0,1)$. Analogous notations will be used for Riesz and Lodato semi-uniformities, as well for merotopies and contiguities.

Theorem. *A family of semi-uniformities in a symmetric closure space always has extensions; $\mathcal{U}^0$ is the coarsest and $\mathcal{U}^1$ the finest extension.*

Proof. 1° *$\mathcal{U}^0$ is coarser than $\mathcal{U}^1$.* It is enough to show that $\mathcal{B} \subset \mathcal{U}^1$.

$U_i^0 x = X$ if $x \in X_i^r$; otherwise $U_i^0 x = U_i x \cup X_i^r$, which is clearly a c-neighbourhood of x, since $U_i x \in \mathsf{s}_i(x)$. For $j \in I$,

$$U_i^0|X_j = (U_i|X_{ij}) \cup (X_j^2 \setminus X_{ij}^2) = (U_j|X_{ij}) \cup (X_j^2 \setminus X_{ij}^2)$$

holds with a suitable $U_j \in \mathcal{U}_j$ (by the accordance); hence $U_i^0|X_j \supset U_j$, and so $U_i^0|X_j \in \mathcal{U}_j$. This means that U_i^0 satisfies (3) and (4), i.e. $U_i^0 \in \mathcal{U}^1$.

$U_{x,B}y \in \mathsf{v}(y)$ for $y \in X$; this follows from $U_{x,B}x = B^r$ for $y = x$, from $U_{x,B} = X$ for $x \neq y \in B^r$, and from $U_{x,B}y = \{x\}^r$ for $y \in B$, because in the last case $c(\{y\}) \subset c(B)$, $x \notin c(\{y\})$, $y \notin c(\{x\})$. $U_{x,B}$ satisfies (4), too: $U_{x,B}|X_i = X_i^2$ if $x \in X_i^r$, and in case $x \in X_i$ we can choose a $U_i \in s\mathcal{U}_i$ with $U_i x \cap (B \cap X_i) = \emptyset$ (as $x \notin c(B) \supset c_i(B \cap X_i)$), and then $U_i \subset U_{x,B}|X_i$.

2° *$\mathcal{U}^1|X_i$ is coarser than $\mathcal{U}_i$.* This is evident from (4).

3° *$\mathcal{U}_i$ is coarser than $\mathcal{U}^0|X_i$.* If $U_i \in \mathcal{U}_i$ then $U_i = U_i^0|X_i \in \mathcal{B}|X_i \subset \subset \mathcal{U}^0|X_i$.

4° *$c(\mathcal{U}^1)$ is coarser than c.* This is clear from (3).

5° *c is coarser than $c(\mathcal{U}^0)$.* Observe that

$$\mathsf{v}(x) = \{U_{x,B}x : x \notin c(B)\}. \tag{5}$$

6° *$\mathcal{U}^0$ and $\mathcal{U}^0$ are extensions.* It follows from 1°, 4° and 5° that $\mathcal{U}^0$ and $\mathcal{U}^1$ are compatible, respectively from 1°, 2° and 3° that $\mathcal{U}^0|X_i = = \mathcal{U}_i = \mathcal{U}^1|X_i$.

7° *$\mathcal{U}^0$ is the coarsest extension.* Let $\mathcal{U}$ be another extension; we have to show that $\mathcal{B} \subset \mathcal{U}$.

For $U_i \in s\mathcal{U}_i$, choose $U \in \mathcal{U}$ such that $U|X_i = U_i$; now $U \subset U_i^0$, thus $U_i^0 \in \mathcal{U}$. If $x \notin c(B)$ then $Ux \cap B = \emptyset$ for some $U \in s\mathcal{U}$, and therefore $U \subset U_{x,B}$.

8° *$\mathcal{U}^1$ is the finest extension.* If $\mathcal{U}$ is another extension then each $U \in \mathcal{U}$ satisfies (3), because $\mathcal{U}$ is compatible, and (4), because $\mathcal{U}|X_i = = \mathcal{U}_i$. Hence $\mathcal{U} \subset \mathcal{U}^1$. $\diamondsuit$

2.2 a) Formulas analogous to 1.3 (1) and (2) are valid for semi-uniformities (and also for merotopies and contiguities). The proofs are essentially the same as the ones given in 1.3 for proximities. We are going to set out the categorical background of these formulas.

Let **C** and **D** be topological categories, and $F : \mathbf{D} \to \mathbf{C}$ a concrete functor. (In contrast to the situation outlined in the introduction of Part I, it is not necessary to assume here that F commutes with the restriction to subsets.) We denote the **C**-structures by c, and the **D**-structures by d (with indices when necessary), and use the conventions introduced in §0, except that a family of **D**-structures in a **C**-space is not required to be either compatible or accordant; in particular, $F(d)$ will also be written as $c(d)$. $c < c'$ denotes that c is coarser than c'.

Consider a family of **D**-structures in a **C**-space. We say that the **D**-structure d on X is a

00-overextension if it is just an extension;

01-overextension if $c = c(d)$ and $d_i < d|X_i \quad (i \in I)$;

10-overextension if $c < c(d)$ and $d_i = d|X_i \quad (i \in I)$;

11-overextension if $c < c(d)$ and $d_i < d|X_i \quad (i \in I)$.

A *pq-underextension* $(p, q = 0, 1)$ is defined in the same way, replacing $<$ by $>$.

Let now a non-empty family of **D**-structures in a **C**-space be fixed.

1° If $d^0[i]$ is the coarsest pq-overextension of $\{d_i\}$, and there exists a pq-overextension of the whole family then $\sup_i d^0[i]$ is the coarsest pq-overextension. (The proof is straightforward.)

2° Assume that (i) the empty family in (X, c) has a coarsest pq-overextension $d^0(c)$ (if $p = 0$ then this means that $d^0(c)$ is the coarsest compatible structure); (ii) each $\{d_i\}$ has a coarsest $1q$-overextension $d^{00}[i]$ *with respect to the indiscrete* **C**-*structure on* X; (iii) the whole family has a pq-overextension. Then

$$\sup\{d^0(c), \sup_i d^{00}[i]\} \tag{1}$$

is the coarsest pq-overextension. (The statement is more symmetrical than it looks to be: $d^0(c)$ is the coarsest $p1$-overextension if each d_i is replaced by the indiscrete **D**-structure on X_i.)

3° The analogue of 1° is valid for pq-underextensions.

4° In the analogue of 2° for pq-underextensions, the condition corresponding to (ii) is superfluous, since $d^{11}[i]$ always exists in a topological category: take the coproduct of d_i and the discrete structure on X_i^{τ}. (The reason for the difference is that 2° and 4° are not dual: subspaces have not been replaced by quotient spaces.)

Observe that Definition 2.1 gives $\mathcal{U}^0$ in the form (1), and $\mathcal{U}^1$ similarly as an infimum.

b) It is possible to deduce Theorems 1.1 and 1.2 from Theorem 2.1; this will be discussed in Part III, where a result on extending semi-uniformities in a proximity space will enable us to do the converse, too, i.e. to partly prove Theorem 2.1 in two steps, first extending the proximities $\delta(\mathcal{U}_i)$ in (X, c), and then the semi-uniformities in (X, δ^0) or (X, δ^1).

B. RIESZ SEMI-UNIFORMITIES IN A CLOSURE SPACE

2.3 If a family of semi-uniformities in a closure space has a Riesz extension then each semi-uniformity is Riesz, the closure is weakly separated, and the trace filters are Cauchy (because the neighbourhood filters in a Riesz semi-uniform space are Cauchy). We are going to prove that these conditions are sufficient, too.

Definition. For a family of semi-uniformities in a closure space, let

$$\mathcal{U}^1_R = \{U \in \mathcal{U}^1 : \Delta \subset \operatorname{Int} U\}. \ \Diamond \tag{1}$$

$\mathcal{U}^1_R$ is clearly a semi-uniformity.

Theorem. *A family of semi-uniformities in a weakly separated closure space has a Riesz extension iff the trace filters are Cauchy; if so then $\mathcal{U}^0$ is the coarsest and $\mathcal{U}^1_R$ the finest Riesz extension.*

Proof. Assume that the trace filters are Cauchy.

$1°$ *$\mathcal{U}^0$ is coarser than $\mathcal{U}^1_R$.* In view of $1°$ from the proof of Theorem 2.1, it is enough to show that $\Delta \subset \operatorname{Int} U$ holds for each $U \in \mathcal{B}$.

$\Delta \subset \operatorname{Int} U^0_i$, because for $x \in X$, there is an $A \in \mathsf{s}_i(x)$ such that $A^2 \subset U_i$, and then $B = A \cup X^r_i \in \mathsf{v}(x)$, thus $(x, x) \in \operatorname{Int} B^2$ and $B^2 \subset U^0_i$.

Similarly, $\Delta \subset \operatorname{Int} U_{x,B}$, because for $y \in X$, there is an $A \in \mathsf{v}(y)$ such that $A^2 \subset U_{x,B}$, namely

$$A = \begin{cases} B^r & \text{if } y \in c(\{x\}), \\ \{x\}^r & \text{if } y \notin c(\{x\}). \end{cases}$$

(If $y \in c(\{x\})$ then, c being weakly separated, from $x \notin c(B)$ we have $y \notin c(B)$, thus $B^r \in \mathsf{v}(y)$ indeed.)

$2°$ *$\mathcal{U}^1_R$ is a Riesz extension.* By $1°$, the evident statement $\mathcal{U}^1_R \subset \mathcal{U}^1$, and Theorem 2.1, $\mathcal{U}^1_R$ is an extension. The compatibility of $\mathcal{U}^1_R$ implies that it is Riesz, as Int in (1) is now the $c(\mathcal{U}^1_R) \times c(\mathcal{U}^1_R)$-interior.

$3°$ $\mathcal{U}^0$ is Riesz, too, because it is coarser than a Riesz semi-uniformity inducing the same closure. Given a Riesz extension $\mathcal{U}$, we have $\mathcal{U} \subset \mathcal{U}^1$ by Theorem 2.1, and the elements of $\mathcal{U}$ satisfy (1), thus $\mathcal{U} \subset \mathcal{U}^1_R$. On the other hand $\mathcal{U}^0 \subset \mathcal{U}$, again by Theorem 2.1. $\Diamond$

If $\{\mathrm{int}X_i : i \in I\}$ covers X and each $\mathcal{U}_i$ is Riesz then it is not necessary to assume that the trace filters are Cauchy: For $x \in X$ and $U_i \in \mathcal{U}_i$, take $j \in I$ with $x \in \mathrm{int}X_j$, and $U_j \in \mathcal{U}_j$ such that $U_j|X_{ij} = U_i|X_{ij}$. As $\mathcal{U}_j$ is Riesz, there is an $A \in \mathsf{s}_j(x)$ with $A^2 \subset U_j$. $x \in \mathrm{int}X_j$ implies that $A \in \mathsf{v}(x)$, thus $A \cap X_i \in \mathsf{s}_i(x)$, and $(A \cap X_i)^2 \subset U_i$.

Corollary. *A family of Riesz-semi-uniformities in a weakly separated closure space has a Riesz extension iff* $\mathcal{U}_i \subset \mathcal{U}^1_R(c)|X_i$ $(i \in I)$.

Proof. Just like the proof of Corollary 1.4. $\diamondsuit$

C. LODATO SEMI-UNIFORMITIES IN A CLOSURE SPACE

2.4 If a family of semi-uniformities in a closure space has a Lodato extension then each semi-uniformity is Lodato, the closure is an S_1-topology, and the trace filters are Cauchy. A modification of Example 1.8 shows that these conditions are not sufficient: replace δ_0 by the Euclidean uniformity $\mathcal{U}_0$ on X_0, and $\delta^1_R(c)$ by a Lodato semi-uniformity $\mathcal{V}$ compatible with it (e.g. by $\mathcal{U}^1_R(c)$); now $\mathcal{U}_0$ and $\mathcal{U}_1 = \mathcal{V}|X_1$ satisfy the necessary conditions, but if they had a Lodato extension $\mathcal{U}$ then the Lodato proximity $\delta(\mathcal{U})$ would extend δ_0 and δ_1. In Example 2.10, we shall define Lodato semi-uniformities in a closure space that do not have a Lodato extension, although the Lodato proximities induced by them do have one.

Notation. In a closure space (X, c), put

$$V_{x,B} = V_{x,B;X} = c(\{x\})^{r2} \cup c(B)^{r2} \tag{1}$$

for $x \in X$, $B \subset X$, $x \notin c(B)$. $\diamondsuit$

Lemma. *If c is weakly separated then $V_{x,B} = \mathrm{Int}\, U_{x,B}$; so if $\mathcal{U}$ is a compatible Lodato semi-uniformity then $V_{x,B} \in \mathcal{U}$.*

Proof. $V_{x,B} \subset \mathrm{Int}\, U_{x,B}$ is evident. Conversely, let $(y, z) \in \mathrm{Int}\, U_{x,B}$. If $y, z \notin c(\{x\})$ then clearly $(y, z) \in V_{x,B}$. If, say, $y \in c(\{x\})$ then take M, N such that $y \in \mathrm{int}\, M$, $z \in \mathrm{int}\, N$, and $M \times N \subset U_{x,B}$. Now $x \in M$ implies $N \subset B^r$, thus $z \in c(B)^r$. On the other hand, $y \in c(B)^r$ follows from the weak separatedness. Hence $(y, z) \in V_{x,B}$ again.

The second statement follows from the first one, using Theorem 2.1 applied to $I = \emptyset$. $\diamondsuit$

2.5 Definition. For a family of Lodato semi-uniformities in an S_1-space, let

(1) $$\mathcal{U}_L^1 = \{U \in \mathcal{U}^1 : \operatorname{Int} U \in \mathcal{U}^1\}.$$

In other words, the $c \times c$-open elements of $\mathcal{U}^1$ form a base for $\mathcal{U}_L^1$. ◇

Lemma. *For a family of Lodato semi-uniformities in an S_1-space, $\mathcal{U}_L^1$ is a compatible Lodato semi-uniformity; it is the finest one among those Lodato semi-uniformities $\mathcal{U}$ on X that induce a closure coarser than c, and for which $\mathcal{U}|X_i$ is coarser than $\mathcal{U}_i$ ($i \in I$).*

Proof. $\mathcal{U}_L^1$ is clearly a semi-uniformity. $\mathcal{U}_L^1 \subset \mathcal{U}^1$, so it follows from Theorem 2.1 that $\mathcal{U}_L^1|X_i$ is coarser than $\mathcal{U}_i$ and $c(\mathcal{U}_L^1)$ is coarser than c.

1° *$c(\mathcal{U}_L^1)$ is finer than c.* It suffices to see that $U_{x,B} \in \mathcal{U}_L^1$ ($x \in X$, $B \subset X$, $x \notin c(B)$). $V_{x,B}$ is clearly a $c \times c$-open entourage contained by $U_{x,B}$, so we have only to check that $V_{x,B} \in \mathcal{U}^1$.

2.1 (3) is satisfied, since $V_{x,B}y$ is open for $y \in X$.

To prove 2.1 (4), fix an $i \in I$. If $c(\{x\}) \cap X_i = \emptyset$ then $X_i^2 \subset V_{x,B}$, thus $V_{x,B}|X_i \in \mathcal{U}_i$ is now evident. Otherwise, pick a point $y \in c(\{x\}) \cap$ $\cap X_i$; then (c being an S$_1$-topology) $c(\{y\}) = c(\{x\})$ and $y \notin c(B) \supset$ $\supset A = c(B) \cap X_i$. As A is c_i-closed, Lemma 2.4 gives

$$V_{y,A;X_i} = (X_i \setminus c_i(\{y\}))^2 \cup (X_i \setminus A)^2 \in \mathcal{U}_i.$$

Now $V_{x,B}|X_i = V_{y,A;X_i}$ follows from $c_i(\{y\}) = c(\{y\}) \cap X_i = c(\{x\}) \cap$ $\cap X_i$. Thus $V_{x,B}|X_i \in \mathcal{U}_i$ again.

2° *$\mathcal{U}_L^1$ is Lodato.* We have established that $\mathcal{U}_L^1$ is compatible, so it is Lodato by (1) (since c is a topology).

3° *$\mathcal{U}_L^1$ is finest.* Let $\mathcal{U}$ be another Lodato semi-uniformity with $\mathcal{U}|X_i \subset \mathcal{U}_i$ ($i \in I$) and $c(\mathcal{U})$ coarser than c; we have to show that $\mathcal{U} \subset \mathcal{U}_L^1$. $\mathcal{U} \subset \mathcal{U}^1$ is evident; moreover, $\mathcal{U}$ has a base consisting of $c(\mathcal{U}) \times c(\mathcal{U})$-open entourages, which are then $c \times c$-open, too. ◇

2.6 Definition. For a family of Lodato semi-uniformities in an S$_1$-space, let $\mathcal{U}_L^0$ be the filter on X^2 generated by the subbase $\mathcal{B}_L$ consisting of the following sets:

$$\begin{array}{ll} \operatorname{Int} U_i^0 & (i \in I,\ U_i \in s\mathcal{U}_i); \\ V_{x,B} & (B \subset X,\ x \in c(B)^r). \ \Diamond \end{array}$$

The elements of $\mathcal{B}_L$ are symmetric, thus $\mathcal{U}_L^0$ is a semi-uniformity iff each $\operatorname{Int} U_i^0$ is an entourage, i.e. iff the trace filters are Cauchy. It does not change $\mathcal{U}_L^0$ iff $s\mathcal{U}_i$ is replaced by $\mathcal{U}_i$ and/or $V_{x,B}$ by $c(\{x\})^{r2} \cup B^{r2}$. Observe that

(1) $$\mathcal{B}_L = \{\operatorname{Int} U : U \in \mathcal{B}\},$$

$\{\operatorname{Int} U \in \mathcal{U}^0\}$ is a base for $\mathcal{U}_L^0$, and $\mathcal{U}^0 \subset \mathcal{U}_L^0$.

Lemma. *If a family of Lodato semi-uniformities is given in an S_1-space, and the trace filters are Cauchy then $\mathcal{U}_L^0$ is the coarsest one among those compatible Lodato semi-uniformities $\mathcal{U}$ on X for which $\mathcal{U}|X_i$ is finer than $\mathcal{U}_i$ $(i \in I)$.*

Proof. Theorem 2.1 and $\mathcal{U}^0 \subset \mathcal{U}_L^0$ imply that $\mathcal{U}_L^0|X_i$ is finer than $\mathcal{U}_i$ and $c(\mathcal{U}_L^0)$ is finer than c. $c(\mathcal{U}_L^0)$ is also coarser than c, since the elements of the subbase $\mathcal{B}_L$ are open; hence $\mathcal{U}_L^0$ is compatible and Lodato.

Let $\mathcal{U}$ be another compatible Lodato semi-uniformity with $\mathcal{U}_i \subset \subset \mathcal{U}|X_i$. Now $U_i^0 \in \mathcal{U}$, and so $\operatorname{Int} U_i^0 \in \mathcal{U}$ (as $\mathcal{U}$ is Lodato). $V_{x,B} \in \mathcal{U}$ by Lemma 2.4. ◇

2.7 Lemma. *A family of Lodato semi-uniformities in an S_1-space has a Lodato extension iff $\mathcal{U}_L^0 \subset \mathcal{U}_L^1$; if so then both $\mathcal{U}_L^0$ and $\mathcal{U}_L^1$ are Lodato extensions.*

Proof. Lemmas 2.5 and 2.6, using that if $\mathcal{U}_L^0 \subset \mathcal{U}_L^1$ then the elements of $\mathcal{U}_L^0$ are entourages, thus the trace filters are Cauchy. ◇

Theorem. *A family of Lodato semi-uniformities in an S_1-space has a Lodato extension iff the trace filters are Cauchy, and for any $i, j \in I$,*

$$(1) \qquad (\operatorname{Int} U_i^0)|X_j \in \mathcal{U}_j \qquad (U_i \in \mathcal{U}_i);$$

if so then $\mathcal{U}_L^0$ is the coarsest and $\mathcal{U}_L^1$ is the finest Lodato extension.

Remark. The accordance can be written in the following equivalent form: $U_i^0|X_j \in \mathcal{U}_j$ for $i, j \in I$, $U_i \in \mathcal{U}_i$, of which (1) is clearly a strengthening. For $i = j$, (1) is equivalent to the statement that $\mathcal{U}_i$ is Lodato, so it was in fact superfluous to assume that the semi-uniformities are Lodato.

Proof. The necessity is obvious. By Lemma 2.6, the sufficiency will also follow if we show that $\mathcal{U}_L^0|X_j \subset \mathcal{U}_j$, i.e. that $\mathcal{B}_L|X_j \subset \mathcal{U}_j$ $(j \in I)$. For $\operatorname{Int} U_i^0$, this is just (1). $V_{x,B} \in \mathcal{U}^1$ was checked in 1° of the proof of Lemma 2.5, so $V_{x,B}|X_j \in \mathcal{U}_j$ by Theorem 2.1. The remaining statements follow from Lemmas 2.7, 2.6 and 2.5. ◇

Corollary. *A family of semi-uniformities in an S_1-space has a Lodato extension iff $\{\mathcal{U}_i, \mathcal{U}_j\}$ has a Lodato extension for any $i, j \in I$.* ◇

2.8 Corollary. *A single Lodato semi-uniformity given in an S_1-space has a Lodato extension iff the trace filters are Cauchy.* ◇

2.9 Theorem. *Let a family of Lodato semi-uniformities be given in*

an S_1-space. Assume that either each X_i is open and the trace filters are Cauchy or each X_i is closed. Then there exists a Lodato extension.
Proof. We are going to check that if $U_i \in \mathcal{U}_i$ is open then

$$\text{(2)} \qquad \operatorname{Int} U_i^0 \supset \operatorname{Int}_j (U_i^0 | X_j);$$

this is sufficient for 2.7 (1), because the open entourages form a base for $\mathcal{U}_i$ (as $\mathcal{U}_i$ is Lodato), and the right hand side of (1) belongs to $\mathcal{U}_j$ (as the semi-uniformities are accordant, and $\mathcal{U}_j$ is Lodato). Take (x, y) from the right hand side of (1).

1° Let X_i be closed. If $x, y \in X_{ij}$ then, U_i being open, we can pick c-open sets $A \ni x$ and $B \ni y$ such that $(A \times B)|X_i \subset U_i$, which implies that $A \times B \subset U_i^0$; thus $(x, y) \in \operatorname{Int} U_i^0$. If, say, $x \in X_j \setminus X_i$ then $(x, y) \in X_i^r \times X$, which is a $c \times c$-open set contained by U_i^0.

2° If X_j is open then there are c-open sets $A \ni x$ and $B \ni y$ such that $A, B \subset X_j$ and $A \times B \subset U_i^0 | X_j \subset U_i^0$. ◇

The analogue of Theorem 1.13 is not valid for semi-uniformities (although it holds for merotopies and contiguities, see Theorems 3.8 and 4.5), not even under the stronger assumption

$$\text{(2)} \qquad c(X_i \setminus X_j) \cap c(X_j \setminus X_i) = \emptyset \qquad (i, j \in I):$$

Example. Let $H =]0,1[$, $T = \{0\} \cup \{1/n : n \in \mathbb{N}\}$,

$$X = (T \cup [2,3[) \times H, X_0 = X \setminus (\{0\} \times H), X_1 = X \setminus (]2,3[\times H).$$

Let c be the Euclidean topology on X, and $\{U_i(\varepsilon) : \varepsilon > 0\}$ a base for $\mathcal{U}_i$ on X_i $(i = 0, 1)$, where, with $P \otimes Q$ denoting $(P \times Q) \cup (Q \times P)$,

$$U_0(\varepsilon) = U(\varepsilon)|X_0 \cup \bigcup_{n \in \mathbb{N}} ((\{1/n\} \times]0, \varepsilon[) \otimes ([2, 2 + 1/n[\times]0, \varepsilon[)),$$
$$U_1(\varepsilon) = U(\varepsilon)|X_1 \cup ((T \times]0, \varepsilon[) \otimes (\{2\} \times]0, \varepsilon[)),$$

and, for $x, y \in X$, $x\, U(\varepsilon)\, y$ iff the Euclidean distance of x and y is $< \varepsilon$. $\{\mathcal{U}_0, \mathcal{U}_1\}$ is a family of Lodato semi-uniformities in (X, c), the trace filters are Cauchy, and (2) holds. But there is no Lodato extension: 2.7 (1) fails for $i = 0$, $j = 1$, $U_i = U_0(1)$, since

$$((0, \varepsilon/2), (2, \varepsilon/2)) \in U_1(\varepsilon) \setminus (\operatorname{Int} U_0(1)^0)|X_1. \ \Diamond$$

2.10 By Theorem 1.13, the induced proximities have a Lodato extension in the above example. We can give, however, a much simpler example with this property:

Example. With X, X_0, X_1 and c as in Example 1.8, let $\mathcal{U}_1$ be the

Euclidean uniformity on X_1, $\mathcal{U}_0$ the precompact uniformity compatible with the Euclidean proximity on X_0. Now the trace filters are Cauchy, and the induced proximities have a Lodato extension (namely the Euclidean proximity on X), but $\mathcal{U}_0$ and $\mathcal{U}_1$ do not have one, since 2.7 (1) fails for $i = 1$ and $j = 0$. $\diamondsuit$

2.11 Concerning extensions of a single *uniformity*, see [6], [5], [7], [2] §5, [3], [4] §2. The same can be said about simultaneous extensions of uniformities as in the case of Efremovich proximities, cf. 1.16; see [4] Remark 1.13 c) and Example 1.13 b) for details.

3. Extending a family of merotopies in a closure space

A. WITHOUT SEPARATION AXIOMS

3.1 If a family of merotopies can be extended in a closure space then the closure is symmetric; this condition will be proved to be sufficient, too. Definitions, results and proofs are very similar to those in §2.

Definitions. For a family of merotopies in a closure space,

a) Let M^0 be the merotopy on X for which the following covers form a subbase B:

$$\text{(1)} \qquad \mathsf{c}_i^0 = \{C_i^0 = C_i \cup X_i^r : C_i \in \mathsf{c}_i\} \qquad (i \in I, \mathsf{c}_i \in \mathsf{M}_i);$$
$$\text{(2)} \qquad \mathsf{c}_{x,B} = \{\{x\}^r, B^r\} \qquad (B \subset X, x \in c(B)^r).$$

b) Let M^1 consist of the covers c of X that satisfy the following conditions:

$$\text{(3)} \qquad \mathrm{St}\,(x, \mathsf{c}) \in \mathsf{v}(x) \qquad (x \in X);$$
$$\text{(4)} \qquad \mathsf{c}|X_i \in \mathsf{M}_i \qquad (i \in I)\ \diamondsuit.$$

Theorem. *A family of merotopies in a symmetric closure space always has extensions;* M^0 *is the coarsest and* M^1 *the finest extension.*

Proof. 1° M^0 *is coarser than* M^1. It is enough to show that $\mathsf{B} \subset \mathsf{M}^1$.

If $x \in X_i^r$ then $\mathrm{St}(x, \mathsf{c}_i^0) = X \in \mathsf{v}(x)$; otherwise $\mathrm{St}(x, \mathsf{c}_i^0) = \mathrm{St}(x, \mathsf{c}_i) \cup X_i^r \in \mathsf{v}(x)$, since $\mathrm{St}(x, \mathsf{c}_i) \in \mathsf{s}_i(x)$. It follows easily from the accordance that c_i^0 satisfies (4), too. Thus $\mathsf{c}_i^0 \in \mathsf{M}^1$.

$\mathrm{St}(y, \mathsf{c}_{x,B})$ is equal to B^r if $y = x$, to X if $x \neq y \in B^r$, and to

$\{x\}^r$ if $y \in B$; thus it belongs to $\mathsf{v}(y)$, in the last case by the symmetry of c. $\mathsf{c}_{x,B}$ satisfies (4), too: if $x \in X_i^r$ then $\{X_i\} \in \mathsf{c}_{x,B}|X_i$; otherwise pick $\mathsf{c}_i \in \mathsf{M}_i$ with $\mathrm{St}(x, \mathsf{c}_i) \cap B = \emptyset$, and then c_i refines $\mathsf{c}_{x,B}|X_i$.

2° M^0 *and* M^1 *are extensions.* Just like in the proof of Theorem 2.1, replacing 2.1 (5) by

$$(5) \qquad \mathsf{v}(x) = \{\mathrm{St}(x, \mathsf{c}_{x,B}) : x \notin c(B)\}.$$

3° M^0 *is coarsest,* M^1 *is finest.* Check that if M is an extension then $\mathsf{B} \subset \mathsf{M}$, and, on the other hand, each $\mathsf{c} \in \mathsf{M}$ satisfies (3) and (4). ◊

B. RIESZ MEROTOPIES IN A CLOSURE SPACE

3.2 If a family of merotopies in a closure space has a Riesz extension then the merotopies are Riesz, the closure is weakly separated, and the trace filters are Cauchy. These conditions are also sufficient.

Definition. For a family of merotopies in a closure space, let

$$(1) \qquad \mathsf{M}^1_R = \{\mathsf{c} \in \mathsf{M}^1 : \mathrm{int}\,\mathsf{c} \text{ is a cover of } X\}.$$

Observe that

$$(2) \qquad \mathrm{int}\,\mathsf{c}_{x,B} = \{c(\{x\})^r, c(B)^r\}.$$

Theorem. *A family of merotopies in a weakly separated closure space has a Riesz extension iff the trace filters are Cauchy; if so then* M^0 *is the coarsest and* M^1_R *the finest Riesz extension.*

Proof. Assume that the trace filters are Cauchy.

1° M^0 *is coarser than* M^1_R. By $\mathsf{M}^0 \subset \mathsf{M}^1$, it is enough to show that $\mathrm{int}\,\mathsf{c}$ is a cover for $\mathsf{c} \in \mathcal{B}$.

$\mathrm{int}\,\mathsf{c}_i^0$ is a cover, because the trace filters are Cauchy. $\mathrm{int}\,\mathsf{c}_{x,B}$ is also a cover, since c is weakly separated, and so $c(\{x\}) \cap c(B) = \emptyset$.

2° The remaining statements can be proved in the same way as in 2° and 3° in the proof of Theorem 2.3, replacing entourages by covers and Int by int. ◊

Corollary. *A family of Riesz merotopies in a weakly separated closure space has a Riesz extension iff* $\mathsf{M}_i \subset \mathsf{M}^1_R(c)|X_i$ $(i \in I)$. ◊

C. LODATO MEROTOPIES IN A CLOSURE SPACE

3.3 If a family of merotopies in a closure space has a Lodato extension then the merotopies are Lodato, the closure is an S_1-topology, and the trace filters are Cauchy. Example 1.8 can be modified for merotopies in the same way as for semi-uniformities (cf. 2.4) showing that the above conditions are not sufficient; a better example will be given in 3.8.

Notation. $\mathsf{d}_{x,B} = \mathsf{d}_{x,B;X} = \operatorname{int} \mathsf{c}_{x,B}$ for $B \subset X$ and $x \in c(B)^r$ (cf. 3.2 (2)). ◇

Lemma. *If* M *is a compatible Lodato merotopy then* $\mathsf{d}_{x,B} \in \mathsf{M}$. ◇

3.4 Definition. For a family of Lodato merotopies in an S_1-space, let

$$\mathsf{M}^1_L = \{\mathsf{c} \in \mathsf{M}^1 : \operatorname{int} \mathsf{c} \in \mathsf{M}^1\}.$$

In other words, the open covers contained by M^1 form a base for M^1_L. ◇

Lemma. *For a family of Lodato merotopies in an* S_1*-space,* M^1_L *is a compatible Lodato merotopy; it is the finest one among those Lodato merotopies* M *on* X *that induce a closure coarser than* c*, and for which* $\mathsf{M}|X_i$ *is coarser than* M_i $(i \in I)$.

Proof. The argument runs along the same lines as the proof of Lemma 2.5, therefore we confine ourselves to showing that $c(\mathsf{M}^1_L)$ is finer than c. It is enough to see that $\mathsf{d}_{x,B} \in \mathsf{M}^1$, because then $\mathsf{c}_{x,B} \in \mathsf{M}^1_L$, and 3.1 (5) can be applied. 3.1 (3) is satisfied, since $\mathsf{d}_{x,B}$ is an open cover.

If $c(\{x\}) \cap X_i = \emptyset$ then $\{X_i\} \in \mathsf{d}_{x,B}|X_i$ thus $\mathsf{d}_{x,B}|X_i \in \mathsf{M}_i$. Otherwise, pick a point $y \in c(\{x\}) \cap X_i$. Now $c(\{x\}) = c(\{y\})$, thus $\mathsf{d}_{x,B} = \mathsf{d}_{y,B}$. But

$$\mathsf{d}_{y,B}|X_i = \mathsf{d}_{y,c(B)}|X_i = \mathsf{d}_{y,c(B)\cap X_i;X_i} \in \mathsf{M}_i$$

by Lemma 3.3. So $\mathsf{d}_{x,B}|X_i \in \mathsf{M}_i$ again, i.e. 3.1 (4) is fullfilled, too. ◇

3.5 Definition. Given a family of Lodato merotopies in an S_1-space such that the trace filters are Cauchy, let M^0_L be the merotopy on X for which the following covers form a subbase B_L:

$$\begin{array}{ll} \operatorname{int} \mathsf{c}^0_i & (i \in I, \mathsf{c}_i \in \mathsf{M}_i); \\ \mathsf{d}_{x,B} & (B \subset X, x \in c(B)^r). \ \diamond \end{array}$$

B_L is indeed a subbase for a merotopy ($\operatorname{int} \mathsf{c}^0_i$ is a cover, because the trace filters are Cauchy; this condition could be dropped as in Definition 2.6, but then the notion of a subbase had to be generalized from covers to arbitrary collections). We have $\mathsf{B}_L = \{\operatorname{int} \mathsf{c} : \mathsf{c} \in \mathsf{B}\}$. $\{\operatorname{int} \mathsf{c} : \mathsf{c} \in \mathsf{M}^0\}$ is a base for M^0_L.

Lemma. *If a family of Lodato merotopies is given in an* S_1*-space, and the trace filters are Cauchy then* M^0_L *is the coarsest one among those* *compatible Lodato merotopies* M *on* X *for which* $\mathsf{M}|X_i$ *is finer than* M_i $(i \in I)$.

Proof. Similar to the proof of Lemma 2.6. ◊

3.6 Lemma. *A family of Lodato merotopies in an* S_1*-space has a Lodato extension iff the trace filters are Cauchy and* $\mathsf{M}_L^0 \subset \mathsf{M}_L^1$*; if so then both* M_L^0 *and* M_L^1 *are Lodato extensions.* ◊

Theorem. *A family of Lodato merotopies in an* S_1*-space has a Lodato extension iff the trace filters are Cauchy, and, for any* $i, j \in I$,

$$(1) \qquad (\operatorname{int} \mathsf{c}_i^0)|X_j \in \mathsf{M}_j \qquad (\mathsf{c}_i \in \mathsf{M}_i);$$

if so then M_L^0 *is the coarsest and* M_L^1 *is the finest Lodato extension.*

Remark. The accordance of merotopies can be written in the following form: $\mathsf{c}_i^0|X_j \in \mathsf{M}_j$.

Proof. Similar to the proof of Theorem 2.7, using that $\mathsf{d}_{x,B} \in \mathsf{M}^1$ was established in the proof of Lemma 3.4. ◊

Corollary. *A family of merotopies in an* S_1*-space has a Lodato extension iff* $\{\mathsf{M}_i, \mathsf{M}_j\}$ *has a Lodato extension for any* $i, j \in I$. ◊

3.7 Corollary. *A single Lodato merotopy in an* S_1*-space has a Lodato extension iff the trace filters are Cauchy.* ◊

3.8 Theorem. *Let a family of Lodato merotopies be given in an* S_1*-space, assume that the trace filters are Cauchy, and*

$$(1) \qquad c(X_i \setminus X_j) \cap (X_j \setminus X_i) = \emptyset \qquad (i, j \in I).$$

Then there exists a Lodato extension.

Proof. To prove 3.6 (1), it is enough to show that if $\mathsf{c}_i \in \mathsf{M}_i$ is open (which may be assumed, as M_i is Lodato) then $\operatorname{int}_j(\mathsf{c}_i^0|X_j)$ is a refinement of $(\operatorname{int} \mathsf{c}_i^0)|X_j$, because the former belongs to M_j by the accordance and the Lodato property of M_j. The above statement is a consequence of

$$(2) \qquad \operatorname{int}_j(G_i^0 \cap X_j) \subset \operatorname{int} G_i^0,$$

where G_i is c_i-open.

For the proof of (2), take a point x from the left hand side of it. If $x \in X_{ij}$ then $x \in G_i$, implying $x \in \operatorname{int} G_i^0$. If $x \in X_j \setminus X_i$ then pick a c-open set H such that $x \in H$ and $H \cap X_j \subset G_i^0$; we may assume by (1) that $H \cap (X_i \setminus X_j) = \emptyset$, thus $H \cap X_i \subset G_i^0$, implying $H \subset G_i^0$. ◊

Corollary. *Let a family of Lodato merotopies be given in an* S_1*-space.*

Assume that either each X_i is open and the trace fileters are Cauchy or each X_i is closed. Then there exists a Lodato extension. $\diamondsuit$

Example. Take $S = \{1/n : n \in \mathbb{N}\}$, $X = S \times (\{0\} \cup S)$, $X_0 = S \times \{0\}$, $X_1 = X_0^r$. Let c be the Euclidean topology (inherited from $\mathbb{R}^2$) on X, and M_0 the merotopy on X_0 that consists of all the covers containing at least one cofinite set. For $\varepsilon > 0$, consider the cover

$$(3) \qquad \begin{aligned} \mathfrak{c}_1(\varepsilon) = \{(]p, p+\varepsilon[\times]q, q+\varepsilon[) \cap X_1 : 0 \le p < 1,\ 0 < q < 1\} \cup \\ \cup \{(\{1/n\} \times]0, \varepsilon[) \cap X_1 : n \in \mathbb{N}\}, \end{aligned}$$

and let $\{\mathfrak{c}_1(\varepsilon) : \varepsilon > 0\}$ form a base for the merotopy M_1 on X_1. Both merotopies are compatible and Lodato; they are evidently accordant; the trace filters are Cauchy by the second line of (3). $\mathcal{U}(\mathsf{M}_0)$ and $\mathcal{U}(\mathsf{M}_1)$ have a common extension, namely the Euclidean uniformity on X. Let M be the Euclidean merotopy on X, which means that $\{\mathfrak{c}(\varepsilon) : \varepsilon > 0\}$ is a base for M, where

$$\mathfrak{c}(\varepsilon) = \{(]p, p+\varepsilon[\times]q, q+\varepsilon[) \cap X : p, q \in \mathbb{R}\}.$$

Now $\Gamma(\mathsf{M})$ is an extension of $\Gamma(\mathsf{M}_0)$ and $\Gamma(\mathsf{M}_1)$. And yet, M_0 and M_1 cannot be extended, as 3.6 (1) is not fulfilled for $i = 1$, $j = 0$ and $\mathfrak{c}_i = \mathfrak{c}_1(1)$. $\diamondsuit$

4. Extending a family of contiguities in a closure space

A. WITHOUT SEPARATION AXIOMS

4.1 The exact counterparts of the results from §3 hold for contiguities. It is in fact possible to do the proofs all over again, inserting the word "finite" in appropriate places; it will be, however, simpler to deduce the results for contiguities from those for merotopies. We shall need some elementary (and well-known) facts about the connexion between contiguities and merotopies (the special case for $I = \emptyset$ of an extension problem to be discussed in Part IV):

Each contiguity Γ can be induced by a coarsest merotopy $\mathsf{M}^0(\Gamma)$, for which Γ (or any base for Γ) is a base; a merotopy of this form (i.e. one that has a base consisting of finite covers) is called *contigual.* Γ is Riesz or Lodato iff $\mathsf{M}^0(\Gamma)$ has the same property. The function

$\Gamma \mapsto \mathsf{M}^0(\Gamma)$ gives a one-to-one correspondence between contiguities and contigual merotopies, keeps the relation finer/coarser, and commutes with the restriction to a subset as well as with taking the induced closure.

If a family of contiguities can be extended in a closure space then the closure is symmetric; similarly to the case of merotopies (and other structures), this condition is sufficient, too.

Definitions. For a family of contiguities in a closure space,

a) Let Γ^0 be the contiguity on X for which the covers f_i^0 ($i \in \in I$, $\mathsf{f}_i \in \Gamma_i$) and $\mathsf{c}_{x,B}$ ($B \subset X, x \in c(B)^r$) form a subbase.

b) Let Γ^1 consist of the finite covers f of x that satisfy the following conditions: $\mathrm{St}(x, \mathsf{f}) \in \mathsf{v}(x)$ ($x \in X$) and $\mathsf{f}|X_i \in \Gamma_i$ ($i \in I$). ◇

In other words

$$(1) \qquad \Gamma^k = \Gamma(\mathsf{M}^k(c, \mathsf{M}^0(\Gamma_i))) \qquad (k = 0, 1).$$

Theorem. *A family of contiguities in a symmetric closure space always has extensions; Γ^0 is the coarsest and Γ^1 the finest extension.*

Proof. It follows from (1) and the foregoing observations that Γ^0 and Γ^1 are extensions. If Γ is an extension then $\mathsf{M}^0(\Gamma)$ is an extension of the merotopies $\mathsf{M}^0(\Gamma_i)$, thus

$$\mathsf{M}^0(c, \mathsf{M}^0(\Gamma_i)) \subset \mathsf{M}^0(\Gamma) \subset \mathsf{M}^1(c, \mathsf{M}^0(\Gamma_i)),$$

implying $\Gamma^0 \subset \Gamma \subset \Gamma^1$. ◇

B. RIESZ CONTIGUITIES IN A CLOSURE SPACE

4.2 If a family of contiguities in a closure space has a Riesz extension then the contiguities are Riesz, the closure is weakly separated, and the trace filters are Cauchy. These conditions are also sufficient.

Definition. For a family of contiguities in a closure space, let

$$\Gamma_R^1 = \{\mathsf{f} \in \Gamma^1 : \operatorname{int} \mathsf{f} \text{ is a cover of } X\}. \ ◇$$

This means that $\Gamma_R^1 = \Gamma(\mathsf{M}_R^1(c, \mathsf{M}^0(\Gamma_i)))$.

Theorem. *A family of contiguities in a weakly separated closure space has a Riesz extension iff the trace filters are Cauchy; if so then Γ^0 is the coarsest and Γ_R^1 the finest Riesz extension.*

Proof. Γ_i-Cauchy means the same as $\mathsf{M}^0(\Gamma_i)$-Cauchy. ◇

C. LODATO CONTIGUITIES IN A CLOSURE SPACE

4.3 If a family of contiguities in a closure space has a Lodato extension then the contiguities are Lodato, the closure is an S_1-topology, and the trace filters are Cauchy. These conditions are not sufficient: modify again Example 1.8, or see 4.5 for a better example.

Definitions. For a family of Lodato contiguities in an S_1-space,

a) Let $\Gamma^1_L = \{\mathsf{f} \in \Gamma^1 : \operatorname{int} \mathsf{f} \in \Gamma^1\}$.

b) Assuming that the trace filters are Cauchy, let Γ^0_L be the contiguity on X for which $\{\operatorname{int} \mathsf{f} : \mathsf{f} \in \Gamma^0\}$ is a base. $\Diamond$

Observe that $\Gamma^k_L = \Gamma(\mathsf{M}^k_L(c, \mathsf{M}^0(\Gamma_i)))$ $(k = 0, 1)$.

Lemma. *A family of Lodato contiguities in an S_1-space has a Lodato extension iff the trace filters are Cauchy and $\Gamma^0_L \subset \Gamma^1_L$; if so then both Γ^0_L and Γ^1_L are Lodato extensions.* $\Diamond$

Theorem. *A family of Lodato contiguities in an S_1-space has a Lodato extension iff the trace filters are Cauchy, and, for any $i, j \in I$,*

$$(\operatorname{int} \mathsf{f}^0_i)|X_j \in \Gamma_j \qquad (\mathsf{f}_i \in \Gamma_i); \tag{1}$$

if so then Γ^0_L is the coarsest and Γ^1_L the finest Lodato extension. $\Diamond$

Corollary. *A family of contiguities in an S_1-space has a Lodato extension iff $\{\Gamma_i, \Gamma_j\}$ has a Lodato extension for any $i, j \in I$.* $\Diamond$

4.4 Corollary. *A single Lodato contiguity given in an S_1-space has a Lodato extension iff the trace filters are Cauchy.* $\Diamond$

4.5 Theorem. *Let a family of Lodato contiguities be given in an S_1-space, assume that the trace filters are Cauchy, and 3.8 (1) holds. Then there exists a Lodato extension.* $\Diamond$

Corollary. *Let a family of Lodato contiguities be given in an S_1-space. Assume that either each X_i is open and the trace filters are Cauchy or each X_i is closed. Then there exists a Lodato extension.* $\Diamond$

Example. Let X, X_0, X_1, c and M_0 be as in Example 3.8. Take $\Gamma_0 = \Gamma(\mathsf{M}_0)$, and let $\{\mathsf{f}_1(k) : k \in \mathbb{N}\}$ be a subbase for Γ_1 on X_1, where

$$\begin{aligned}\mathsf{f}_1(k) = \{\{(1/m, 1/n) : m, n \geq k, m \not\equiv (\operatorname{mod} 3)\} : \mu = 0, 1, 2\} \cup \\ \cup\{\{(\{1/m, 1/n) : n \geq k\} : m < k\} \cup \\ \cup\{\{(1/m, 1/n) : m \geq k\} : n < k\} \cup \\ \cup\{\{(1/m, 1/n)\} : m, n < k\}.\end{aligned}$$

Now $\{\Gamma_0, \Gamma_1\}$ is a family of Lodato contiguities, the trace filters are

Cauchy, the induced proximities have a Lodato extension (the Euclidean one on X), but Γ_0 and Γ_1 do not have one, as 4.3 (1) fails for $i = 1$, $j = 0$, $\mathsf{f}_i = \mathsf{f}_1(1)$. $\diamondsuit$

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Mathematica Pannonica
2/1 (1991), 37 – 48

CENTRALIZER NEAR-RINGS ACTING ON SE-GROUPS

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Received November 1989

AMS Subject Classification: 16 A 76

Keywords: centralizer near-rings, matrix near-rings

Abstract: Let $N=C(A,G)$ be a centralizer near-ring determined by a group A of automorphisms of the group G such that the identity of N is the sum of a finite number of mutually orthogonal primitive idempotents, e_i. A group M is called an SE-group for N if N acts as a semigroup of endomorphisms on M with an aditional "strong" property for the idempotents, e_i. In this paper we investigate the structure of the centralizer near-ring $C(N,M)$ and as an application obtain a near-ring analogue to a well known matrix theory result.

1. Introduction

Let R be a ring with 1. Then R forms a left unital R-module ${}_RR$ and a right unital R-module R_R. With each of these R-modules we have the ring $\mathrm{End}_R({}_RR)$ of R-endomorphisms of ${}_RR$ and the ring $\mathrm{End}_R(R_R)$ of R-endomorphisms of R_R. It is easy to see that $\mathrm{End}_R({}_RR)$ is anti-isomorphic to R and $\mathrm{End}_R(R_R)$ is isomorphic to R.

This left-right duality for rings fails for near-rings. Let N be a (right) zero-symmetric near-ring with 1 which does not satisfy the left distributive law. Then ${}_NN$ is a left unital N-module (see Pilz [6] or Meldrum [4] for near-ring terminology and basic facts). However, since N does not satisfy the left distributive law, there exist elements n_1, n_2 and n_3 in N such that $n_1(n_2+n_3) \neq n_1n_2 + n_1n_3$, violating a right N-module axiom.

The near-ring analogy to $\mathrm{End}_R({}_RR)$ is the set $\mathrm{Map}_N({}_NN)$ consisting of all maps f from N into N such that $f(nm) = nf(m)$ for all n in N and all m in ${}_NN$. As in the ring case $\mathrm{Map}_N({}_NN)$ consists precisely of right multiplication maps by elements of N, but if N does not satisfy the left distributive law then, under function addition and function composition, $\mathrm{Map}_N({}_NN)$ does not form a near-ring since it is not closed under addition.

Using N_N we see that $\mathrm{Map}_N(N_N)$ consists of left multipliation maps by elements of N and it forms a (right) near-ring under function addition and function composition. The near-ring N acts on N_N as a semigroup (under function composition) of endomorphisms of the group $(N_N, +)$. For if n_1, n_2 are in N_N and n is in N then $(n_1+n_2)n = = n_1n + n_2n$. Since the left multiplication maps $\mathrm{Map}_N(N_N)$ are precisely the functions on N that commute with the right multiplication maps so $\mathrm{Map}_N(N_N)$ is the centralizer near-ring $C(N, N_N)$ where N acts on N_N as a semigroup of endomorphisms via right multiplication. (See [2] for details about centralizer near-rings $C(S, G)$ where S is a semigroup of endomorphisms of the group G.)

The prototype of a finite-ring is the centralizer near-ring $C(A, G)$ where G is a finite group and A is a group of automorphisms of G (see [2]). If $N = C(A, G)$ then the identity 1 of N is the sum of mutually orthogonal primitive idempotents, $1 = e_1 + e_2 + \cdots + e_t$. Moreover we have, for every i, j with $i \neq j, n(e_i + e_j) = ne_i + ne_j$ for all n in N, and $e_i + e_j = e_j + e_i$. This implies that if n belongs to N such that $ne_i = 0$ for every i then $n = 0$. With this in mind we have the following definition where $N = C(A, G)$.

Definition. If $N = C(A, G)$ with $1 = e_1 + e_2 + \cdots + e_t$ as above, then a group $(M, +)$ is an *SE-group (strong endomorphism group)* for N if

there is a composition $M \times N$ to M such that
(a) $(m_1 + m_2)n = m_1 n + m_2 n$ for every m_1, m_2 in M and n in N,
(b) $(mn_1)n_2 = m(n_1 n_2)$ for every m in M and n_1, n_2 in N,
(c) $m1 = m$ for every m in M,
(d) $m0 = 0$ for every m in M, and
(e) if m in M is sucht that $me_i = 0$ for all i, then $m = 0$.

Thus, when M is an SE-group for $N = C(A, G)$, the first four axioms require that N acts as a semigroup of endomorphisms on M with 1 acting as the identity map and 0 as the zero map, while the "strong" property (e) leads to $m(e_i + e_j) = me_i + me_j$ for all i, j with $i \neq j$.

We note that if M_1 and M_2 are SE-groups for N then so is $M_1 + M_2$. Also if R is a right ideal of N then R is an SE-group for N. In particular, N_N is an SE-group for N. Moreover, with each SE-group M for N we have the corresponding centralizer near-ring $C(N, M)$.

It is the purpose of this article to investigate the structure of the near-ring $C(N, M)$ where N is a finite centralizer near-ring of the type $C(A, G)$, A is a group of automorphisms of G and M is an SE-group for N. In the next section we focus on the case where $N = C(A, G)$ is a simple near-ring. In section 3 we present two general results and in the final section we use a theorem of A.P.J. van der Walt ([8]) to obtain a near-ring analogue of a well-known matrix theory result.

2. Structure of $C(N, M), N$ simple

In this section N represents a finite centralizer near-ring $C(A, G)$ where A is a group of automorphisms of the finite group G and $1 = e_1 + e_2 + \cdots + e_t$ where the e_i's are mutually orthogonal primitive idempotents. We recall that if N is simple then there exists a group G and a fixed point free group A of automorphisms of G such that N is isomorphic to $C(A, G)$.

Lemma 1. *Let $N = C(A, G)$ with $1 = e_1 + e_2 + \cdots + e_t$ and let M be an SE-group for N. If f belongs to $C(N, M)$ then*
(a) *$f(Me_i)$ is a subset of Me_i for every i and*
(b) *$f(m_1 e_1 + m_2 e_2 + \cdots + m_t e_t) = f(m_1 e_1) + f(m_2 e_2) + \cdots + f(m_t e_t)$ for all m_i in M.*

Proof. (a) For m in $M, f(me_i) = f(me_ie_i) = f(me_i)e_i$ which is in Me_i.

(b) $f(m_1e_1+m_2e_2+\cdots+m_te_t) = f(m_1e_1+m_2e_2+\cdots+m_te_t)(e_1+ +e_2+\cdots+e_t) = f(m_1e_1+m_2e_2+\cdots+m_te_t)e_1+f(m_1e_1+m_2e_2+\cdots+ +m_te_t)e_2+\cdots+f(m_1e_1+m_2e_2+\cdots+m_te_t)e_t = f(me_1)+f(me_2)+ +\cdots+f(me_t)$. $\diamondsuit$

Lemma 2. *If N is simple with $N = C(A,G)$ where A is fixed point free then f in $C(N,M)$ is completely determined by its action on the set Me_1.*

Proof. For $i \neq 1$ there exist elements e_{i1} and e_{1i} in N such that $e_{i1}e_{1i} = e_i$, $e_{1i}e_{i1} = e_1$, $e_{i1}e_1 = e_{i1}$ and $e_{1i}e_i = e_{1i}$. We have $Me_i = = Me_{i1}e_{1i}$ and so $f(me_i) = f(me_{i1}e_{1i}) = f(me_{i1})e_{1i}$. Since me_{i1} belongs to Me_1 so $f(me_{i1})$ is known and f is determined on Me_i. Since M is a sum of the Me_i's, f is determined on M by Lemma 1, part (b). We note that the extension of f is unique, for if $f(Me_1) = \{0\}$ then $f(me_i) = f(me_{i1})e_{1i} = 0$ and so f is the zero map. $\diamondsuit$

Our first theorem characterizes $C(N,M)$ when N is simple.

Theorem 1. *Let $N = C(A,G)$ be a finite simple near-ring where A is a fixed point free group of automorphisms of G. Let M be an SE-group for N. Then $C(N,M)$ is isomorphic to $C(N_{11}^*, Me_1)$ where N_{11}^* is the set of nonzero elements in e_1Ne_1 and acts on Me_1 by right multiplication.*

Proof. Define ψ from $C(N,M)$ to $C(N_{11}^*, Me_1)$ by $\psi(f) = f$ restricted to the set Me_1. By Lemma 1, $\psi(f)$ is a function on the group Me_1. Since f belongs to $C(N,M)$, $f(me_1)n_{11} = f(me_1n_{11})$ where n_{11} is in N_{11}^*. This means $\psi(f)$ belongs to $C(N_{11}^*, Me_1)$. The function ψ is one-to-one by Lemma 3. That ψ preserves sums and products in $C(N,M)$ is easily checked.

It remains to show that ψ is onto. To this end, select g in $C(N_{11}^*, Me_1)$. The function g is already defined on Me_1 and we need to extend g to all of M. Define g on Me_i as follows: $g(me_i) = g(me_{i1})e_{1i}$. We show that g is well defined. For suppose $m_1e_i = m_2e_i, m_1, m_2$ in M. Then $(m_1e_{i1} - m_2e_{i1})e_{1i} = 0$. Hence $(m_1e_{i1} - m_2e_{i1})e_1 = 0$ and since $(m_1e_{i1} - m_2e_{i1})e_j = 0$ for $j = 2, \ldots, t$, we have $m_1e_{i1} - m_2e_{i1} = 0$ by property (e) of the definitionn of SE-group. Extend g to all of M additively, that is $g(m) = g(me_1+me_2+\cdots+me_t) = g(me_1)+ +g(me_2)+\cdots+g(me_t)$. It remains to show that the extended function g belongs to $C(N,M)$, i.e. that $g(mn) = g(m)n$ for every m in M and

n in N. We have $g(mn) = g(mne_1) + g(mne_2) + \cdots + g(mne_t)$ and $g(m)n = g(m)ne_1 + g(m)ne_2 + \cdots + g(m)ne_t$, so it suffices to show that $g(mne_i) = g(m)ne_i$ for each i. Since N is a centralizer near-ring, $ne_i = e_jne_i$ for some index j (which depends on n). So we have

$$\begin{aligned} g(mne_i) &= g(me_jne_i) \\ &= g(me_{j1}e_{1j}ne_{i1}e_{1i}) \\ &= g(me_{j1}(e_{1j}ne_{i1}))e_{1i} \qquad \text{(definition of } g) \\ &= g(me_{j1}(e_{1j}ne_{i1})(ne_{1i} \qquad (g \text{ belongs to } C(N_{11}^*, Me_1)) \\ &= g(me_{j1})(e_{1j}ne_i) \\ &= g(me_{j1}e_{1j})ne_i \qquad \text{(definition of } g \text{ on } M_{e_j}) \\ &= g(me_j)ne_i \\ &= [g(me_1) + g(me_2) + \cdots + g(me_t)]ne_i \text{ (since } ne_i = e_jne_i \text{ and} \\ &\quad\ g(me_j) \text{ belongs to } Me_j \text{ for each } j) \\ &= g(m)ne_i. \ \diamondsuit \end{aligned}$$

We remark that since centralizer near-rings are zero-symmetric, $C(N_{11}^*, Me_1) = C(e_1Ne_1, Me_1)$. In the sequel we will often use this observation.

By specialzing M we obtain several applications of Theorem 1. Our first application is obtained by letting $M = N_N$.

Lemma 3. *Let N be the finite simple near-ring $C(A, G)$ where A is a fixed point free group of automorphisms of G and $1 = e_1 + e_2 + \cdots + e_t$, mutually orthogonal primitive idempotents. Then*

(a) *N_{11}^* is a multiplicative group anti-isomorphic to A,*

(b) *Ne_1 is an additive group isomorphic to G,*

(c) *N_{11}^* acts on Ne_1 by right multiplication as a fixed point free group of automorphisms, and*

(d) *$C(N_{11}^*, Ne_1)$ is isomorphic to $C(A, G)$.*

Proof. (a) Let v_1 be a nonzero element in G sucht that $e_1(v_1) = v_1$. Then v_1 belongs to Av_1 and $N_{11}^* = \{f \text{ in } N | f(v_1) \text{ belongs to } Av_1 \text{ and } f(w) = 0 \text{ for all } w \text{ not in } Av_1\}$. Since A is fixed point free it follows that for every σ in A there exists a unique f in N_{11}^* such that $f(v_1) = \sigma v_1$. Define ψ from N_{11}^* to A by $\psi(f) = \sigma$ where $f(v_1) = \sigma v_1$. If f, g are in N_{11}^* with $f(v_1) = \sigma v_1$ and $g(v_1) = \sigma' v_1$ then $(gf)(v_1) = g(\sigma v_1) = \sigma g(v_1) = \sigma\sigma' v_1$. So $\psi(gf) = \sigma\sigma' = \psi(f)\psi(g)$. The function ψ is clearly one-to-one and onto.

(b) Since A is fixed point free, if $\sigma \neq 1$ belongs to A and if v is nonzero in G then $\sigma(v) \neq v$. This implies (see [2]) that if v_1 is as in part (a) then for every w in G there exists a function f in N such that $f(v_1) = w$ and $f(v) = 0$ for all v not Av_1. Moreover f is unique with this property and f belongs to Ne_1. Define φ from Ne_1 to G by $\varphi(f) = w$. Now φ is easily seen to be an isomorphism of Ne_1 onto G.

(c) If n_{11} belongs to N_{11}^* then define the map $R_{n_{11}}$ from Ne_1 to Ne_1 by $R_{n_{11}}(ne_1) = ne_1n_{11}$. The map $R_{n_{11}}$ is clearly an endomorphism of the group $(Ne_1, +)$. Moreover it is an automorphism, for if $R_{n_{11}}(ne_1 = 0$ then $ne_1n_{11} = 0$. But N_{11}^* is a group with identity e_1 under multiplication so n_{11} has an inverse n_{11}^{-1} and $0 = 0n_{11}^{-1} = (ne_1n_{11})n_{11}^{-1} = ne_1$, which implies $R_{n_{11}}$ is one-to-one. Since Ne_1 is finite the map is onto.

To show N_{11}^* acts fixed point freely on Ne_1 suppose $R_{n_{11}}(ne_1) = ne_1 \neq 0$. Then $ne_1 = ne_1n_{11}$. We have $ne_1 = e_jne_1$ for some j. Since N is simple there exists m_{ij} in e_1Ne_j such that $m_{1j}ne_1 = e_1$. This means $e_1 = m_{1j}ne_1 = m_{1j}ne_1n_{11}$, and $R_{n_{11}} = R_{e_1}$ which is the identity map on N_{11}.

The correspondence $R_{n_{11}}$ to n_{11} is an anti-isomorphic of $\{R_{n_{11}} | n_{11}$ belongs to $N_{11}^*\}$ with N_{11}^* and since the latter is anti-isomorphic to A, $\{R_{n_{11}} | n_{11}$ belongs to $N_{11}^*\}$ is isomorphic to A.

(d) To show $C(N_{11}^*, Ne_1)$ and $C(A, G)$ are isomorphic it suffices to show that the pair (N_{11}^*, Ne_1) is isomorphic to the pair (A, G) by way of a semi-linear transformation ψ from Ne_1 onto G (see Maxson and Smith [3] or Ramakotaiah [7]). (Here N_{11}^* is identified with the right multiplication maps by elements of the set N_{11}^*.) Let ψ be the isomorphism from Ne_1 to G defined as in (b) and let β from $\{R_{n_{11}} | n_{11}$ belongs to $N_{11}^*\}$ to A be the isomorphism as developed in (c). Then $\psi(R_{n_{11}}f) = \psi(fn_{11}) = fn_{11}(v_1) = f(n_{11}v_1) = f(\beta v_1) = \beta f(v_1) = R_{n_{11}}^{\beta}\psi(f)$, and ψ is our desired one-to-one semi-linear transformation. $\diamondsuit$

This leads to the following application of Theorem 1.

Corollary 1. *Let N be a finite simple near-ring with $N = C(A, G)$ where A is fixed point free. Then $C(N, N_N)$ is isomorphic to N.*

Proof. By Theorem 1, $C(N, N_N)$ is isomorphic to $C(N_{11}^*, Ne_1)$ which is isomorphic to N by Lemma 3. $\diamondsuit$

Corollary 2. *Let N be a finite simple near-ring with $N = C(A, G)$ where A is fixed point free. If k is a positive integer let $N^k =$*

$= N \oplus N \oplus \cdots \oplus N$ *(k direct summands). Then* $C(N, N^k)$ *is isomorphic to* $C(A, G^k)$. *In particular* $C(N, N^k)$ *is simple.*

Proof. By Theorem 1, $C(N, N^k)$ is isomorphic to $C(N_{11}^*, N^k e_1)$. As in the proof of Lemma 3, $N^k e_1$ is isomorphic to G^k and N_{11}^* acts on $N^k e_1$ fixed point freely by right multiplication. Also as in the proof of Lemma 1 the pairs $(N_{11}^*, N^k e_1)$ and (A, G^k) are isomorphic via a semi-linear transformation. So $C(N, N^k)$ is isomorphic to $C(A, G^k)$ and since A acts fixed point freely on G^k, $C(N, N^k)$ is simple. $\diamond$

If $N = C(A, G)$ and if R is a right ideal of N then $M = R$ is an SE-group for N. Our next application of Theorem 1 deals with this situation. First we describe the right ideals in the simple near-ring $N = C(A, G)$.

Lemma 4. *Let* N *be a finite simple near-ring with* $N = C(A, G)$ *where* A *is fixed point free. A nonempty subset* R *of* N *is a right ideal of* N *if and only if there exists an* A*-invariant subgroup* H *of* G *such that* $R = e_H N$ *where* e_H *in* N *is the idempotent map on* G *which is the identity on* H *and zero off* H.

Proof. If H is an A-invariant subgroup of G and if $R = e_H N$ it is easily verified that R is a right ideal on N.

Now assume R is a right ideal of N. Let $H = \{w \text{ in } G \mid \text{ there is a } v \text{ in } G \text{ and an } f \text{ in } R \text{ with } f(v) = w\}$. To show H is an A-invariant subgroup of G select $w \neq 0$ in H. Then there exists a $v \neq 0$ in G and an f in R such that $f(v) = w$. For β in A we have $f(\beta v) = \beta f(v) = \beta w$, an element of H. So H is A-invariant. Since N is simple and A is fixed point free, it follows that for every $v \neq 0$ in G and any u in G there exists an n in N such that $n(v) = u$. So if w belongs to H with $g(u) = w$ where g is in R then $g(u) = gn(v) = w$ and gn belongs to R since R is a right ideal. This means $H = Rv$ for every nonzero v in G. But Rv is clearly a group, so H is a subgroup of G.

Let e_H be the idempotent in N which is the identity on H and 0 off H. We show now that e_H belongs to R. For $h \neq 0$ in H let e_h be the idempotent in N which is the identity on Ah and 0 elsewhere. Since $Rh = h$ and $e_h(h) = h$ we have $Re_h h = H$. The elements (maps) of Re_h are all 0 off Ah, so there exists an re_h in Re_h such that $re_h(h) = h$ and re_h is 0 off Ah. This means $re_h = e_h$ and e_h belongs to R. Since e_h belongs to R for all nonzero h in H and since H is finite, e_H belongs to R (e_H is the sum of e_h's, one h for each nonzero A-orbit in H). $\diamond$

Corollary 2. *Let* N *be a finite simple near-ring with* $N = C(A, G)$

where A is a fixed point free group of automorphisms. Let $R = e_H N$ be a right ideal of N. Then $C(N, R)$ is isomorphic to $C(N_{11}^*, Re_1)$ which in turn is isomorphic to $C(A, h)$, a simple near-ring.

Proof. That $C(N, R)$ is isomorphic to $C(N_{11}^*, Re_1)$ is clear from Theorem 1. To see that N_{11}^* acts fixed point freely on the group Re_1 by right multiplication it is enough to use Lemma 3, part (c) since Re_1 is a subset of Ne_1.

We have Re_1 isomorphic to H and as in the proof of Lemma 3, part (d) the pairs (N_{11}^*, Re_1) and (A, H) are isomorphic via a semi-linear transformation. So the simple near-ring $C(N_{11}^*, Re_1)$ is isomorphic to $C(A, H)$. $\diamondsuit$

Corollary 4. *Let N be a finite simple near-ring with $N = C(A, G)$ where A is fixed point free. If $R = e_H N$ is a right ideal of N then for any positive integer $k, C(N, R^k)$ is isomorphic to $C(N_{11}^*, (Re_1)^k)$ which in turn is isomorphic to $C(A, H^k)$, a simple near-ring.*

Proof. Similarly to that of Corollary 2. $\diamondsuit$

Corollary 5. *Let N be a finite simple near-ring with $N = C(A, G)$ where A is fixed point free. If $R = e_H N$ is a right ideal of N such that $(R, +)$ is a normal subgroup of $(N, +)$, then N acts on N/R by $(a + R)n = an + R$ and $C(N, N/R)$ is isomorphic to $C(N_{11}^*, (N/R)e_1)$ which in turn is isomorphic to the near-ring $C(A, G/H)$. Moreover $C(A, G/H)$ is simple.*

Proof. The first isomorphism is from Theorem 1. Since $(R, +)$ is normal in $(N, +)$ so H is normal in G. One checks that $(N_{11}^*, (N/R)e_1)$ and $(A, G/H)$ are isomorphic via a semi-linear transformation.

To see that $C(A, G/H)$ is simple it suffices to see that A acts fixed point freely on G/H. Suppose $\beta \neq 1$ belongs to A and that $\beta(v + H) = v + H$. This means $-v + \beta v$ belongs to H. We recall (see [1]) that a fixed point free automorphism β on a finite group G has the property that every x in G has the unique form $-x + \beta x$. Since β acts fixed point freely on H and since $-v + \beta v$ belongs to H we have $-v + \beta v = -w + \beta w$ for some w in H. This implies $v = w$ and $v + H = H$, i.e. $v + H$ is the identity element of G/H. $\diamondsuit$

Corollary 6. *Let N be a finite simple near-ring with $N = C(A, G)$ where A is fixed point free. If $R = e_H N$ is a right ideal of N such that $(R, +)$ is a normal subgroup of $(N, +)$ and if k is a positive integer then $C(N, (N/R)^k)$ is isomorphic to $C(N_{11}^*, (N/R)^k)$ which in turn is isomorphic to $C(A, (G/H)^k)$.*

Proof. Similar to that of Corollary 2. ◊

If N is simple and if M is an SE-group then $C(N,M)$ need not be simple as the following example shows.

Example. Let N be $GF(4)$, the finite field with 4 elements. The field N is clearly a simple near-ring and $N = C(A,G)$ where G is the group $(GF(4),+)$ and A is the fixed point free automorphism group on $GF(4)$ consisting of the right multiplication maps by the three nonzero elements of $GF(4)$.

Let $N = \{0,1,a,a^2\}$, then $A = \{1, R_a, R_{a^2}\}$. Let $M = S_3$, the symmetric group on three elements. Define the action of N on M as follows: if β is in S_3 then

$$\begin{aligned} \beta 0 &= (1) \\ \beta 1 &= \beta \\ \beta a &= (123)^{-1}\beta(123) \\ \beta a^2 &= (132)^{-1}\beta(132). \end{aligned}$$

So right multiplication by 0 is the zero endomorphism of S_3, by 1 is the identity map, by a is the automorphism which is conjugation by (123), and by a^2 is the automorphism which is conjugation by (132). With this action of N on M, M forms an SE-group for N. But $C(N,M) = C(N^*, S_3)$ is not simple since (123) in $M = S_3$ is fixed by all the nonzero elements N^* in N and (12) is not (so there is stabilizer containment, see [2]).

Let N be a finite near-ring with $N = C(A,G)$ where A is a fixed point free group of automorphisms of G. In N we have $1 = e_1 + e_2 + \cdots + e_u$ where the e_i's are mutually orthogonal primitive idempotents. Suppose the positive integer s is a proper divisor of u, say $u = ts$. Let

$$\begin{aligned} f_1 &= e_1 + e_2 + \cdots + e_s \\ f_2 &= e_{s+1} + e_{s+2} + \cdots + e_{2s} \\ &\cdots \\ f_t &= e_{(t-1)s+1} + e_{(t-1)s+2} + \cdots + e_{ts}, \end{aligned}$$

then $1 = f_1 + f_2 + \cdots + f_t$ where the f_i's are mutually orthogonal idempotents. If M is an SE-group for N and if m is any element in M then $m = m(f_1 + f_2 + \cdots + f_t) = mf_1 + mf_2 + \cdots + mf_1$. (For if m belongs to M, then for each i,$(m(f_1+f_2+\cdots+f_t) - mf_t - \cdots - mf_2 - mf_1)e_i = 0$. Since M is an SE-group, $m(f_1 + f_2 + \cdots + f_t) - mf_t - \cdots - mf_2 - mf_1 = 0$).

Theorem 2. *If N and M are as above then $C(N,M)$ is isomorphic to $C(f_1Nf_1, Mf_1)$.*
Proof. Since N is simple and A is fixed point free then for every i,j such that $i \neq j$ there exist elements e_{ij} in e_iNe_j such that $e_{ij}e_{ji} = e_i$. The proof of Theorem 2 is the same as that of Theorem 1 replacing e_i with f_i, and replacing $e_{1i}e_{i1}$ by

$$f_{1i} = e_{1,(i-1)s+1} + e_{2,(i-1)s+2} + \cdots + e_{s,is},$$
$$f_{i1} = e_{(i-1)s+1,1} + e_{(i-1)s+2,2} + \cdots + e_{is,s},$$

where we have $f_{1i}f_{i1} = f_1$ and $f_{i1}f_{1i} = f_i$. ◇

We mention two special situations for Theorem 2.

(a) *Let $M = N$, then $C(N, N_N)$ is isomorphic to each of the following: $C(e_1Ne_1, Ne_1)$, $C(f_1Nf_1, Nf_1)$ and N.*

(b) *Let $M = N^k$ (k, a positive integer), then $C(N, N^k)$ is isomorphic to each of the following: $C(e_1Ne_1, (Ne_1)^k)$ and $C(f_1Nf_1, (Nf_1)^k)$.*

3. Structure of $C(N,M)$, N not simple

Assume N is a finite semisimple near-ring where $N = N_1 \oplus N_2 \oplus \oplus \cdots \oplus N_s$ (direct sum) with each N_i simple. If f_i is the identity of N_i for each i then $1 = f_1 + f_2 + \cdots + f_s$ in N. Let M be an SE-group for N. Then if m is in M we have $m = m(f_1 + f_2 + \cdots + f_s) = mf_1 + +mf_2 + \cdots + mf_s$.
Theorem 3. *If N and M are as above then $C(N,M) = C(N_1, Mf_1) \oplus \oplus C(N_2, Mf_2) \oplus \cdots \oplus C(N_s, Mf_s)$ (direct sum).*
Note. Since each N_i is simple, it follows that if the conditions of Theorem 1 are satisfied (which will be the case if N_i is not a ring) then $C(N_i, Mf_i)$ is isomorphic to $C(e_i^1N_ie_i^1, Mf_ie_i^1) = C(e_i^1N_ie_i^1, Me_i^1)$ where in N_i, $f_i = e_1^1 + e_i^2 + \cdots + e_i^{t_i}$ (primitive idempotents).
Proof of Theorem 3. Clearly $M = Mf_1 \oplus Mf_2 \oplus \cdots \oplus Mf_s$ (direct sum). If g belongs to $C(N,M)$ then $g(Mf_i)$ is a subset of Mf_i and $g(mf_1 + mf_2 + \cdots + mf_s) = gmf_1 + gmf_2 + \cdots + gmf_s$. The map φ from $C(N,M)$ to $C(N_1, Mf_1) \oplus C(N_2, Mf_2) \oplus \cdots \oplus C(N_s, Mf_s)$ defined by $\varphi(g) = g_1 + g_2 + \cdots + g_s$ is our isomorphism where g_i is g restricted to Mf_i. ◇

The following is valid for an arbitrary finite centralizer near-ring of the form $C(A,G)$ where A is a group of automorphisms of G.

Theorem 4. *Let N be the finite centralizer near-ring $C(A,G)$. If H is an A-invariant subgroup of G let $R = \{f$ in $N | f(G)$ is a subset of $H\}$. Then $C(N,R)$ is isomorphic to R.*

Proof. R is easily seen to be a right ideal of N. If e_H is the idempotent in N which is the identity on H and 0 off H then $R = e_H N$. We have

$$\begin{aligned} C(N,R) &= \{f | f(rn) = f(r)n \text{ for all } r \text{ in } R \text{ and } n \text{ in } N\} \\ &= \{f | f(e_H n) = f(e_H)n \text{ for all } n \text{ in } N\} \\ &= \{L_r | r \text{ is in } R\} \text{ (where } L_r \text{ is the left multiplication} \\ &\quad \text{map by } r \text{ on } R) \text{ which is ismorphic to } R. \diamondsuit \end{aligned}$$

4. Applications to matrix near-rings

J.D.P. Meldrum and A.P.J. van der Walt have introduced the concept of a matrix near-ring (see [5]) which we now recall. Let N be a near-ring with 1 and let t be a positive integer. For an element r in N and for integers i,j with $1 \leq i,\ j \leq t$ define the function f_{ij}^r on N as follows:

$$f_{ij}^r(n_1, \ldots, n_i, \ldots, n_j, \ldots, n_t) = (0, \ldots, rn_j, \ldots, 0, \ldots, 0)$$

(where rn_j is in the i^{th} position). The $t \times t$ matrix near-ring over $N, M_t(N)$, is the subnearring of $\text{Map}(N^t)$ generated by $\{f_{ij}^r | r$ is in N and $1 \leq i,\ j \leq t\}$. We note that f_{ij}^r belongs to $C(N,N^t)$. Therefore $M_t(N)$ is a subnearring of $C(N,N^t)$. The following result was proven by van der Walt in [8].

Theorem (van der Walt). *Let N be a finite simple near-ring such that $N = C(A,G)$ where A is a fixed point free group of automorphisms on G. Then $M_t(N)$ is isomorphic to $C(A,G^t)$.*

Our information on SE-groups for a finite simple near-ring N can be used together with van der Walt's theorem to prove a near-ring analogue to a familiar matrix ring result in ring theory.

Theorem 5. *Let N be a finite simple near-ring with $N = C(A,G)$ where A is a fixed point free group of automorphisms on G. Let s and t be positive integers. Then $C(C(N,N^s), C(N,N^s)^t)$ is isomorphic to*

$C(N, N^{st})$.

Proof. Since N is simple we have seen that $C(N, N^s)$ is a simple near-ring and $C(N, N^s)$ is isomorphic to $C(A, G^s)$. Using Corollary 2, $C(C(N, N^s), C(N, N^s)^t)$ is isomorphic ot $C(A, (G^s)^t)$ which is isomorphic to $C(A, G^{st})$ and therefore isomorphic to $C(N, N^{st})$. $\diamondsuit$

Corollary 7. *If N is a finite simple near-ring with $N = C(A, G)$ where A is a fixed point free group of automorphisms on G then $M_t(M_s(N))$ is isomorphic to $M_{ts}(N)$.*

Proof. From van der Walt's theorem $C(A, G^{ts})$ is isomorphic to $M_{ts}(N)$ and $M_t(C(A, G^s)$ is isomorphic to $M_t(M_s(N))$. $\diamondsuit$

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Mathematica Pannonica
2/1 (1991), 49 – 61

AN INITIAL AND BOUNDARY VALUE PROBLEM FOR NONLINEAR COMPOSITE TYPE SYSTEMS OF THREE EQUATIONS*

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Received December 1989

AMS Subject Classification: 35 M 05, 30 G 20, 30 E 25

Keywords: composite type systems, elliptic systems, a priori estimate, Schauder imbedding

Abstract: Boundary value problems for systems of composite type were investigated by A. Dzhuraev, see [1]. Using the theory of singular integral equations in [1] linear problems for linear systems of three and of four equa-

* This work was done while the second and third authors visited the Free University Berlin in fall 1988 on exchange programs between the Educational State Commission of P. R. China and DFG and DAAD, respectively.

tions having one and two real characteristics, respectively are treated. Here a nonlinear problem for a nonlinear system of three equations is studied by utilizing a method from the theory of elliptic systems (see e.g. [3],[4]) based on Schauder imbedding. The case of three equations is important in particular because every elliptic second order equation in two independent variables may be reduced to a first order composite type system of three equations.

1. Formulation of the initial and boundary value problem

In this paper, we consider the nonlinear system of first order composite type equations

$$(1.1)\qquad \begin{cases} \omega_{\overline{z}} = F(z,\omega,\omega_z,s), \\ F = Q_1\omega_z + Q_2\overline{\omega_{\overline{z}}} + A_1\omega + A_2\overline{\omega} + A_3 s + A_4, \end{cases}$$

$$(1.2)\qquad \begin{cases} s_y = G(z,\omega,s), \\ G = B_1\omega + B_2\overline{\omega} + B_3 s + B_4, \end{cases}$$

in a bounded simply connected domain D, where

$$\begin{aligned} &Q_j = Q_j(z,\omega,\omega_z,s), \qquad j=1,2, \quad A_j = A_j(z,\omega,s), \\ &B_j = B_j(z,\omega,s), \qquad j=1,\dots,4, \end{aligned}$$

and $\omega(z), Q_j, A_j, B_j (j=1,2), A_4$ are complex valued functions, $B_2 = \overline{B_1}$, $s(z)$, A_3, $B_j (j=3,4)$ are real valued functions. For the sake of convenience, we may assume that D is the unit disk and the lower boundary of D is $\gamma = \{|z|=1,\, y\le 0\}$. We suppose that system (1.1) and (1.2) satisfy the following condition.

Condition C

(1) $Q_j(z,\omega,U,s), j=1,2,$ $A_j(z,\omega,s)$, $j=1,\dots,4$ are measurable in $z\in D$ for all continuous functions $\omega(z)$, $s(z)$ and all measurable functions $U(z)$ on $\overline{D}$, satisfying

$$(1.3)\qquad \begin{aligned} &L_p[A_j(z,\omega(z),s(z)),\overline{D}] \le k_0 < \infty, \quad j=1,2,4, \\ &L_p[A_3(z,\omega(z),s(z)),\overline{D}] \le \varepsilon, \end{aligned}$$

where $p(>2)$, $k_0(>0)$ and $\varepsilon(>0)$ are positive constants.

(2) The above mentioned functions are continuous in $\omega \in \mathbb{C}$ (the complex plane) and $s \in \mathbb{R}$ (the real axis) for almost every point $z \in D$ and $U \in \mathbb{C}$.

(3) The complex equation (1.1) satisfies the uniform ellipticity condition

$$|F(z,\omega,U_1,s) - F(z,\omega,U_2,s)| \leq q_0|U_1 - U_2|, \tag{1.4}$$

for almost every point $z \in D$ and $\omega, U_1, U_2 \in \mathbb{C}$, $s \in \mathbb{R}$, in which $q_0(<1)$ is a non-negative constant.

(4) $B_j(z,\omega,s)$ $(j=1,\ldots,4)$, $G(z,\omega,s)$ are continuous for $z \in \overline{D}$ for all Hölder continuous functions $\omega_j(z), s_j(z) \in C_\beta(\overline{D})$ $(j=1,2)$ satisfying

$$\begin{cases} C_\beta[B_j(z,\omega_1,s_1),\overline{D}] & \leq k_0 < \infty, \quad j=1,\ldots,4, \\ G(z,\omega_1,s_1) - G(z,\omega_2,s_2) & = B_1^*(\omega_1-\omega_2) + B_2^*(\overline{\omega_1-\omega_2}) + \\ & \quad + B_3^*(s_1-s_2), \end{cases} \tag{1.5}$$

in which $C_\beta[B_j^*,\overline{D}] \leq k_0, \beta\,(0<\beta<1)$ is real, for $j=1,2,3$.

For system (1.1) and (1.2) we discuss the following nonlinear initial and boundary value problem.

Problem A

$$\operatorname{Re}[\overline{\lambda(t)}\omega(t)] = P(t,w,s)t, \in \Gamma = \partial D, \tag{1.6}$$

$$a(t)s(t) = Q(t,\omega,s), t \in \gamma. \tag{1.7}$$

Here $\lambda(t), P(t,\omega,s)$ are Hölder continuous functions, $|\lambda(t)| = 1$, and $\lambda(t), P_0(t) = P(t,0,0), P\,(t,\omega,s)$ satisfy

$$\begin{cases} C_\alpha[\lambda\,[t\,(\zeta)],L] \leq k_0, C_\alpha\,[P_0\,[t\,(\zeta)],L] \leq k_1, L = \zeta\,(\Gamma), \\ C_\alpha[P\,(t\,(\zeta),\omega_1,s_1) - P\,(t\,(\zeta),\omega_2,s_2),L] \leq \\ \leq \varepsilon\{C_\alpha[\omega_1-\omega_2,L] + C_\alpha[s_1-s_2,\ell]\}, \ell = \zeta(\gamma), \end{cases} \tag{1.8}$$

for all $\omega_j[t\,(\zeta)] \in C_\alpha\,(L), s_j\,(t) \in C_\alpha(\ell)$, $j=1,2$, where $\zeta\,(z)$ is the homeomorphic solution to the Beltrami equation $\zeta_{\overline{z}} = q\,(z)\zeta_z$ with a proper $q\,(|q\,(z)| \leq q_0 < 1)$ which maps D onto the unit disk H such that $\zeta\,(0) = 0$, $\zeta\,(1) = 1; z\,(\zeta)$ is the inverse function of $\zeta\,(z)$, k_1 and ε are positive constants. Moreover, $|a\,(t)| = 1$, $Q_0\,(t) = Q\,(t,0,0)$ and $Q\,(t,\omega,s)$ satisfy

$$\left\{\begin{array}{l} C_\beta[Q_0(t),\gamma] \le k_2, \\ C_\beta[Q(t,\omega_1,s_1) - Q(t,\omega_2,s_2),\gamma] \le \\ \le k_2 C_\beta(\omega_1-\omega_2,\gamma) + \varepsilon C_\beta(s_1-s_2,\gamma), \end{array}\right. \tag{1.9}$$

in which k_2 is a positive constant. Obviously `Problem A` is not necessarily solvable. Hence we consider the modified initial-boundary value problem (**Problem B**) where (1.6) is replaced by

$$\operatorname{Re}[\overline{\lambda(t)}\,\omega(t)] = P(t,\omega,s) + h(t),\ t\in\Gamma, \tag{1.10}$$

with

$$h(t) = \left\{\begin{array}{l} 0, t\in\Gamma,\ \text{if } K\ge 0, \quad K = \frac{1}{2\pi}\Delta_\Gamma \arg\lambda(t), \\ h_0 + \operatorname{Re}\sum_{m=1}^{-K-1}(h_m^+ + ih_m^-)t^m,\ t\in\Gamma,\ \text{if } K<0, \end{array}\right. \tag{1.11}$$

where $h_0, h_m^\pm$ $(m = 1,\dots,-K-1)$ are unknown real constants to be determined appropriately. If $K\ge 0$, we assume that the solution $\omega(z)$ to `Problem A` satisfies the side conditions

$$\operatorname{Im}[\overline{\lambda(a_j)}\omega(a_j)] = b_j, \quad j = 1,\dots,2K+1, \tag{1.12}$$

where a_j $(j = 1,\dots,2K+1)$ are distinct points on Γ, and $b_j (j = = 1,\dots,2K+1)$ are real constants with the condition $|b_j| \le k_1$.

In the following, we first give an a priori estimate of solutions to `Problem B`. Afterwards, we prove `Problem B` and `Problem A` to be solvable by using the *Schauder fixed-point theorem.* Under some more restrictions, we can discuss the uniqueness of the solution to `Problem B`.

2. A propri estimate of solutions to the initial and boundary value problem

First of all, we discuss the system of first order composite type equations

$$\left\{\begin{array}{l} \omega_{\bar z} = F^*(z,\omega,\omega_z,s), \\ F^* = Q_1\omega_z + Q_2\overline{\omega_{\bar z}} + A_1\omega + A_2\overline{\omega} + A, \end{array}\right. \tag{2.1}$$

$$\left\{\begin{array}{l} s_y = G^*(z,\omega,s), \\ G^* = B_3 s + B, \end{array}\right. \tag{2.2}$$

together with the following linear initial and boundary value problem.

Problem B*

$$\mathrm{Re}[\overline{\lambda(t)}\omega(t)] = P_0(t) + h(t),\ t \in \Gamma, \tag{2.3}$$

$$Im[\overline{\lambda(a_j)}\omega(a_j)] = b_j,\ j = 1, \ldots, 2K+1,\ K \geq 0, \tag{2.4}$$

$$a(t)s(t) = Q_0(t),\ t \in \gamma, \tag{2.5}$$

where Q_j, A_j $(j = 1, 2)$, $B_3, \lambda, P_0, h, b_j, a, Q_0$ are defined as in 1, and $A = A(z, \omega, s)$, $B = B(z, \omega, s)$ are similar to A_4, B_4, but satisfsying the conditions

$$L_p[A, \overline{D}] \leq k_3, C_\beta[B, \overline{D}] \leq k_4, \tag{2.6}$$

for any $\omega(z), s(z) \in C_\beta(\overline{D})$, in which k_3, k_4 are non-negative constants.

Lemma 2.1. *If* $[\omega(z), s(z)]$ *is a solution to* Problem B* *for the system* (1.1), (2.2), *then* $[\omega(z), s(z)]$ *satisfies the estimates*

$$C_\beta[\omega, \overline{D}] \leq M_1(k_1 + k_3),\ L_{p_0}[|\omega_z| + |\omega_{\overline{z}}|, \overline{D}] \leq M_2(k_1 + k_3), \tag{2.7}$$

$$C^*_\beta[s, \overline{D}] := C_\beta[s, \overline{D}] + C[s_y, \overline{D}] \leq M_3(k_2 + k_4), \tag{2.8}$$

where $M_j = M_j(q_0, p_0, k_0, \alpha, k, K)$, $j = 1, 2, 3$, $k = (k_1, k_2, k_3, k_4)$, $\beta = \min(\alpha, 1 - \frac{2}{p_0})$, $p_0 = \min(p, \frac{1}{1-\alpha})$.

Proof. Substituting the solution $[\omega, s]$ to Problem B* into the complex system (2.1), (2.2), and assuming that $k' = \max(k_1, k_3) > 0$, $k'' = \max(k_2, k_4) > 0$, we put

$$W(z) = \frac{\omega(z)}{k'}, S(z) = \frac{s(z)}{k''}. \tag{2.9}$$

It is clear that $W(z)$ is a solution to the boundary value problem

$$W_{\overline{z}} = Q_1 W_z + Q_2 \overline{W}_{\overline{z}} + A_1 W + A_2 \overline{W} + \frac{A}{k'}, \tag{2.10}$$

$$\mathrm{Re}[\overline{\lambda(t)}\, W(t)] = \frac{P_0(t) + h(t)}{k'},\ t \in \Gamma, \tag{2.11}$$

$$\mathrm{Im}[\overline{\lambda(a_j)} W(a_j)] = \frac{b_j}{k'},\ ,\quad j = 1, \ldots, 2K+1, K \geq 0. \tag{2.12}$$

Noting that

$$L_p\left[\frac{A}{k'}, \overline{D}\right] \leq 1,\ C_\alpha\left[\frac{P_0(t(\zeta))}{k'}, L\right] \leq 1,\quad |\frac{b_j}{k'}| \leq 1, \tag{2.13}$$

and according to Theorem 5.6 of Chapter 5 in [3] or Theorem 4.3 of Chapter 2 in [4], we know that $W(z)$ satisfies the estimate

$$C_\beta[W, \overline{D}] \leq M_1,\ L_{p_0}[|W_{\overline{z}}| + |W_z|, \overline{D}] \leq M_2. \tag{2.14}$$

Moreover, $S(z)$ is a solution to the initial value problem

$$S_y = B_3 S + \tfrac{B}{k''}, \tag{2.15}$$

$$a(t)S(t) = \tfrac{Q_0(t)}{k''},\ t \in \gamma, \tag{2.16}$$

where $C_\beta\left[\frac{B}{k''}, \overline{D}\right] \le 1$, $C_\beta\left[\frac{Q_0}{k''}, \gamma\right] \le 1$. On the basis of Theorem 2.4 in [2], $S(z)$ can be seen to satisfy the estimate

$$C^*_\beta\,[S, \overline{D}] \le M_3. \tag{2.17}$$

From (2.14), (2.17) it follows that (2.7), (2.8) for $k' > 0$, $k'' > 0$ are true. If $k' = 0$ or $k'' = 0$, then (2.7), (2.8) for $k' = \varepsilon > 0$ or $k'' = \varepsilon > 0$ hold. Letting ε tend to 0, we obtain (2.7), (2.8) for $k' = 0$ or $k'' = 0$. ◇

Theorem 2.2. *Let the complex system* (1.1) *and* (1.2) *satisfy* Condition C *and the constant ε in* (1.3), (1.8) *and* (1.9) *be small enough. Then the solution $[\omega(z), s(z)]$ to* Problem B *for* (1.1), (1.2) *satisfies the estimate*

$$U = C_\beta[\omega, \overline{D}] + L_{p_0}\left[|\omega_{\overline{z}}| + |\omega_z|, \overline{D}\right] \le M_4, \tag{2.18}$$

$$V = C^*_\beta\,[s, \overline{D}] \le M_5, \tag{2.19}$$

where $M_j = M_j(q_0, p_0, k_0, a, k, K)$, $j = 4, 5$.

Proof. Let the solution $[\omega(z), s(z)]$ be inserted into the complex system (1.1), (1.2), the boundary condition (1.10), the side condition (1.12) and the initial condition (1.7). We see that $A = A_3 s + A_4$, $B = B_1\omega + B_2\overline{\omega} + B_4$, $P(t, \omega, s)$, $Q(t, \omega, s), b_j$ satisfy

$$L_p[A, \overline{D}] \le \varepsilon C[s, \overline{D}] + L_p[A_4, \overline{D}] \le \varepsilon C[s, \overline{D}] + k_0, \tag{2.20}$$

$$C_\beta[B, \overline{D}] \le C_\beta[B_1\omega + B_2\overline{\omega}, \overline{D}] + C_\beta[B_4, \overline{D}] \le 2k_0 C_\beta[\omega, \overline{D}] + k_0, \tag{2.21}$$

$$\begin{aligned} C_\alpha[P, L] \le\ & C_\alpha[P_0(t(\zeta)), L] + C_\alpha[P[t(\zeta), \omega, s] - P_0[t(\zeta)], L] \le \\ \le\ & k_1 + \varepsilon\{C_\alpha[\omega, L] + C_\beta[s, \ell]\}, \end{aligned} \tag{2.22}$$

$$|b_j| \le k_1,\ j = 1, \ldots, 2K+1, K \ge 0, \tag{2.23}$$

$$\begin{aligned} C_\beta[Q, \gamma] \le\ & C_\beta[Q_0(t), \gamma] + k_0 C_\beta[\omega, \gamma] + \varepsilon C_\beta[s, \gamma] \le \\ & \le k_2 + k_2 C_\beta[\omega, \overline{D}] + \varepsilon C_\beta[s, \overline{D}]. \end{aligned} \tag{2.24}$$

Using (2.7) and (2.8) we have

$$
\begin{aligned}
&U \le (M_1+M_2)\{\varepsilon C[s,\overline{D}]+k_0+k_1+\varepsilon[C_\alpha(\omega,L)+C_\alpha[s,\ell]\}\le\\
(2.25)\quad &\le (M_1+M_2)[k_0+k_1+\varepsilon C_\beta(\omega,\overline{D})+\varepsilon C_\beta(s,\overline{D})]\le\\
&\le (M_1+M_2)(k_0+k_1+\varepsilon U+\varepsilon V),
\end{aligned}
$$

$$
(2.26)\quad \begin{aligned}
&V\le M_3[2k_0C_\beta(\omega,\overline{D})+k_0+k_2+k_2C_\beta(\omega\,\overline{D})+\varepsilon C_\beta(s,\overline{D}]\le\\
&\le M_3[k_0+k_2+(2k_0+k_2)U+\varepsilon V].
\end{aligned}
$$

Choosing the constant ε so small that

$$(M_1+M_2)\varepsilon\le\tfrac{1}{2},\quad M_3\,[1+2(2k_0+k_2)(M_1+M_2)]\,\varepsilon\le\tfrac{1}{2},$$

one can show

$$(2.27)\qquad U\le\tfrac{(M_1+M_2)(k_0+k_1+\varepsilon V)}{1-(M_1+M_2)\varepsilon}\le 2(M_1+M_2)(k_0+k_1+\varepsilon V),$$

$$
\begin{aligned}
&V\le M_3[k_0+k_2+2(2k_0+k_2)(M_1+M_2)(k_0+k_1+\varepsilon V)+\varepsilon V]\le\\
(2.28)\quad &\le\frac{M_3[k_0+k_2+2(2k_0+k_2)(k_0+k_1)(M_1+M_2)]}{1-M_3[1+2(2k_0+k_2)(M_1+M_2)]\varepsilon}\le\\
&\le 2M_3[k_0+k_2+2(2k_0+k_2)(k_0+k_1)(M_1+M_2)]=M_5,
\end{aligned}
$$

$$(2.29)\qquad U\le 2(M_1+M_2)(k_0+k_1+\varepsilon M_5)=M_4.\ \diamondsuit$$

3. Solvability of the initial and boundary value problem

First we prove the existence of solutions to `Problem B` for the system

$$(3.1)\quad \begin{cases}\omega_{\overline{z}}=F(z,\omega,\omega_z),\ F=Q_1\omega_z+Q_z\overline{\omega}_{\overline{z}}+A_1\omega+A_z\overline{\omega}+A_3,\\ Q_j=Q_j(z,\omega_z),\ j=1,2,A_j=A_j(z),\ j=1,2,3,\end{cases}$$

and (1.2) by using the parameter extension method, and then verify the existence of solutions to `Problem B` for the system (1.1) and (1.2) by using Theorem 2.2 and the *Schauder fixed point theorem.* Finally, we give conditions for `Problem A` for (1.1), (1.2) to be solvable.

Theorem 3.1. *Let the system* (3.1), (1.2) *satisfy* `Condition C` *and the constant ε be small enough. Then* `Problem B` *for* (3.1), (1.2) *is solvable.*

Proof. We consider the following initial boundary value problem with parameter $t\,(0 \le t \le 1)$.

Problem B′

$$\omega_{\bar{z}} = tF(z,\omega,\omega_z) + A(z) \text{ in } D,\ A \in L_{p_0}(\overline{D}), \tag{3.2}$$

$$\operatorname{Re}[\overline{\lambda(z)}\,\omega(z)] = tP(z,\omega,s) + p(z) + h(z), \text{ on } \Gamma, p \in C_\beta(\Gamma), \tag{3.3}$$

$$\operatorname{Im}[\overline{\lambda(a_j)}\,\omega(a_j)] = b_j,\ \ j = 1,\dots,2K+1, K \ge 0, \tag{3.4}$$

$$s_y = tG(z,\omega,s) + B(z) \text{ in } D,\ \ B \in C_\beta(\overline{D}), \tag{3.5}$$

$$a(z)s(z) = tQ(z,\omega,s) + q(z) \text{ on } \gamma, q \in C_\beta(\gamma). \tag{3.6}$$

When $t = 0$, `Problem` B′ has a unique solution $[\omega(z), s(z)]$ with $\omega \in$ $\in C_\beta(\overline{D})$, $s \in C^*_\beta(\overline{D})$ - see [2], [3] and [4].
Assuming that `Problem` B′ for $t_0\,(0 \le t_0 \le 1)$ is solvable, we will prove that there exists a positive constant δ such that `Problem B'` on

$$E = \{t \mid |t - t_0| \le \delta, 0 \le t \le 1\} \tag{3.7}$$

for any $A \in L_{p_0}(\overline{D})$, $B \in C_\beta(\overline{D})$, $p \in C_\beta(\Gamma)$ and $q \in C_\beta(\gamma)$ has a unique solution $[\omega(z), s(z)]$, $\omega \in C_\beta(\overline{D}) \cap W^1_{p_0}(D)$, $s \in C^*_\beta(\overline{D})$.
We rewrite (3.2) – (3.6) as

$$\omega_{\bar{z}} - t_0 F(z,\omega,\omega_z) = (t - t_0)F(z,\omega,\omega_z) + A(z), \tag{3.8}$$

$$\operatorname{Re}[\overline{\lambda(z)}\,\omega(z)] - t_0 P(z,\omega,s) = (t - t_0)P(z,\omega,s) + p(z) + h(z), \tag{3.9}$$

$$\operatorname{Im}[\overline{\lambda(a_j)}\,\omega(a_j)] = b_j,\ j = 1,\dots,2K+1, K \ge 0, \tag{3.10}$$

$$s_y - t_0 G(z,\omega,s) = (t - t_0)G(z,\omega,s) + B(z), \tag{3.11}$$

$$a(z)s(z) - t_0 Q(z,\omega,s) = (t - t_0)Q(z,\omega,s) + q(z). \tag{3.12}$$

Choosing arbitrary functions $\omega_0 \in C_\beta(\overline{D}) \cap W^1_{p_0}(D)$, $s_0 \in C^*_\beta(\overline{D})$, for instance $\omega_0(z) \equiv 0$, $s_0(z) \equiv 0$, we substitute $\omega_0(z), s_0(z)$ into the corresponding positions of the right hand sides in (3.8) – (3.12). By assumption, for t_0 the initial-boundary value problem (3.8) – (3.12) has a unique solution $[\omega_1(z), s_1(z)]$, $\omega_1 \in C_\beta(\overline{D}) \cap W^1_{p_0}(D), s_1 \in C^*_\beta(\overline{D})$. Let us substitute $\omega_1(z), s_1(z)$ into the right hand sides of (3.8) – (3.12) and find unique solution $[\omega_2(z), s_2(z)], \omega_2 \in C_\beta(\overline{D}) \cap W^1_{p_0}(D)$, $s_2 \in$ $\in C^*_\beta(\overline{D})$ to this system. Thus, we obtain $[\omega_n(z), s_n(z)], n = 1, 2, \dots$, satisfying

$$\omega_{n+1\bar{z}} - t_0 F(z,\omega_{n+1},\omega_{n+1\,z}) = (t - t_0)F(z,\omega_n,\omega_{nz}) + A(z), \tag{3.13}$$

$$\begin{aligned}\mathrm{Re}[\overline{\lambda}\,\omega_{n+1}] - t_0 P(z,\omega_{n+1},\, s_{n+1}) &= \\ = (t-t_0)\,P(z,\omega_n s_n) + p(z) + h(z),&\end{aligned} \tag{3.14}$$

$$\mathrm{Im}[\overline{\lambda(a_j)}\,\omega_{n+1}(a_j)] = b_j,\ j = 1,\ldots,2K+1, K \geq 0, \tag{3.15}$$

$$s_{n+1y} - t_0 G(z,\omega_{n+1},s_{n+1}) = (t-t_0)\,G(z,\omega_n,s_n) + B(z), \tag{3.16}$$

$$a(z)s_{n+1} - t_0 Q(z,\omega_{n+1},s_{n+1}) = (t-t_0)\,Q(z,\omega_n,s_n) + q(z). \tag{3.17}$$

Setting $W_{n+1} = \omega_{n+1} - \omega_n$, $S_{n+1} = s_{n+1} - s_n$ from (3.13) – (3.17), we have

$$\begin{aligned}W_{n+1\overline{z}} - t_0[F(z,W_{n+1},W_{n+1z}) - F(z,W_n,W_{nz})] &= \\ = (t-t_0)[F(z,W_n,W_{nz}) - F(z,W_{n-1},W_{n-1z})],&\end{aligned} \tag{3.18}$$

$$\begin{aligned}\mathrm{Re}\,[\overline{\lambda}\,W_{n+1}] - t_0[P(z,\omega_{n+1},s_{n+1}) - P(z,\omega_n,s_n)] &= \\ = (t-t_0)[P(z,\omega_n,s_n) - P(z,\omega_{n-1},s_{n-1})] + h(z),&\end{aligned} \tag{3.19}$$

$$Im\,[\overline{\lambda(a_j)}W_{n+1}(a_j)] = 0, j = 1,\ldots,2K+1, K \geq 0, \tag{3.20}$$

$$\begin{aligned}S_{n+1y} - t_0[G(z,\omega_{n+1},s_{n+1}) - G(z,\omega_n,s_n)] &= \\ = (t-t_0)\,[G(z,\omega_n,s_n) - G(z,\omega_{n-1},s_{n-1}],&\end{aligned} \tag{3.21}$$

$$\begin{aligned}a(z)S_{n+1} - t_0[Q(z,\omega_{n+1},s_{n+1}) - Q(z,\omega_n,s_n)] &= \\ = (t-t_0)\,[Q(z,\omega_n,s_n) - Q(z,\omega_{n-1},s_{n-1})].&\end{aligned} \tag{3.22}$$

By Condition C

$$\begin{aligned}L_{p_0}[F(z,W_n,W_{nz}) - F(z,W_{n-1},W_{n-1z}),\overline{D}] &\leq \\ \leq L_{p_0}[W_{nz},\overline{D}] + 2k_0 C_\beta\,[W_n,\overline{D}],&\end{aligned} \tag{3.23}$$

$$\begin{aligned}&C_\alpha\{P[z(\zeta),\omega_n(z(\zeta)),s_n(z(\zeta))] - \\ &-P[z(\zeta),\omega_{n-1}(z(\zeta)),s_{n-1}(z(\zeta))],L \leq \\ &\leq \varepsilon\,\{C_\alpha[W_n(z(\zeta)),L] + C_\alpha[S_n(z(\zeta)),\ell]\},\end{aligned} \tag{3.24}$$

$$\begin{aligned}C_\beta[G(z,\omega_n,s_n) - G(z,\omega_{n-1},s_{n-1}),\overline{D}] &\leq \\ \leq 2k_0 C_\beta[W_n,\overline{D}] + k_0 C_\beta[S_n,\overline{D}],&\end{aligned} \tag{3.25}$$

$$(3.26)\qquad \begin{aligned} & C_\beta[Q(z,\omega_n,s_n)-Q(z,\omega_{n-1},s_{n-1}),\gamma]\leq \\ & \leq k_2 C_\beta[W_n,\gamma]+\varepsilon C_\beta[S_n,\gamma] \end{aligned}$$

can be obtained.
According to the method in the proof of Theorem 2.2, we can conclude that

$$(3.27)\qquad \begin{aligned} U_{n+1} := C_\beta[W_{n+1},\overline{D}] + L_{p_0}[|W_{n+1\bar{z}}|+|W_{n+1\,z}|,\overline{D}]\leq \\ \leq |t-t_0|\,M_6 U_n, \end{aligned}$$

$$(3.28)\qquad V_{n+1} := C^*_\beta[S_{n+1},\overline{D}]\leq |t-t_0|\,M_6 V_n,$$

where $M_6 = M_6(q_0,p_0,k_0,\alpha,k,K,\varepsilon)\geq 0$.
Choosing $\delta = \frac{1}{2(M_6+1)}$, then for $|t-t_0|\leq\delta$, $0\leq t\leq 1$, and $n>$ $>N+1>1$, we can derive the inequality

$$U_{n+1}\leq \tfrac{1}{2}U_n\leq \tfrac{1}{2^N}U_1,\, V_{n+1}\leq \tfrac{1}{2^N}V_1.$$

Moreover, if $n,m > N+1$, then

$$(3.30)\qquad \begin{aligned} & C_\beta[\omega_n-\omega_m,\overline{D}]+L_{p_0}[|\omega_n-\omega_m)_{\bar z}|+|(\omega_n-\omega_m)_z|,\overline{D}]\leq \\ & \leq \tfrac{1}{2^N}\textstyle\sum_{j=0}^{\infty}\tfrac{1}{2^j}U_1=\tfrac{1}{2^{N-1}}U_1, \\ & C^*_\beta[s_n-s_m,\overline{D}]\leq \tfrac{1}{2^{N-1}}C^*_\beta[s_1,\overline{D}]. \end{aligned}$$

This shows that $C_\beta[\omega_n-\omega_m,\overline{D}]+L_{p_0}[|(\omega_n-\omega_m)_{\bar z}|+|(\omega_n-\omega_m)_z|,$ $\overline{D}]\to 0$, $C^*_\beta[s_n-s_m,\overline{D}]\to 0$, if $n,m\to\infty$. Hence there exist $\omega_*\in$ $\in C_\beta(\overline{D})\cap W^1_{p_0}(D)$, $s_*\in C^*_\beta(\overline{D})$, such that $C_\beta[\omega_n-\omega_*,\overline{D}]+$ $+L_{p_0}\left[|(\omega_n-\omega_*)_{\bar z}|+|(\omega_n-\omega_*)_z|,\overline{D}\right]\to 0$, $C^*_\beta[s_n-s_*,\overline{D}]\to 0$, as $n\to$ $\to\infty$, and $[\omega_n(z),s_n(z)]$ is just a solution to `Problem B'` on E for (3.2) – (3.6). Thus, we know that when $t=0,1,\ldots,[\frac{1}{\delta}]\delta,1$, Problem B′ for (3.2) – (3.6) is solvable. In particular, when $t=1$, $A=0$, $p=0$, $B=0$, $q=0$, `Problem B'` i. e. `Problem B` for (3.1), (1.2) is solvable. ◇

Theorem 3.2. *Under the same hypotheses as in Theorem 2.2,* `Problem B` *for* (1.1), (1.2) *has a solution.*

Proof. We introduce a bounded and closed convex set B_M in the Banach space $C(\overline{D})\times C(\overline{D})$, the elements of which are vectors of functions $w=[\omega,s]$ satisfying the condition

$$(3.31)\qquad C[\omega,\overline{D}]\leq M_4, C[s,\overline{D}]\leq M_5,$$

where M_4, M_5 are the constants stated in (2.18), (2.19). We choose an arbitrary vector of functions $\Omega = [W, S] \in B_M$ and insert $W(z), S(z)$ into the appropiate positions of the complex equation (1.1). Following Theorem 3.1, there exists a solution $[\omega(z), s(z)]$ to the initial boundary value `Problem B'`:

$$\begin{aligned}&\omega_{\bar{z}} = f(z, \omega, W, s, \omega_z),\\ (3.32)\qquad &f = Q_1(z, W, \omega_z, s)\,\omega_z + Q_2(z, W, \omega_z, s)\,\overline{\omega_{\bar{z}}} +\\ &+ A_1(z, W, s)\,\omega + A_2(z, W, s)\,\overline{\omega} + A_3(z, W, s),\end{aligned}$$

and (1.2), (1.6), (1.10), (1.12), (1.7).
According to Theorem 2.2, the solution $[\omega(z), s(z)]$ satisfies the estimates (2.18) and (2.19), obviously $w = [\omega, s] \in B_M$. Denoting this mapping from $\Omega \in B_M$ onto w by $w = \underset{\sim}{S}\,[\Omega]$, it is clear that $\underset{\sim}{S}$ is an operator which maps B_M onto a compact set in B_M.
To prove that $\underset{\sim}{S}$ is continuous in B_M, we select a sequence of vectors $[W_n, S_n](n = 0, 1, 2, \ldots)$ satisfying the condition

$$(3.33)\qquad C[W_n - W_0, \overline{D}] \to 0, C[S_n - S_0, \overline{D}] \to 0 \text{ as } n \to \infty$$

and consider the difference $w_n - w_0 = \underset{\sim}{S}\,(\Omega_n) - \underset{\sim}{S}\,(\Omega_0)$. We have

$$(3.34)\qquad [\omega_n - \omega_0]_{\bar{z}} = f(z, \omega_n, W_n, \omega_{nz}) - f(z, \omega_0, W_0, \omega_{0z}),$$

$$(3.35)\qquad \mathrm{Re}[\overline{\lambda(t)}(\omega_n - \omega_0)] = P(z, \omega_n, s_n) - P(z, \omega_0, s_0) + h(t), t \in \Gamma,$$

$$(3.36)\qquad \mathrm{Im}[\overline{\lambda(a_j)}(\omega_n(a_j) - \omega_0(a_j))] = 0,\ j = 1, \ldots, 2K+1, K \geq 0,$$

$$(3.37)\qquad (s_n - s_0)_y = G(z, w_n, s_n) - G(z, w_0, s_0),$$

$$(3.38)\qquad a(t)[s_n - s_0] = Q(t, w_n, s_n) - Q(t, w_0, s_0),\ t \in \gamma.$$

The complex equation (3.34) can be written as

$$(3.39)\qquad \begin{aligned}&[\omega_n - \omega_0]_{\bar{z}} - [f(z, \omega_n, W_n, \omega_{nz}) - f(z, \omega_0, W_n, \omega_{0z})] = c_n,\\ &c_n = f(z, \omega_0, W_n, \omega_{0z}) - f(z, \omega_0, W_0, \omega_{0z}).\end{aligned}$$

Using the method in the proof of Theorem 2.2 of Chapter 4 in [3] or Theorem 2.6 of Chapter 2 in [4], we can verify that $L_{p_0}[c_n, \overline{D}] \to 0$ as $n \to \infty$. Hence, applying the method used in the proof of Theorem 2.1,

$$(3.40)\qquad C_\beta[\omega_n - \omega_0, \overline{D}],\ C_\beta[s_n - s_0, \overline{D}] \leq M_7\ L_{p_0}[c_n, \overline{D}]$$

can be concluded where M_7 is a non-negative constant. If $n \to \infty$, then $C[\omega_n - \omega_0, \overline{D}] \to 0$, $C[s_n - s_0, \overline{D}] \to 0$. Hence, $w = S(\Omega)$ is a continuous mapping from B_M onto a compact set in B_M. On the basis of the *Schauder fixed point theorem*, there exists a vector $w = [\omega, s] \in B_M$, so that $\omega = S(\omega)$, and $w = [\omega, s]$ is just a solution to `Problem B` for the system (1.1) and (1.2). $\diamondsuit$

Theorem 3.3. *Suppose that the system* (1.1), (1.2) *satisfies the same conditions as in Theorem 2.2, then the following statement holds*

(1) *If* $K \geq 0$, `Problem A` *for* (1.1), (1.2) *is solvable.*

(2) *If* $K < 0$, *there are* $-2K-1$ *conditions for* `Problem A` *to be solvable.*

Proof. Let us substitute the solution $[\omega(z), s(z)]$ to `Problem B` into the boundary condition (1.10). If $h(z) = 0, z \in \Gamma$, then $[\omega(z), s(z)]$ is also a solution to `Problem A` for (1.1), (1.2). The total number of real equalities in $h(z) = 0$ is just the total number of conditions stated in the theorem. $\diamondsuit$

Finally, in order to discuss the uniqueness of the solution to `Problem B` and `Problem A` for (1.1), (1.2) the following additional condition is imposed.

There exist $A_1^*, A_2^* \in L_{p_0}(\overline{D})$, with $L_{p_0}[A_2^*, \overline{D}]$ small enough, such that

$$F(z, \omega_1, U, s_1) - F(z, \omega_2, U, s_2) = A_1^*(\omega_1 - \omega_2) + A_2^*(s_1 - s_2), \tag{3.41}$$

for any functions $\omega_j, s_j \in C_\beta(\overline{D}), j = 1, 2$, and $U \in L_{p_0}(\overline{D})$ $(2 < p_0 < < p)$.

Theorem 3.4. (1.1), (1.2) *satisfies* `Condition C` *and* (3.41), *and the constant* ε *in* (1.3), (1.8), (1.9) *is small enough, then the solutions to* `Problem B` *are unique.*

Proof. Let $[\omega_1(z), s_1(z)]$, $[\omega_2(z), s_2(z)]$ be two solutions to `Problem B` for (1.1), (1.2). It is clear that $[\omega, s] = [\omega_1 - \omega_2, s_1 - s_2]$ is a solution to the initial-boundary value problem

$$\omega_{\overline{z}} = Q\omega_z + A_1^*\omega + A_2^* s,$$

$$Q = \begin{cases} \frac{F(z,\omega_1,\omega_{1z},s_1) - F(z,\omega_1,\omega_{2z},s_2)}{\omega_z}, & \omega_z \neq 0, \\ 0, \quad \omega_z = 0; \end{cases}$$

$$s_y = B_1^*\omega + B_2^*\overline{\omega} + B_3^* s,$$

$$\mathrm{Re}\,[\overline{\lambda(t)}\omega(t)] = P(t, \omega_1, s_1) - P(t, \omega_2, s_2) + h(t), t \in \Gamma,$$

$$Im\,[\overline{\lambda(a_j)}\omega(a_j)] = 0,\ j = 1, 2, \ldots, 2K+1 (0 \leq K);$$

$$a(t)s(t) = Q(t, \omega_1, s_1) - Q(t, \omega_2, s_2), t \in \gamma.$$

With the method used in the proof of Theorem 2.2, we can show

$$C_\beta[\omega, \overline{D}] + L_{p_0}[|\omega_{\overline{z}}| + |\omega_z|, \overline{D}] = 0,$$
$$C^*_\beta(s, \overline{D}) = 0,$$

so that $\omega(z) \equiv 0, s(z) \equiv 0$, i.e. $\omega_1(z) \equiv \omega_2(z) s_1(z) \equiv s_2(z)$ in $\overline{D}$. $\Diamond$

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Mathematica Pannonica
2/1 (1991), 63 – 66

A NOTE ON CYCLE DOUBLE COVERS IN CAYLEY GRAPHS

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Received April 1990

AMS Subject Classification: 05 C 70, 05 C 25, 05 C 10

Keywords: Cayley graph, Hamilton cycle double cover

Abstract: We show that every finite Cayley graph of degree at least two has a cycle double cover.

All graphs in this note are finite. Lovász ([4], Problem 11) conjectured that every connected vertex-transitive graph has a Hamilton path. One well-studied class of vertex-transitive graphs is the class of Cayley graphs. Tarsi [5] and Goddyn [1] proved that every 2-edge-connected graph with a Hamilton path has a cycle double cover. Combining these ideas led the authors to consider cycle double covers in Cayley graphs.

We prove that every Cayley graph of degree at least two has a cycle double cover.

Lemma 1. *Let G be a graph whose edge set can be decomposed into a collection of* 1*-factors and* 2*-factors, where the number of* 1*-factors is not* 1. *Then G has a cycle double cover.*

Proof. This is virtually trivial. We have a decomposition of the edge set of G into a set of cycles $\mathcal{C}$ and a set of 1-factors $F_1, F_2, \ldots, F_k$. Each of the graphs induced by $F_i \cup F_{i+1}$, $i = 1, \ldots, k$ (addition modulo n) is a 2-factor $\mathcal{F}_i$ of G. The collection of cycles in the $\mathcal{F}_i$ together with two copies of each of the cycles in $\mathcal{C}$, constitutes a cycle double cover of G. $\diamondsuit$

Now suppose that Γ is a group, and $S \subseteq \Gamma$, where

(i) $id \notin S$,

(ii) $v \in S \Rightarrow v^{-1} \in S$, and

(iii) S generates Γ

We write $\mathrm{Cay}(\Gamma, S)$ for the Cayley graph of Γ with respect to this set of generators. $\mathrm{Cay}(\Gamma, S)$ has vertex set Γ, and its edge set is $\{(g, gs) : g \in \Gamma, s \in S\}$.

Theorem. *If $G = \mathrm{Cay}(\Gamma, S)$, with $|S| \geq 2$, then G has a cycle double cover.*

Proof. Each element g of S corresponds to a set of edges E_g of G. If order(g) $= 2$, then E_g is a 1-factor of G. If order(g) $\neq 2$, then E_g is a 2-factor of G.

Case 1. If every element of S has order greater than two, or if at least two elements of S have order two, then Lemma 1 applies.

Case 2. If order(x) $= 2$, for exactly one $x \in S$, then there is some $y \in S$ with order(y) > 2. The sets E_g, $g \in S' = S - \{x, y, y^{-1}\}$, correspond to 2-factors of G. We note that $G - \cup\{E_g : g \in S'\}$ may not be connected, and may consist of several disjoint copies of $\mathrm{Cay}(<x, y, y^{-1}>, \{x, y, y^{-1}\})$. Thus, we need only consider the case in which $|S| = 3$. If Γ is abelian, then G has a Hamilton cycle (see Holsztyński and Strube [2]) and therefore a cycle double cover. Thus, we may assume that Γ is non-abelian.

Since $\{x, y\}$ generates Γ, $\{y, xy\}$ generates Γ. Suppose that $y(xy)^j = id$, for some j. Then $y = (xy)^{-j}$, and $\{xy\}$ generates Γ, contradicting the assumption that Γ is non-abelian. Therefore, we may assume that $y(xy)^j \neq id$, for any j. Similarly, we may assume that

$x(yx)^j \neq id$, for any j.

We now describe a cycle double cover of G. For $g \in \Gamma$, let

$$C_g = g_0 g_1 g_2 \cdots g_{2k-1} g_0,$$

denote the closed trail of G with vertices $g_{2i} = g(xy)^i$, $g_{2i+1} = g_{2i}x$, $0 \leq i < k = \text{order}(xy)$. We claim that C_g is, in fact, a cycle of G. If C_g is not a cycle, then $g_\alpha = g_\beta$, for some α, β, where, without loss of generality, $0 \leq \alpha < \beta < 2k$. We consider four cases depending on the parities of α and β.

(i) If α and β are both even, then $(xy)^{(\beta-\alpha)/2} = id$, which contradicts $\text{order}(xy) = k$.

(ii) If α and β are both odd, then $g_{\alpha+1} = g_{\beta+1}$, and case (i) applies.

(iii) If α is odd and β is even, then $y(xy)^{(\beta-\alpha-1)/2} = id$, contradicting the fact that $y(xy)^j \neq id$, for any j.

(iv) If α is even and β is odd, then $y(yx)^{(\beta-\alpha-1)/2} = id$, contradicting the fact that $x(yx)^j \neq id$, for any j.

Therefore, C_g must be a cycle. We note that the cycles C_g and C_{gxy} are really the same cycle, and thus that each cycle has k names.

Let g and h be adjacent vertices in $\text{Cay}(< x, y, y^{-1} >, \{x, y, y^{-1}\})$. Thus, $h = gx$ or $h = gy$ or $h = gy^{-1}$. If $h = gx$, the edge (g, h) is contained in the cycles C_g and C_h. If $h = gy$, the edge (g, h) is contained in the cycle C_{gx} and in E_y. If $h = gy^{-1}$, the edge (g, h) is contained in the cycle C_{hx} and in E_y. We also note that every vertex g is on exactly two of the cycles of the form C_w, specifically, C_g and C_{gx}.

Thus the set $\mathcal{C} = \{C_g : g \in \Gamma\} \cup E_y$ is a cycle double cover of G. $\Diamond$

We leave open the question of whether or not every vertex-transitive graph has a cycle double cover. There have been many papers establishing that certain classes of vertex-transitive graphs have Hamilton paths. We mention only one such result. Lipman [3] proves that every graph with a transitive nilpotent automorphism group, and every vertex-transitive graph on a prime power number of vertices must have a Hamilton path. Thus, by [1,5], each of these graphs has a cycle double cover. It is easy to see that if $|S|$ is even, then the Cayley graph has a cycle cover.

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2/1 (1991), 67 – 70

BERNSTEIN APPROXIMATION OF A FUNCTION WHICH DERIVATIVES SATISFY THE LIPSCHITZ CONDITION ON BOUNDED SQUARE OR TRIANGLE*

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Received August 1990

AMS Subject Classification: 41 A 10, 41 A 17

Keywords: Real function approximation, Bernstein polynomials

Abstract: Let Bf be the Bernstein polynomial of a function f such that its first derivatives satisfy the Lipschitz condition of order 1 on the unit square or on the standard triangle. It is shown that the approximation-error of the function f by the polynomial Bf does not exceed a quantity depending on the Lipschitz constants and the degree of Bf only. This way it is the full analogue to the one-dimensional case observed first by A.O. Geldfond.

* This research was supported by grant CPBP 01.01.1/3 (1987,1989) from the Polish Academy of Sciences.

1. One-dimensional case. Gelfond has shown [1] that

$$\max_{0\le x\le 1}|B_nf(x)-f(x)|\le \tfrac{1}{4n}L$$

for every function $f\in C^1_{<0,1>}$ such that its derivative f' satisfies the Lipschitz condition of degree 1 and constant L. Here B_nf stands for the classical Bernstein polynomial of degree n built for a function f, i.e. $B_nf(x):=\Sigma_{j=0}^n\left(\frac{j}{n}\right)p_{n,j}(x)$, where $p_{n,j}(x):=\binom{n}{j}x^j(1-x)^{n-j}$ and $x\in<0,1>$.

2. Approximation on the unit square. In 1933 Hildebrandt and Schoenberg have extended the notion of Bernstein polynomial to the case when a function f being approximated is defined on the unit square

$$K:=\{(x,y)\in\mathbb{R}^2:0\le x,y\le 1\}.$$

They set

$$B_{m,n}f(x,y):=\sum_{j=0}^m\sum_{k=0}^n f\left(\tfrac{j}{m},\tfrac{k}{n}\right)p_{m,j}(x)p_{n,k}(y)$$

and they proved that $B_{m,n}f$ tends to the function f uniformly in the space C_K (of all functions continuous on the domain K) as m and n increase to the infinity.

The polynomials $B_{m,n}f$ have been investigated a.o. by Butzer and Aramă (for details on bibliography see [2]). In the fifties they have proved the analogues of the classical one-dimensional cases (concerning the approximations of the derivatives of the function f by the derivatives of the polynomial $B_{m,n}f$ and the preserving of the convexity of the function f by its polynomial $B_{m,n}f$, respectively). Here we complete these analogues, namely we give the analogue to the Gelfond's result listed in Part 1, that is we show that there holds the following

Theorem 1. *Let f_1 and f_2 be the first derivatives* (with respect to the first and the second argument, respectively) *of a function $f\in C^1_K$ and let L_1,L_2 be the positive constants such that*

$$(*)\qquad |f_j(x,y)-f(s,t)|\le L_1|x-s|+L_2|y-t|$$

for $j=1,2$ and for every points $(x,y),(s,t)\in K$ (we assume here that $|x-s|$, $|y-t|\le 1$, naturally). *Then*

$$|B_{m,n}f(x,y)-f(x,y)|\le\tfrac14\left(\tfrac{1}{m}L_1+\tfrac{1}{n}L_2\right).$$

Proof. We will use the identities

$$\sum_{j=0}^{m}\left(\frac{j}{m}\right)^r p_{m,j}(x) = x^r \qquad \text{for} \qquad r = 0,1$$

and the following form of the Mean Value Theorem

$$f(x,y) - f(s,t) = f_1(\sigma,y)(x-s) + f_2(x,\tau)(y-t),$$

where σ and τ are the points laying somewhere in the intervals $(x-|x-s|,\, x+|x-s|)$ and $(y-|y-t|,\, y+|y-t|)$, respectively. Applying the above and the Lipschitz condition $(*)$ we obtain

$$\begin{aligned}|f(x,y) - f(s,t) - f_1(s,t)(x-s) - f_2(s,t)(y-t)| \leq \\ \leq L_1(x-s)^2 + L_2(y-t)^2.\end{aligned}$$

Therefore the remainder $R := B_{m,n}f(x,y) - f(x,y)$ can be estimated as follows

$$\begin{aligned}|R| \leq \sum_{j=0}^{m}\sum_{k=0}^{n}\{|f_1(\sigma,\frac{k}{n}) - f_1(x,y)| \cdot |x - \frac{j}{m}| + \\ + |f_2(\frac{j}{m},\tau) - f_2(x,y)| \cdot |y - \frac{k}{n}|\} p_{m,j}(x)p_{n,k}(y) \leq \\ \leq L_1\frac{x(1-x)}{m} + L_2\frac{y(1-y)}{n} \leq \frac{1}{4}(\frac{1}{m}L_1 + \frac{1}{n}L_2). \ \diamond\end{aligned}$$

3. Approximation on the standard simplex.

Now we investigate the case when a function f is defined on the standard simple

$$T := \{(x,y) \in \mathbb{R}^2 : 0 \leq x, y, x+y \leq 1\}.$$

On this triangle there are defined the Steffensen polynomials $p_{m,j,k}$

$$p_{m,j,k}(x,y) := \binom{m}{j}\binom{m-j}{k} x^j y^k (1-x-y)^{m-j-k}.$$

Making use of them one can define (see [3]) the following polynomials $S_m f$,

$$S_m f(x,y) := \sum_{j=0}^{m} \sum_{k=0}^{m-j} f(\tfrac{j}{m}, \tfrac{k}{m}) p_{m,j,k}(x,y).$$

These polynomials, called the Bernstein polynomials on the triangle T, were investigated a.o. by Stancu (1960) and Lupaş (1974) whose obtained the analogical theorems to **Butzer's and Aramă's** results. That analogues can be completed (comp. **Theorem** 1) by the following

Theorem 2. *If $f \in C_T^1$ and if the Lipschitz condition* (*) *holds true for $j = 1, 2$ on the whole triangle T, then*

$$|S_m f(x,y) - f(x,y)| \le \tfrac{1}{4m}(L_1 + L_2).$$

Proof goes similarly to the proof of Theorem 1, one has only use the identities

$$\sum_{j=0}^{m} \sum_{k=0}^{m-j} \quad p_{m,j,k}(x,y) = 1,$$
$$\sum_{j=0}^{m} \sum_{k=0}^{m-j} \tfrac{j}{m} p_{m,j,k}(x,y) = x,$$
$$\sum_{j=0}^{m} \sum_{k=0}^{m-j} \tfrac{k}{m} p_{m,j,k}(x,y) = y. \ \diamondsuit$$

4. Multidimensional cases. Using the same technique as in Part 2 and Part 3 one can easily obtain the *multidimensional analogues* of the Theorems 1 and 2 concerning the Bernstein approximation on the cubes

$$\{(x_1, x_2, \dots, x_d) : 0 \le x_j \le 1 \quad \text{for} \quad j = 1, 2, \dots, d\}$$

and on the simplexes

$$\{(x_1, x_2, \dots, x_d) : 0 \le x_j \le 1 \quad \text{for} \quad j = 1, 2, \dots, d \quad \text{and} \ \sum_{j=1}^{d} x_j \le 1\}.$$

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Mathematica Pannonica
2/1 (1991), 71 – 76

THE ITERATES ARE NOT DENSE IN C

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Received March 1990

AMS Subject Classification: 54 H 20, 26 A 18

Keywords: iterates, density.

Abstract: It is proved that the set of all iterates of continous functions are not dense in C.

Let C denote the set of continuous functions mapping $[0,1]$ into itself endowed with the sup norm. Denot by f^k the k^{th} iterate of the continuous function f. The structure of the set $W^k = \{f^k : f \in C\}$ was examined by M. Laczkovich and P.D. Humke. They proved in [1] and [2] that W^2 is not everywhere dense in C and W^k is an analytic non-Borel subset of C. The author of this paper proved in [3], [4] that the set $\bigcup\limits_{k>1} W^k$ of iterates of continuous functions is a first category set and W^2 is nowhere dense. The aim of this paper is to prove the following

Theorem. *The set* $\bigcup\limits_{k>1} W^k$ *of iterates of continuous functions is not everywhere dense in* C.

In other words: *there exists an open ball* B (see Figure 1) *such that* B *does not contain any iterates of any continuous function.*

The centre of the ball B is the continuous function g which is linear on $[0, \frac{1}{2}]$, $[\frac{1}{2}, 1]$ and $g(0) = 0.03$, $g(\frac{1}{2}) = 0.99$, $g(1) = 0.03$ and the radius of the ball $r = 0.01$.

We introduce the following notations. We denote the lower boundary of B by $g_1(x) = g(x) - 0.01$ and the upper one by $g_2(x) = g(x) + 0.01$. Put $u_1 = \sup_{x\in[0,1]} g_1(x) = 0.98$, $u_2 = \inf\{g_1(x) | g_2(x) > u_1\} = u_1 - 2r = 0.96$, $u_3 = g_2(u_2)$ and $u_4 = g_2(u_3)$. $u_4 < g_2^{-1}(\frac{1}{2})$. Both g_1 and g_2 have only one fixed point, say r_1, r_2 respectively; put $D = [r_1, r_2]$. It is clear that for every $f \in B$ $\mathrm{Fix}(f) \subset D$ holds, where $\mathrm{Fix}(f) = \{x | f(x) = x\}$.

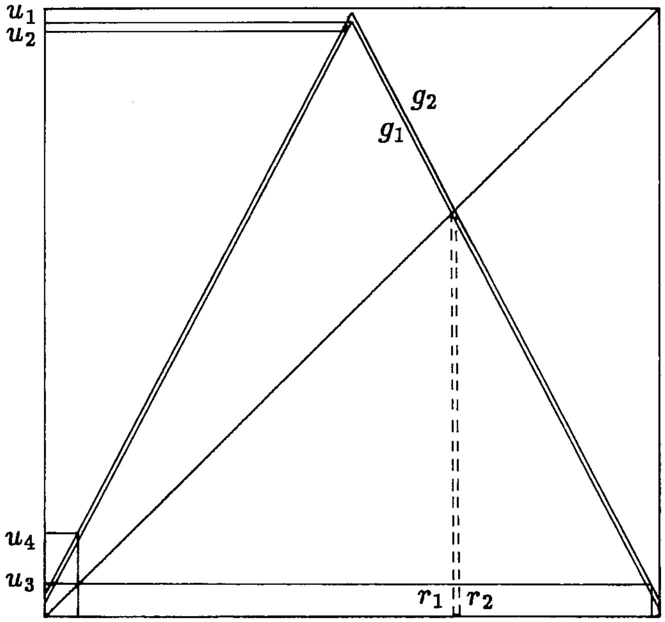

Figure 1

For every $H \subset [0,1]$ we denote by $\overline{H}$ the complement of H. Let $A, B \subset [0,1]$ we shall write $A < B$ if $a < b$ for every $a \in A$ and $b \in B$.

Proof of the Theorem. Assume that there exists $f \in B \cap \bigcup_{k>1} W^k$ say $f = \varphi^n$ for $\varphi \in C$ and $n > 1$. We define $I = \{x | g_2(x) > u_1\}$. We choose a, b such that $I = (a, b)$. It is easy to see that $u_4 < a$. For every $y \in \overline{I}$ $\varphi(y) \neq \varphi(\frac{1}{2})$ since $\varphi(y) = \varphi(\frac{1}{2})$ implies $f(y) = f(\frac{1}{2})$ which contradicts the definition of I.

There are four cases to consider:

Case 1. (i) $\varphi(\frac{1}{2}) < \varphi(\overline{I})$,
Case 2. (ii) $\varphi([0,a]) > \varphi(\frac{1}{2}) > \varphi([b,1])$,
Case 3. (iii) $\varphi([0,a]) < \varphi(\frac{1}{2}) < \varphi([b,1])$,
Case 4. (iv) $\varphi(\frac{1}{2}) > \varphi(\overline{I})$.

We prove that each of them leads to a contradiction.

Case 1: (i) holds. Now $\varphi(\frac{1}{2}) > a$ since otherwise $\mathrm{Fix}(\varphi) \cap [0,b] \neq \emptyset$ and thus $\mathrm{Fix}(f) \cap [0,b] \neq \emptyset$ which is impossible since $\mathrm{Fix}(f) \subset D$. Hence

$\varphi(\bar{I}) > \varphi(\frac{1}{2}) > a$ and, in particular, $\varphi(a) > a$. On the other hand, $f([0,1]) \cap [0,a) \neq \emptyset$ and hence there is y with $\varphi(y) < I$. Then $y \in I$ and $\varphi(y) < y$. It follows that $\mathrm{Fix}(\varphi) \cap I \neq \emptyset$. Therefore $\mathrm{Fix}(f) \cap I \neq \emptyset$ which is impossible since $\mathrm{Fix}(f) \subset D$ and $D \cap I = \emptyset$. $\Diamond$
Case 2: (ii) holds. First we prove

$$\varphi([0,u_3]) \cap [u_1,1] \neq \emptyset. \tag{1}$$

It is clear that (ii) implies

$$\min_{x\in[u_3,b]} \varphi(x) < \min_{x\in[0,u_3]} \varphi(x). \tag{2}$$

We also know that $\varphi([0,1]) \cap [u_1,1] \neq \emptyset$, hence by (ii), $\varphi([0,b]) \cap [u_1,1] \neq \neq \emptyset$. Thus if (1) doesn't hold then

$$\max_{x\in[u_3,b]} \varphi(x) > \max_{x\in[0,u_3]} \varphi(x) \tag{3}$$

must hold. From (2) and (3) we get $\varphi([0,u_3]) \subset \varphi([u_3,b])$ and thus $f([0,u_3]) \subset f([u_3,b])$ which is false and (1) follows. Pick $x_0 \in \in \varphi^{-1}([u_1,1]) \cap [0,u_3]$. Then

$$\varphi(f(x_0)) = f(\varphi(x_0)) < u_3 \tag{4}$$

holds by the definition of u_3. Since $f(x_0) < u_4$ (implied by $x_0 \in [0,u_3]$) from (4) we have

$$\min_{x\in[0,u_4]} \varphi(x) < u_3$$

and further (1) implies $\max_{x\in[0,u_4]} \varphi(x) > u_1$. Thus $\mathrm{Fix}(\varphi) \subset \varphi([0,u_4])$ since $\mathrm{Fix}(\varphi) \subset D \subset [u_3,u_1]$. Hence $f([0,u_4]) \cap \mathrm{Fix}(f) \neq \emptyset$ which contradicts $f \in B$. $\Diamond$
Case 3: (iii) holds. Let $d = \max \mathrm{Fix}(\varphi)$. First, we show

$$\varphi([0,d]) \leq u_1. \tag{5}$$

Suppose instead that

$$\exists m \leq d \text{ such that } \varphi(m) > u_1. \tag{6}$$

Since

$$(7) \qquad \varphi(x) < x \quad \text{for every} \quad x > d,$$

we have $\varphi([d,u_1]) < u_1$. From the assumptions (6) and (iii) it follows that $\varphi([a,m]) \supset \varphi([d,u_1])$ and thus $f([a,m]) \supset f([d,u_1])$ which contradicts $f \in B$. Thus (5) holds. Choose m such that $\varphi(m) > u_1$. From (5) and (7) we have $m > u_1$ whence $f(m) < u_4$. Thus $\exists 0 \le j \le n-1$ for which $[\varphi^j(m)\varphi^{j+1}(m)] \cap \mathrm{Fix}(\varphi) = \emptyset$. Let z be an arbitrary element of the set $[\varphi^j(m), \varphi^{j+1}(m)] \cap \mathrm{Fix}(\varphi)$. Then $z \in \varphi^j([\varphi(m), m])$ and hence

$$z = \varphi^{n-j}(z) \in f([\varphi(m), m]) \subset f([u_1, 1])$$

which is impossible as $z \in D$. $\diamondsuit$

Case 4: (iv) holds. Assume first that $n \ge 3$. We need 3 Lemmas.

Lemma 1. *Put* $j = \min\{x \in [a,b] | \varphi(x) \ge u_1\}$. *(It follows from* (iv) *that such a* j *exists.) Then the following inequalities hold:*

$$(8) \qquad \varphi(f(j)) < u_3,$$
$$(9) \qquad \varphi(f^2(j)) < u_4.$$

Proof. The relations $\varphi(j) \ge u_1$ and $f([u_1,1]) < u_3$ imply that $f(\varphi(j)) = \varphi(f(j)) < u_3$ which proves (8), while (9) follows from the definition of u_4:

$$\varphi(f^2(j)) = f^2(\varphi(j)) = f(f(\varphi(j))) \in f([0,u_3]) < u_4. \ \diamondsuit$$

Lemma 2. $\varphi([u_2,1]) < j$.

Proof. Assume that $\varphi([u_2,1]) < j$ doesn't hold. It follows from (8) that

$$\min_{x \in [u_2,1]} \varphi(x) < u_3 < j$$

thus $\exists x_0 \in [u_2,1]$ such that $\varphi(x_0) = j$. Hence

$$(10) \qquad \varphi^2(x_0) \ge u_1.$$

On the other hand: $\varphi([0,u_3]) \cap \mathrm{Fix}(\varphi) = \emptyset$ since otherwise $f([0,u_3]) \cap$ $\cap \mathrm{Fix}(f) \neq \emptyset$ would hold which is impossible. Thus from $f^2(j) \in [0,u_3]$ and from (9) we have $\min \varphi([0,u_3]) < \mathrm{Fix}(\varphi)$. Using (8) we find

$$\varphi^2(f(j)) = \varphi(\varphi(f(j))) \in \varphi([0,u_3]) < \mathrm{Fix}(\varphi).$$

From this and (10) we get

$$\min_{x \in [u_2,1]} \varphi^2(x) < \mathrm{Fix}(\varphi) < u_1 \le \max_{x \in [u_2,1]} \varphi^2(x),$$

thus $\varphi^2([u_2,1]) \supset \text{Fix}(\varphi)$, that is $f([u_2,1]) \cap \text{Fix}(f) \neq \emptyset$ contradicting $f \in B$. $\diamondsuit$

Since $f(j) > u_1$ and $f^2(j) < u_3$, it follows from (9) that $\varphi([0,u_3]) \cap$ $\cap [0,u_4] \neq \emptyset$. On the other hand, $\varphi(j) \geq u_1 > j$ and, as $u_3 < u_4 < j$, it follows that there is a $v_2 < j$ such that $\varphi(v_2) = j$.

Lemma 3. $\varphi^{-1}(v_2) \cap (0,j) \neq \emptyset$.

Proof. We first show that $u_4 < v_2$. Assume that $v_2 \leq u_4$. Then

$$\varphi([0,u_4]) \supset [u_4,j] \tag{11}$$

since $\varphi(v_2) = j$ and $\varphi(f^2(j)) < u_4$. Further $\varphi([0,u_4]) < \text{Fix}(\varphi)$, since otherwise $\text{Fix}(f) \cap f([0,u_4]) \neq \emptyset$ which is impossible. Now by (11) $\varphi^2([0,u_4]) \supset \varphi([u_4,j]) \supset [\varphi(u_4),\varphi(j)] \supset \text{Fix}(\varphi)$ but this implies

$$f([0,u_4]) \cap \text{Fix}(f) \neq \emptyset,$$

a contradiction. Thus $u_4 < v_2$. We know that $\varphi(f^2(j)) \in [0,u_4]$ so using (8) and the definition of v_2 we get a point z such that $\min f^2(j) < z < v_2$ and $\varphi(z) = v_2$. Thus $\varphi^{-1}(v_2) \cap (0,j) \neq \emptyset$. $\diamondsuit$

Choose $v_1 \in \varphi^{-1}(v_2) \cap (0,j)$; the action of the first 3 iterates on v_1 is shown bellow:

$$v_1 \overset{\varphi}{\Rightarrow} v_2 \overset{\varphi}{\Rightarrow} j \overset{\varphi}{\Rightarrow} \varphi(j) \in [u_1,1].$$

Then $\varphi^3(v_1) \geq u_1$ and by Lemma 2 $\varphi^3(\varphi_2) = \varphi(v^3(\varphi_1)) < j$. Thus we get

$$\varphi^3([v_1,v_2]) \supset [j,u_1]. \tag{12}$$

But $\varphi(j) \geq u_1$ and it follows from Lemma 2 that $\varphi(u_1) < j$ whence

$$\varphi([j,u_1]) \supset [j,u_1]. \tag{13}$$

From (12) and (13) we get

$$\begin{gathered} f([v_1,v_2]) = \varphi^{n-3}(\varphi^3[v_1,v_2]) \supset \varphi^{n-3}([j,u_1]) \supset \\ \supset \varphi^{n-4}([j,u_1]) \supset \ldots \supset [j,u_1] \end{gathered}$$

and it follows from the definition of I that $[v_1,v_2] \cap I \neq \emptyset$, whence $v_2 \in I$. Thus

$$\varphi([v_2,j]) \supset [\varphi(v_2),\varphi(j)] \supset \text{Fix}(\varphi)$$

and further $[v_2,j] \supset I$ since $v_2 \in I$. Thus we get $f(I) \cap \text{Fix}(f) \neq \emptyset$ which is a contradiction.

It remains to consider Case 4 with $n = 2$. We keep the definition of j from Lemma 1. We have

$$\varphi(j) \in [u_2, 1] \quad \text{and} \quad \varphi(\varphi(j)) \in [u_2, 1]. \tag{14}$$

Whence $\varphi([u_2,1]) \cap [u_2,1] \neq \emptyset$ but $\varphi(f(j)) = f(\varphi(j)) < u_3$ shows that $\varphi([u_2,1]) \cap [0,u_3] \neq \emptyset$ and hence $f([u_2,1]) \cap \mathrm{Fix}(f) \neq \emptyset$ which is a contradiction. $\diamondsuit$

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2/1 (1991), 77 – 94

ON k-HAMILTON GEOMETRY

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Dedicated to Professor Radu Miron.

Received February 1990

AMS Subject Classification: 53 C 60, 53 B 40

Keywords: vector bundle, Hamilton geometry, differential structures, nonlinear connections, d-connections, metrical structures.

Abstract: The theory of Hamilton geometry $(k=1)$ has been developed by R. MIRON ([15], [16]). In this paper we study the theory of k-Hamilton geometry $(k>1)$ using Miron's theory of Hamilton geometry as a pattern. First we show the reasons for undertaking this work and the previous results in the theory of k-Lagrange geometry ([8], [9], [14]). Next, we consider the vector bundle $\xi = (\bigoplus_1^k T^*M, \pi^*, M)$ and describe the geometry of the total space $E^* = \bigoplus_1^k T^*M$ called k-Hamilton geometry.

1. Introduction

It is well-known that parameter-invariant problems (i.e. homogeneous cases) in the calculus of variations lend themselves well to geometrical interpretation and this has given rise to metric differential geometries such as that of Finsler and its special cases: Riemannian and Minkowskian geometry. But it is also known from the classical calculus of variations (c.f. [3], [17], [18]) that there exist several problems for which the fundamental integral is dependent on the choice of the parameter. This dependence implies that the Lagrangian cannot possess certain homogenity properties.

It was J. Kern [6] who introduced the term *Lagrange geometry* with a regular Lagrangian but without homogenity condition. It is obvious that this geometry is more general than the Finslerian.

Although the introduction of the notion of Lagrange geometry belongs to J. Kern, the whole theory of Lagrange geometry has been developed by Romanian geometers led by R. Miron (c.f. [1], [2], [11], [12], [13]). In the models for Lagrange geometry the basic manifold is the total space TM of the tangent bundle to a manifold M.

In a series of papers ([8], [9], [14]) *we have constructed a geometrical model for variational problems of multiple integrals called k-Lagrange geometry.* The formulation of variational problems of multiple integrals (c.f. [19], [20]) suggests that a *geometrical model* could be the *total space* $E = \bigoplus\limits_1^k TM$ of the *vector bundle* $\bigoplus\limits_1^k TM \to M$.

We note that this vector bundle was used by C. Günther [5] too. *Our theory,* on the contrary, *is based on the study of a metric which is derived from the Lagrangian. We have used as a pattern the geometry of the total space of a vector bundle* as it was developed by R. Miron [10].

We have described differential structures, nonlinear connections, d-connections and metrical structures on $E = \bigoplus\limits_1^k TM$. We have pointed out that E carries several tensorial structures and studied conditions for their integrability. Furthermore we have given an application of k-Langrange geometry considering the Moór equivalence problem ([17], [18]) in the calculus of variations of multiple integrals.

In the papers [15] and [16] R. Miron has introduced a new concept: *Hamilton geometry* which corresponds to the notion of Lagrange geometry under the duality of the tangent ($TM \to M$) and the cotangent ($T^*M \to M$) bundles. He studied also its applications in theoretical physics.

This article has been inspired by R. Miron's papers mentioned above and by the theory of k-Lagrange geometry.

Let us consider the 1-jet bundle $J^1(\mathbb{R}^k, TM) \to M$ together with the 1-cojet bundle $J^1(TM,\mathbb{R}^k) \to M$. The 1-jet bundle has typical fiber $L(\mathbb{R}^k,\mathbb{R}^n)$ while the 1-cojet bundle has typical fiber $L(\mathbb{R}^n,\mathbb{R}^k)$. We recall that $J^1(\mathbb{R}^k, TM) \simeq \mathrm{Hom}(\mathbb{R}^k, TM) \simeq TM \otimes (\mathbb{R}^k)^*$ as vector bundles ([5], [8]). Moreover we have $J^1(TM,\mathbb{R}^k) \simeq \mathrm{Hom}\,(TM,\mathbb{R}^k) \simeq$ $\simeq T^*M \otimes \mathbb{R}^k$ as vector bundles too. Here $\mathrm{Hom}\,(\mathbb{R}^k, TM)$ denotes the total space of the vector bundle defined by all linear maps $\mathbb{R}^k \to$ $\to T_qM, q \in M$. Since there exist the isomorphisms $\mathrm{Hom}\,(\mathbb{R}^k, TM) \simeq$ $\simeq \bigoplus\limits_1^k TM$ and $\mathrm{Hom}\,(TM,\mathbb{R}^k) \simeq \bigoplus\limits_1^k T^*M$, it follows that $\mathrm{Hom}(TM,\mathbb{R}^k)$ is the dual of $\mathrm{Hom}\,(\mathbb{R}^k, TM)$. We shall use these isomorphisms in the sequel. Let π^* be the projection on M, i.e. $\pi^* : \bigoplus\limits_1^k T^*M \to M$. We shall consider the vector bundle $\zeta^* = (\bigoplus\limits_1^k T^*M,\ \pi^*,\ M)$ and the geometry of the total space $E^* = \bigoplus\limits_1^k T^*M$.

First we describe differential structures and nonlinear connections on E^*. We define tensorial structures on E^* and give conditions for their integrability.

Moreover, we shall study d-connections, metrical structures on E^* and the Legendre transformation between E and E^*.

Acknowledgement. The author wishes to express her gratitude to Professors Radu Miron and Mihai Anastasiei for their valuable comments and suggestions. She would like to thank also Professor Hans Sachs for his advice.

2. Differential structure on $E^* = \bigoplus\limits_1^k T^*M$

Let (U, ψ) be a local chart on M. Then $(U, \varphi^*, \mathbb{R}^{kn})$ is a bundle chart of the vector bundle ζ^* where

$$\varphi^* : (\pi^*)^1(U) \to U \times \mathbb{R}^{kn} \tag{2.1}$$

and for $\underset{(\alpha)}{\xi}{}_q \in \bigoplus_1^k T_q^* M \quad (q \in M, \alpha = \overline{1,k})$ we have

$$\varphi^*(\underset{(1)}{\xi}{}_q, \ldots, \underset{(k)}{\xi}{}_q) = (p_i^\alpha). \tag{2.2}$$

We can see that

$$\underset{(\alpha)}{\xi}{}_q = p_i^\alpha dq^i = p_1^\alpha dq^1 + \ldots + p_n^\alpha dq^n \tag{2.3}$$

which is *a linear form* for every α.

Puttig $(q^i) = \psi(q)$ we define

$$\begin{aligned} &\text{(a)} \quad h^* : (\pi^*)^1(U) \to \psi(U) \times \mathbb{R}^{kn} \\ &\text{(b)} \quad h^*(\underset{(1)}{\xi}{}_q, \ldots, \underset{(k)}{\xi}{}_q) = (q^i, p_i^\alpha) \end{aligned} \tag{2.4}$$

then we get the canonical coordinates (q^i, p_i^α) on $(\pi^*)^1(U)$. The set of charts $((\pi^*)^1(U), h^*)$ defines a differentiable atlas on $E^* = \bigoplus_1^k T^*M$.

The transition maps on E^* are as follows:

$$\begin{aligned} &\text{(a)} \quad \bar{q}^i = \bar{q}^i(q^1, \ldots, q^n) \\ &\text{(b)} \quad \bar{p}_i^\alpha = (\bar{\partial}_i q^j) p_j^\alpha \end{aligned} \tag{2.5}$$

where $\bar{\partial}_i := \partial / \partial \bar{q}^i$. The transformation law shows that (p_i^α) can be considered as a *covariant* vector. (In the following we denote p_i^α by p_a where $\binom{\alpha}{i} := a$ and use a shorter notation $a, b, c \ldots$ instead of double covariant indices $\binom{\alpha}{i}$ or contravariant indices $\binom{i}{\alpha}$ if the computation allows it for us.)

A local natural basis of the tangent space $T_{u^*}(E^*)$ in $u^* \in E^*$ is $(\partial_i, \partial_\alpha^i) = (\partial_i, \partial^a)$ where $\partial_i := \partial / \partial_q^i$ and $\partial_\alpha^i := \partial / \partial p_i^\alpha$. Its dual basis is $(dq^i, dp_i^\alpha) := (dq^i, dp_a)$. Under a change of coordinates in (2.5) we obtain

$$\begin{aligned} &\text{(a)} \ \bar{\partial}_i = \bar{\partial}_i q^j \partial_j - p_k^\alpha \bar{\partial}_i \bar{\partial}_j q^k \partial_h \bar{q}^j \partial_\alpha^h \\ &\text{(b)} \ \bar{\partial}_\alpha^i = \partial_j \bar{q}^i \partial_\alpha^j \qquad (\bar{\partial}_\alpha^i := \partial / \partial \bar{p}_i^\alpha) \end{aligned} \tag{2.6}$$

and

$$\begin{aligned} &\text{(a)} \ d\bar{q}^j = \partial_i \bar{q}^j dq^i \\ &\text{(b)} \ d\bar{p}_i^\alpha = p_k^\alpha \bar{\partial}_i \bar{\partial}_j q^k \partial_h \bar{q}^j dq^h + \bar{\partial}_i q^j dp_j^\alpha, \end{aligned} \tag{2.7}$$

respectively.

Hence we have a generalization of a result given by R. Miron in the case $k = 1$ in [15]:

Proposition 2.1. *Setting*

$$\widetilde{\mathrm{p}} = (\overset{1}{p}, \dots, \overset{k}{p}) \tag{2.8}$$

we get an $\mathbb{R}^k$*-valued 1-form on* E^* *whose components*

$$\overset{\alpha}{p} = p_i^\alpha dq^i \qquad (\alpha = \overline{1,k})$$

will be called fundamental forms.

The differential of $\widetilde{\mathrm{p}}$ is obtained by differentiating its components. So we have the following $\mathbb{R}^k$-valued 2-form on E^*:

$$\omega := d\widetilde{\mathrm{p}} \tag{2.9}$$

where

$$\begin{aligned} &\text{(a)} \quad \omega = (\overset{1}{\omega}, \dots, \overset{k}{\omega}), \\ &\text{(b)} \quad \overset{\alpha}{\omega} = dp_i^\alpha \wedge dq^i \qquad (\alpha = \overline{1,k}). \end{aligned} \tag{2.10}$$

Any 2-form $\overset{\alpha}{\omega}$ is nondegenerate and $d\overset{\alpha}{\omega} = 0$, hence

$$d\omega = (d\overset{1}{\omega}, \dots, d\overset{k}{\omega}) = 0. \tag{2.11}$$

This means that the $\mathbb{R}^k$-valued 2-form ω is closed. Therefore ω is a *polysymplectic form* and (E^*, ω) is a *polysymplectic manifold* in C. Günther's sense [5].

3. Nonlinear connections on $\mathbf{E}^* = \bigoplus_1^k \mathbf{T}^*\mathbf{M}$

The kernel of $D\pi^*$ (the differential of π^*) is a subbundle of the cotangent bundle $\bigoplus_1^k T^*M \to M$. It will be denoted by $VE^* \to E^*$ and will be called the vertical bundle. A map $u^* \to V_{u^*}(E^*)$ where $u^* \in E^*$ and $V_{u^*}(E^*)$ is the fiber of the vertical bundle, defines a distribution $\mathbf{V}$ on $E^* = \bigoplus_1^k T^*M$ which will be called the vertical distribution.

Definition 3.1. A *nonlinear connection* on E^* is a distribution $\mathbf{N}$: $u^* \in E^* \to N_{u^*} \subset T_{u^*}(E^*)$ which is supplementary to the vertical distribution $\mathbf{V}$, i.e.

$$T_{u^*}(E^*) = N_{u^*} \oplus V_{u^*} \tag{3.1}$$

holds for any $u^* \in E^*$.

The vertical subspaces $V_{u^*}(E^*)$ are spanned by $(\partial^i_\alpha) = (\partial^a)$. The *horizontal distribution* $\mathbf{N}$ is locally determined by

$$\delta^*_i = \partial_i + N^\alpha_{ji}(q,p)\partial^j_\alpha \qquad (\delta^*_i := \delta/\delta q^i). \tag{3.2}$$

Hence $(\delta^*_i, \partial^i_\alpha) = (\delta^*_i, \partial^a)$ is a local frame, adapted to the decomposition of $T_{u^*}(E^*)$. The real functions $N^\alpha_{ji}(q,p)$ defined on $(\pi^*)^{-1}(U)$ are called the local coefficients of the nonlinear connection $\mathbf{N}$ and characterize it. The dual frame is $(dq^i, \delta^* p^\alpha_i) = (dq^i, \delta^* p_a)$ where

$$\delta^* p_a = dp_a - N_{aj} dq^j. \tag{3.3}$$

Under a change of coordinates in (2.5) we get their transformation laws:

$$\begin{aligned} &(a)\ \ \bar\delta^*_j = \bar\partial_j q^i \delta^*_i &&(b)\ \ \bar\partial^i_\alpha = \partial_j \bar q^i \partial^j_\alpha \\ &(c)\ \ d\bar q^i = \partial_j \bar q^i dq^j &&(d)\ \ \delta^* \bar p^\alpha_i = \bar\partial_i q^j \delta^* p^\alpha_j. \end{aligned} \tag{3.4}$$

With respect to the transformation (2.5) the coefficients $N^\alpha_{ji}(q^s, p^\beta_s)$ of a nonlinear connection $\mathbf{N}$ have the following transformation law:

$$\bar N^\alpha_{ji}(\bar q, \bar p) = \bar\partial_j q^k \bar\partial_i q^h N^\alpha_{kh}(q,p) + p^\alpha_k \bar\partial_j \bar\partial_i q^k \tag{3.5}$$

for every α.

A direct calculation gives

$$\begin{aligned} &(a)\ \ [\delta^*_i, \delta^*_j] = (\delta^*_i N_{aj} - \delta^*_j N_{ai})\partial^a \\ &(b)\ \ [\delta^*_i, \partial^a] = -(\partial^a N_{bi})\partial^b \\ &(c)\ \ [\partial^a, \partial^b] = 0. \end{aligned} \tag{3.6}$$

We can associate to a nonlinear connection on E^* the following geometrical objects:

$$\begin{aligned} &(a)\quad \tau_{ia} = N_{ia} - N_{ai} &&(a := \tbinom{\alpha}{j}), \\ &(b)\quad R_{aij} = \delta^*_i N_{aj} - \delta^*_j N_{ai} &&(a := \tbinom{\alpha}{k}). \end{aligned} \tag{3.7}$$

They give us antisymmetric d-tensor fields in i and j. Moreover we obtain

Proposition 3.1. *The horizontal distribution* $\mathbf{N}$ *is integrable iff* $R_{aij} = 0$.

The decomposition of the tangent space $T_{u^*}(E^*)$ at the point $u^* \in E^*$ is the following

$$T_{u^*}(E^*) = H_{u^*}(E^*) \oplus V_{u^*}(E^*). \tag{3.8}$$

This decomposition defines a decomposition of the cotangent space $T^*_{u^*}(E^*)$ in $u^* \in E^*$:

$$T^*_{u^*}(E^*) = H^{\perp}_{u^*}(E^*) \oplus V^{\perp}_{u^*}(E^*). \tag{3.9}$$

The coframe $(dq^i, \delta^* p^\alpha_i)$ is adapted to this decomposition.

The elements of $H^{\perp}_{u^*}(E^*)$ are 1-forms which vanish for vertical vector fields and the element of $V^{\perp}_{u^*}(E^*)$ are also 1-forms which vanish for horizontal vector fields.

An easy computation shows that dq^i are *horizontal 1-forms* and $\delta^* p_a$ are *vertical 1-forms.*

By using $\delta^* p^\alpha_i$ we find that

$$\overset{\alpha}{\omega}{}^{\perp} = \tfrac{1}{2}\tau^\alpha_{ji} dq^j \wedge dq^i + \delta^* p^\alpha_i \wedge dq^i \qquad (\alpha = \overline{1,k}) \tag{3.11}$$

which is compatible with the decomposition (3.9). This formula introduces the 2-forms

$$\overset{\alpha}{\Theta} = \delta^* p^\alpha_i \wedge dq^i \qquad (\alpha = \overline{1,k}) \tag{3.12}$$

considered by R. Miron [15] in the case $k = 1$.

The $\mathbb{R}^k$-valued 2-form $\Theta = (\overset{1}{\Theta}, \dots, \overset{k}{\Theta})$ defines an *almost polysymplectic structure* on E^*. As we have seen above this is a polysymplectic form if $\tau^\alpha_{ji} = 0$ for every α. In this case the nonlinear connection $\mathbf{N}(N^\alpha_{ji})$ is symmetric.

As in the case of *Hamilton geometry* (for $k = 1$) we have the following relations between the adapted frames on $T_{u^*}(E^*)$ and $T^*_{u^*}(E^*)$ respectively:

$$\begin{array}{lllll} & \text{(a)} & < dq^i, \delta^*_j > = \delta^i_j & \text{(b)} & < \delta^* p^\alpha_i, \delta^*_j > = 0 \\ (3.13) & & & & \\ & \text{(c)} & < dq^i, \partial^j_\alpha > = 0 & \text{(d)} & < \delta^* p^\alpha_i, \partial^j_\beta > = \delta^j_i \delta^\alpha_\beta. \end{array}$$

We can associate to $\mathbf{N}$ an *almost product structure* $\mathbf{P}$ on E^* defined as follows:

$$\text{(a)}\ \mathbf{P}(\delta^*_i) = \delta^*_i \qquad \text{(b)}\ \mathbf{P}(\partial^i_\alpha) = -\partial^i_\alpha. \tag{3.14}$$

It is easy to check that

$$\overset{\alpha}{\Theta}(PX, PY) = -\overset{\alpha}{\Theta}(X, Y) \qquad (\alpha = \overline{1, k}) \tag{3.15}$$

for any $X, Y \in \mathcal{X}(E^*)$. This can be written as

$$\Theta(PX, PY) = -\Theta(X, Y) \tag{3.16}$$

where $\Theta = (\overset{1}{\Theta}, \ldots, \overset{k}{\Theta})$. We shall call $(\Theta, \mathbf{P})$ *almost hyperbolic structure* on E^*.

4. Tensorial structures on $\mathbf{E}^* = \bigoplus\limits_{1}^{k} \mathbf{T}^*\mathbf{M}$

If we set

(4.1) (a) $\overset{\alpha}{F}(\delta_i^*) = -\partial_\alpha^i$ (b) $\overset{\alpha}{F}(\partial_\alpha^i) = \delta_i^*$ (c) $\overset{\alpha}{F}(\partial_\beta^i) = 0, (\forall \beta \neq \alpha)$

we obtain k *f-structures* for which

$$\overset{\alpha}{F}{}^3 + \overset{\alpha}{F} = 0 \qquad (\alpha = \overline{1, k}). \tag{4.2}$$

Analogously, we can define as in the case of *k-Lagrange geometry [8]* the following *tensorial structures* $\overset{\alpha}{Q}(\alpha = \overline{1, k})$

(4.3) (a) $\overset{\alpha}{Q}(\delta_i^*) = \partial_\alpha^i$ (b) $\overset{\alpha}{Q}(\partial_\alpha^i) = \delta_i^*$ (c) $\overset{\alpha}{Q}(\partial_\beta^i) = 0 \quad (\forall \beta \neq \alpha)$

and we obtain

$$\overset{\alpha}{Q}{}^3 - \overset{\alpha}{Q} = 0 \qquad (\alpha = \overline{1, k}). \tag{4.4}$$

Moreover we have

$$\begin{aligned} &\text{(a)} \;\; \overset{\alpha}{\Theta}(\overset{\beta}{F}X, \overset{\beta}{F}Y) = \overset{\alpha}{\Theta}(X, Y), \\ &\text{(b)} \;\; \overset{\alpha}{\Theta}(\overset{\beta}{Q}X, \overset{\beta}{Q}Y) = \overset{\alpha}{\Theta}(X, Y) \end{aligned} \tag{4.5}$$

for any α, β and any $X, Y \in \mathcal{X}(E^*)$.

Now we study the *integrability of the structures* $\overset{\alpha}{F}$ and $\overset{\alpha}{Q}$ respectively.

We have the following conditions

$$(4.6)\qquad \begin{array}{ll}\text{(a)} & \operatorname{rank}(\overset{\alpha}{F}) = n < nk \\ \text{(b)} & \operatorname{rank}(\overset{\alpha}{Q}) = n < nk\end{array} \qquad (k > 1)$$

It is easy to see that $F_1 = -\overset{\alpha}{F}{}^2$ and $F_2 = \overset{\alpha}{F}{}^2 + I$ are two supplementary projectors associated to $\overset{\alpha}{F}$. It is said that $\overset{\alpha}{F}$ is integrable if the distributions associated to F_1 and F_2 are integrable. As V. Duc [4] proved these distributions are integrable iff $N_{\overset{\alpha}{F}{}^2} = 0$ where $N_{\overset{\alpha}{F}{}^2}$ means the Nijenhuis tensor field of $\overset{\alpha}{F}{}^2$ $(\alpha = \overline{1,k})$.

Further we associate with $\overset{\alpha}{Q}$ the set of projectors $Q_1 = I - \overset{\alpha}{Q}{}^2$, $Q_2 =$
$= \frac{1}{2}(\overset{\alpha}{Q} + \overset{\alpha}{Q}{}^2)$ and $Q_3 = \frac{1}{2}(-\overset{\alpha}{Q} + \overset{\alpha}{Q}{}^2)$. Let $D_i (i = 1,2,3)$ be the distributions defined by these projectors. The structure $\overset{\alpha}{Q}$ is said to be integrable if the distributions D_i and $D_i + D_j (j = 1,2,3)$ are integrable. V. Duc [4] proved that $\overset{\alpha}{Q}$ is integrable iff $N_{\overset{\alpha}{Q}} = O$ where $N_{\overset{\alpha}{Q}}$ is the Nijenhuis tensor field of $\overset{\alpha}{Q} (\alpha = \overline{1,k})$.

Using the definition of $\overset{\alpha}{F}$ we get

$$(4.7)\qquad \text{(a)}\ \overset{\alpha}{F}{}^2(\delta_i^*) = -\delta_i^* \quad \text{(b)}\ \overset{\alpha}{F}{}^2(\partial_\alpha^i) = -\partial_\alpha^i \quad \text{(c)}\ \overset{\alpha}{F}{}^2(\partial_\beta^i) = 0 \quad (\beta \neq \alpha).$$

The relation (4.2) implies that

$$\overset{\alpha}{F}{}^4 = -\overset{\alpha}{F}{}^2.$$

Hence we have

$$(4.9)\qquad N_{\overset{\alpha}{F}{}^2}(X,Y) = [\overset{\alpha}{F}{}^2X, \overset{\alpha}{F}{}^2Y] + \overset{\alpha}{F}{}^4[X,Y] - \overset{\alpha}{F}{}^2[\overset{\alpha}{F}{}^2X, Y] - \overset{\alpha}{F}{}^2[X, \overset{\alpha}{F}{}^2Y] = [\overset{\alpha}{F}{}^2X, \overset{\alpha}{F}{}^2Y] - \overset{\alpha}{F}{}^2[X,Y] - \overset{\alpha}{F}{}^2[\overset{\alpha}{F}{}^2X, Y] - \overset{\alpha}{F}{}^2[X, \overset{\alpha}{F}{}^2Y].$$

To find conditions for the integrability of $\overset{\alpha}{F}$ which are equivalent to V. Duc's conditions we shall compute $N_{\overset{\alpha}{F}{}^2}$ in the adapted frame $(\delta_i^*, \partial_\alpha^i)$ using the relations (3.6) and (3.7) (b). We obtain for fixed α

$$(4.10)\qquad \begin{array}{lll}\text{(a)} & N_{\overset{\alpha}{F}{}^2}(\delta_j^*, \delta_k^*) = R_{ijk}^\beta \partial_\beta^i & \text{(summing over } \beta \neq \alpha) \\ \text{(b)} & N_{\overset{\alpha}{F}{}^2}(\delta_j^*, \partial_\alpha^k) = -\partial_\alpha^k(N_{ij}^\beta)\partial_\beta^i & \text{(summing over } \beta \neq \alpha) \\ \text{(c)} & N_{\overset{\alpha}{F}{}^2}(\delta_i^*, \partial_\beta^k) = 0 & (\beta \neq \alpha)\end{array}$$

$$\text{(d)}\quad N_{\overset{\alpha}{F}^2}(\partial^j_\alpha, \partial^k_\alpha) = 0$$

$$\text{(e)}\quad N_{\overset{\alpha}{F}^2}(\partial^j_\alpha, \partial^k_\beta) = 0 \qquad (\beta \neq \alpha)$$

$$\text{(f)}\quad N_{\overset{\alpha}{F}^2}(\partial^j_\beta, \partial^k_\gamma) = 0 \qquad (\beta \neq \alpha,\ \gamma \neq \alpha).$$

Theorem 4.1. *$\overset{\alpha}{F}$ is integrable iff*

$$\text{(4.11) (a)}\quad R^\beta_{ijk} = 0 \quad (\forall \beta \neq \alpha), \qquad \text{(b)}\quad \partial^k_\alpha(N^\beta_{ij}) = 0 \quad (\forall \beta \neq \alpha).$$

Corollary 4.1. *All $\overset{\alpha}{F}(\alpha = \overline{1,k})$ are integrable iff R^α_{ijk} vanish and N^α_{ik} depend only on $(p^\alpha_i)(\alpha = \overline{1,k})$.*

Remark. Condition (4.11) (a) shows that integrability of the horizontal distribution is a necessary and sufficient condition for the integrability of $\overset{\alpha}{F}$.

Now we proceed similarly for $\overset{\alpha}{Q}$ using its definition. In general we have

$$\text{(4.13)}\qquad N_{\overset{\alpha}{Q}}(X,Y) = [\overset{\alpha}{Q}X, \overset{\alpha}{Q}Y] + \overset{\alpha}{Q}{}^2[X,Y] - \overset{\alpha}{Q}[\overset{\alpha}{Q}X, Y] - \overset{\alpha}{Q}[X, \overset{\alpha}{Q}Y], \quad X, Y \in \mathcal{X}(E^*),$$

hence for the adapted frame we obtain

$$\text{(a)}\quad N_{\overset{\alpha}{Q}}(\delta^*_j, \delta^*_k) = R^\alpha_{ijk}\partial^i_\alpha - \sum_i (\partial^k_\alpha N^\alpha_{ij} - \partial^j_\alpha N^\alpha_{ik})\delta^*_i$$
$$\text{(not summing over } \alpha)$$

$$\text{(b)}\quad N_{\overset{\alpha}{Q}}(\delta^*_j, \partial^k_\alpha) = \partial^j_\alpha N^\gamma_{ik}\partial^i_\gamma - \partial^k_\alpha N^\alpha_{ij}\partial^i_\alpha - \sum_i R^\alpha_{ijk}\delta^*_i = \partial^j_\alpha N^\gamma_{ik}\partial^i_\gamma - \sum_i R^\alpha_{ijk}\delta^*_i \quad (\gamma \neq \alpha)$$

$$\text{(4.14) (c)}\quad N_{\overset{\alpha}{Q}}(\delta^*_j, \partial^k_\beta) = -\sum_i (\partial^k_\beta N^\alpha_{ij})\delta^*_i \quad (\beta \neq \alpha)$$

$$\text{(d)}\quad N_{\overset{\alpha}{Q}}(\partial^j_\alpha, \partial^k_\alpha) = R^\gamma_{ijk}\partial^i_\gamma + \sum_i (\partial^k_\alpha N^\alpha_{ij} - \partial^j_\alpha N^\alpha_{ik})\delta^*_i$$
$$\text{(not summing over } \alpha)$$

$$\text{(e)}\quad N_{\overset{\alpha}{Q}}(\partial^j_\alpha, \partial^k_\beta) = \sum_i \partial^k_\beta N^\alpha_{ij}\delta^*_i \quad (\beta \neq \alpha)$$

$$\text{(f)}\quad N_{\overset{\alpha}{Q}}\,\partial^j_\beta, \partial^k_\gamma) = 0 \quad (\beta \neq \alpha, \gamma \neq \alpha).$$

So we have proved

Theorem 4.2. *$\overset{\alpha}{Q}$ is integrable iff*

$$(4.15)\quad \begin{aligned} &(a)\ R^{\alpha}_{ijk} = 0\ (\textit{i.e. the horizontal distribution is integrable}) \\ &(b)\ \partial^j_\alpha N^\gamma_{ik} = 0 \qquad (\gamma \neq \alpha) \\ &(c)\ \partial^k_\alpha N^\alpha_{ij} = \partial^j_\alpha N^\alpha_{ik} \\ &(d)\ \partial^k_\beta N^\alpha_{ij} = 0 \qquad (\beta \neq \alpha). \end{aligned}$$

Corollary 4.2. *All* $\overset{\alpha}{Q}(\alpha = \overline{1,k})$ *are integrable iff*

$$(4.16)\quad \begin{aligned} &(a)\ R^{\alpha}_{ijk} = 0 \\ &(b)\ N^\alpha_{ij}\ \textit{depends only on}\ (p^\alpha_i)(\alpha\ \textit{is fixed}). \end{aligned}$$

We can see that the condition (4.14) (c) is equivalent to (4.15) (a) and the condition (4.15) (d) follows from (4.15) (b) and (c).

Corollary 4.3. *If* $\overset{\alpha}{Q}$ *is integrable then* $\overset{\alpha}{F}$ *is integrable. If all* $\overset{\alpha}{Q}$ *are integrable then all* $\overset{\alpha}{F}$ *are integrable.*

5. *d*-connections on $E^* = \bigoplus_1^k T^*M$

A *distinguished connection* – shortly *d-connection* – on E^*, endowed with a nonlinear connection, is a linear connection D on E^* which preserves by parallel displacements the horizontal and the vertical distributions.

Now we are interested in its *local* representation. We put as usual:

$$(5.1)\quad (a)\ D^h_X Y = D_{hX} Y \quad (b)\ D^\nu_X = D_{\nu X} Y \quad (X, Y \in \mathcal{X}(E^*))$$

and with respect to the adapted frame $(\delta^*_i, \partial^i_\alpha)$ we set

$$(5.2)\quad \begin{aligned} &(a)\ D^h_{\delta^*_k}\delta^*_j = \overset{*}{L}{}^i_{jk}\delta^*_i \qquad &(b)\ D^h_{\delta^*_k}\partial^j_\beta = \overset{*}{L}{}^{j\alpha}_{\beta ik}\partial^i_\alpha \\ &(c)\ D^\nu_{\partial^k_\beta}\delta^*_j = \overset{*}{C}{}_j{}^i{}_\beta^k\delta^*_i \qquad &(d)\ D^\nu_{\partial^k_\beta}\partial^j_\gamma = \overset{*}{C}{}^{\alpha jk}_{i\gamma\beta}\partial^i_\alpha. \end{aligned}$$

Hence we have obtained a set of functions

$$\text{(5.3)} \qquad \overset{*}{\Gamma} D = (\overset{*}{L}{}^{i}_{jk}(q,p),\ \overset{*}{L}{}^{j\alpha}_{\beta ik}(q,p),\ \overset{*}{C}{}_{j}{}^{i}{}_{\beta}^{k}(q,p),\ \overset{*}{C}{}^{\alpha jk}_{i\gamma\beta}(q,p)).$$

The set $\overset{*}{L}{}^{i}_{jk}$ changes like the components of a linear connection on M and the set $\overset{*}{L}{}^{j\alpha}_{\beta ik}$ changes like the components of a linear connection in a vector bundle if we consider $\binom{j}{\beta}$, $\binom{\alpha}{i}$ as contravariant and covariant indices ([8]). The $\overset{*}{C}{}_{j}{}^{i}{}_{\beta}^{k}$ and $\overset{*}{C}{}^{\alpha jk}_{i\gamma\beta}$ change like the components of the d-tensor fields on E^*.

The set $\overset{*}{\Gamma} D$ characterizes a d-linear connection, i.e. if $\overset{*}{\Gamma} D$ is given, there exists a unique d-connection such that its local coefficients are just $\overset{*}{\Gamma} D$.

Now the h- and ν-*covariant derivatives* can also be considered with respect to $\overset{*}{\Gamma} D$. Since later we need the h- and ν-covariant derivatives of a double covariant tensor field $g = g_{ij}(q,p)dq^i \otimes dq^j$ on M and of a double contravariant tensor field $\widetilde{g} = g^{ij}_{\alpha\beta}(q,p)\delta^* p^\alpha_i \otimes \delta^* p^\beta_j$ on E^* now we give them

$$\text{(5.4)} \quad \begin{aligned} &\text{(a)} \quad g_{ijk} = \delta^*_k g_{ij} - \overset{*}{L}{}^{s}_{ik} g_{sj} - \overset{*}{L}{}^{s}_{jk} g_{is}, \\ &\text{(b)} \quad g_{ij\alpha}^{\ \ \ k} = \partial^k_\alpha g_{ij} - \overset{*}{C}{}^{sk}_{i\alpha} g_{sj} - \overset{*}{C}{}^{sk}_{j\alpha} g_{is}, \\ &\text{(c)} \quad g^{i\ j}_{\alpha\beta k} = \delta^*_k g^{i\ j}_{\alpha\beta} + \overset{*}{L}{}^{i\ \gamma}_{\alpha mk} g^{mj}_{\gamma\beta} + \overset{*}{L}{}^{j\ \gamma}_{\beta mk} g^{im}_{\alpha\gamma}, \\ &\text{(d)} \quad g^{i\ j\ k}_{\alpha\beta\gamma} = \partial^k_\beta g^{i\ j}_{\alpha\beta} + \overset{*}{C}{}^{\epsilon\ i\ k}_{m\alpha\gamma} g^{im}_{\alpha\epsilon} + \overset{*}{C}{}^{\epsilon\ i\ k}_{m\beta\gamma} g^{im}_{\alpha\epsilon}. \end{aligned}$$

The vector field $Z = p^\alpha_i \partial^i_\alpha$ which is globally defined on E^* will be called the *Liouville* vector field on E^*.

A d-connection is of *Cartan* type if

$$\text{(5.5)} \qquad D^h_X Z = 0, \qquad D^\nu_X Z = X \qquad (\forall X \in \mathcal{X}(E^*)).$$

Expressing locally this condition we obtain:

Theorem 5.1. *A d-connection is of Cartan type iff*

$$\text{(5.6)} \qquad D^\alpha_{ik} = -p^\beta_j \overset{*}{L}{}^{j\alpha}_{\beta ik} + N^{\ \alpha}_{ik} = 0, \qquad p^\gamma_j \overset{*}{C}{}^{\alpha jk}_{i\gamma\beta} = 0.$$

The tensor field D^α_{ik} is called *h-deflection* tensor field associated to D. (cf. [15]).

The *torsions* of D are defined as usual. Their local coefficients are the following:

$$
(5.7)\quad
\begin{array}{llll}
(a) & \overset{*}{T}{}^{i}_{kj} = \overset{*}{L}{}^{i}_{kj} - \overset{*}{L}{}^{i}_{jk} & (b) & \overset{*}{R}{}^{\alpha}_{ijk} = \delta^{*}_{j}(N^{\alpha}_{ik}) - \delta^{*}_{k}(N^{\alpha}_{ij}) \\
(c) & \overset{*}{C}{}_{j}{}^{i}{}_{\beta}^{k} & (d) & \overset{*}{P}{}^{\alpha}_{ik\beta}{}^{j} = \partial^{j}_{\beta} N^{\alpha}_{ik} - \overset{*}{L}{}^{j\alpha}_{\beta ik}, \\
(e) & \overset{*}{S}{}^{\alpha jk}_{i\gamma\beta} = \overset{*}{C}{}^{\alpha jk}_{i\gamma\beta} - \overset{*}{C}{}^{\alpha kj}_{i\beta\gamma}. & &
\end{array}
$$

6. Metrical structures on $E^{*} = \bigoplus\limits_{1}^{k} T^{*}M$

Definition 6.1. A function $\mathbf{H} : \bigoplus\limits_{1}^{k} T^{*}M \to \mathbb{R}$ is called a *Hamilton function* (or a *Hamiltonian*). If $\mathbf{H} : (q^{i}, p^{\alpha}_{i}) \to H(q^{i}, p^{\alpha}_{i})$ and the matrix with the elements

$$
(6.1)\qquad g^{i\,j}_{\alpha\beta} = \partial^{i}_{\alpha}\partial^{i}_{\beta}\,\mathbf{H} \qquad (g^{i\,j}_{\alpha\beta} = g^{j\,i}_{\beta\alpha})
$$

is nondegenerate, i.e. its rank is nk, then the *Hamiltonian* $\mathbf{H}$ will be called *regular.*

Theorem 6.1. *Any regular Hamiltonian* $\mathbf{H}(q,p)$ *defines a metrical structure called Hamilton structure in the vertical bundle* VE^{*}.

Proof. Define the map

$$
(6.2)\quad
\begin{array}{ll}
(a) & g_{u^{*}} : V_{u^{*}}(E^{*}) \times V_{u^{*}}(E^{*}) \to \mathbb{R} \quad (u^{*} \in E^{*}) \text{ as} \\
(b) & g_{u^{*}}(\mathbf{X}, \mathbf{Y}) = g^{i\,j}_{\alpha\beta} X^{\alpha}_{i} Y^{\beta}_{j} \text{ where} \\
(c) & \mathbf{X} = X^{\alpha}_{i}\partial^{i}_{\alpha} \text{ and } \mathbf{Y} = Y^{\beta}_{j}\partial^{j}_{\beta}
\end{array}
$$

are vertical vector fields. This map is well-defined and obviously linear with respect to $\mathbf{X}$ and $\mathbf{Y}$. By Definition 6.1. it is nondegenerate. $\Diamond$

Now if $\mathbf{g} = g_{ij}(q,p)dq^{i} \otimes dq^{j}$ is a tensor field on E^{*} such that $\det||g_{ij}|| \neq 0$ the following metrical structure can be considered on E^{*}:

$$
(6.3)\qquad \mathbf{G} = g_{ij}(q,p)dq^{i} \otimes dq^{j} + g^{i\,j}_{\alpha\beta}(q,p)\delta^{*}p^{\alpha}_{i} \otimes \delta^{*}p^{\beta}_{j}.
$$

As usual we say that a d-connection is *metrical* with respect to G if

$$DG_X G = 0 \tag{6.4}$$

for any $X \in \mathcal{X}(E^*)$. This condition is equivalent to the following: (6.5) $g_{ij|k} = 0, \quad g_{ij}|^k = 0, \quad g^{i\,j}_{\alpha\beta|k} = 0, \quad g^{i\,j}_{\alpha\beta}|^k_\gamma = 0.$

We can prove by direct calculation using (5.4) and the symmetric property of the Hamilton structure:

Theorem 6.2. *The following d-connection is metrical and it torsions T and S vanish:*

(6.3)

$$\text{(a) } \overset{*}{L}{}^i_{jk} = \frac{1}{2} g^{im}(\delta^*_j g_{km} + \delta^*_k g_{jm} - \delta^*_m g_{jk})$$

$$\text{(b) } \overset{*}{L}{}^{j\alpha}_{\beta ik} = \partial^j_\beta N^\alpha_{ik} + \frac{1}{2} g^{\alpha\gamma}_{im}(\delta^*_k g^{jm}_{\beta\gamma} - \partial^j_\beta(N^\epsilon_{rk}) g^{rm}_{\epsilon\gamma} - \partial^m_\gamma(N^\epsilon_{rk}) g^{rj}_{\epsilon\beta})$$

$$\text{(c) } \overset{*}{C}{}_i{}^{ik}_{\ \beta} = \frac{1}{2} g^{is} \partial^k_\beta(g_{js})$$

$$\text{(d) } \overset{*}{C}{}^{\alpha jk}_{i\gamma\beta} = \frac{1}{2} g^{\alpha\epsilon}_{ir}(\partial^j_\gamma g^{kr}_{\beta\epsilon} + \partial^k_\beta g^{jr}_{\gamma\epsilon} - \partial^r_\epsilon g^{jk}_{\gamma\beta}).$$

Here g^{ij} is the inverse of g_{jk} and $g^{\alpha\gamma}_{im}$ is the inverse of $g^{mj}_{\gamma\beta}$ i.e.

$$g^{ij} g_{jk} = \delta^i_k \quad \text{and} \quad g^{\alpha\gamma}_{im} g^{mj}_{\gamma\beta} = \delta^j_i \delta^\alpha_\beta := \delta^{\alpha j}_{i\beta}$$

hold.

An interesting particular case is obtained when g_{ij} do not depend on p. In such a case $g_{ij}(q)$ can be thought as defining a metrical structure on M and we have

$$\text{(a) } \overset{*}{L}{}^i_{jk} = \frac{1}{2} g^{im}(\delta^*_j g_{km} + \delta^*_k g_{jm} - \delta^*_m g_{jk})$$

$$\text{(b) } \overset{*}{C}{}_j{}^{i\,k}_{\ \beta} = 0$$

(6.7)

$$\text{(c) } \overset{*}{L}{}^{j\alpha}_{\beta ik} = \partial^j_\beta N^\alpha_{ik} + \frac{1}{2} g^{\alpha\gamma}_{im}(\delta^*_k g^{jm}_{\beta\gamma} - \partial^j_\beta(N^\epsilon_{rk}) g^{rm}_{\epsilon\gamma} - \partial^m_\gamma(N^\epsilon_{rk}) g^{rj}_{\epsilon\beta}$$

$$\text{(d) } \overset{*}{C}{}^{\alpha jk}_{i\gamma\beta} = \frac{1}{2} g^{\alpha\epsilon}_{in}(\partial^j_r g^{kr}_{\beta\epsilon} + \partial^k_\beta g^{jr}_{\gamma\epsilon} - \partial^r_\epsilon g^{jk}_{\gamma\beta}).$$

Furthermore taking into account the Hamilton metric in (6.1) we get

$$\overset{*}{C}{}^{\alpha jk}_{i\gamma\beta} = \frac{1}{2} g^{\alpha\epsilon}_{i\ r} \frac{\partial^3 \mathbf{H}}{\partial p^\gamma_j \partial p^\beta_k \partial p^\epsilon_r} \tag{6.8}$$

and so the contravariant part of $\overset{*}{C}$

$$\overset{*}{C}{}^{jmk}_{\gamma\delta\beta} := g^{mi}_{\delta\alpha}\overset{*}{C}{}^{j\alpha k}_{\gamma i\beta} = \frac{1}{2}\frac{\partial^3 \mathbf{H}}{\partial p^\delta_m \partial p^\gamma_j \partial p^\beta_k} \tag{6.9}$$

is symmetric in the indices $\binom{j}{\gamma}$, $\binom{m}{\delta}$, $\binom{k}{\beta}$.

Remark. $\overset{*}{C}{}^{jmk}_{\gamma\delta\beta}$ corresponds to the tensor $\widetilde{C}^{jmk}$ from the Hamilton geometry ($k = 1$).

7. Legendre transformation

Let us consider $E = J^1(\mathbb{R}^k, TM) \simeq \bigoplus_1^k TM \xrightarrow{\pi} M$ and $\overset{*}{E} = J^1(TM, \mathbb{R}^k) \simeq \bigoplus_1^k T^*M \xrightarrow{\pi^*} M$. A Lagrangian $\mathbf{L}$ is a real-valued function on E (c.f. [14]). The vertical derivative of $\mathbf{L}$ is written as $d_\nu \mathbf{L}|_u = d(\mathbf{L}|_{E_{\pi(u)}})$ where d means differential of functions. This is a vertical 1-form because $\mathbf{L}|_{E_{\pi(u)}}$ means locally that $(x^1, \ldots, x^n)$ are fixed so $d(\mathbf{L}|_{E_{\pi(u)}}) = \partial^\alpha_i \mathbf{L} dy^i_\alpha$, i.e. an element $\partial^\alpha_i(\mathbf{L})$ of E^* is obtained. Hence a map $\mathcal{L} : E \to E^*$ can be defined as follows:

$$\mathcal{L}(x^i, y^i_\alpha) = (q^i, p^\alpha_i = \partial^\alpha_i \mathbf{L}(x, y)) \qquad ((x^i) = (q^i) \in M). \tag{7.1}$$

The map $\mathcal{L}$ is called *Legendre* map. Generally it is not a bundle morphism but it preserves the fibers.

Definition 7.1. The Lagrangian $\mathbf{L}$ is said to be *regular* if $\mathcal{L}$ is a local diffeomorphism and it is said to be *hyperregular* if $\mathcal{L}$ is a global diffeomorphism. In the latter case $\mathcal{L}$ will be called Legendre transformation.

By (7.1) $\mathbf{L}$ is regular iff the matrix $(g^{\alpha\beta}_{i\,j}) := (\partial^\alpha_i \partial^\beta_j \mathbf{L})$ is nondegenerate in any system of coordinates, i.e. the second order differential of $\mathbf{L}|_{E_{\pi(u)}}$ is nondegenerate for every $u \in E$.

Let us put the relation between a Hamiltonian and a Lagrangian under the Legendre transformation $\mathcal{L}$:

$$\mathbf{H} = p^\beta_j y^j_\beta - \mathbf{L}. \tag{7.2}$$

We prove:

Proposition 7.1. *The inverse $\mathcal{L}^{-1}$ of the Legrende transformation $\mathcal{L}$ is*

$$\mathcal{L}^{-1}(q^i, p_i^\alpha) = (x^i, y_\alpha^i = \partial_\alpha^i \mathbf{H}(x,p)). \tag{7.3}$$

Proof. We shall show that $\mathcal{L}^{-1} o \mathcal{L} = id|_E$ and conversely, $\mathcal{L} o \mathcal{L}^{-1} = id|_{E^*}$. Consider the Defintion 7.1. Since $\mathbf{L}(x,y)$ does not depend on $\partial_i^\alpha \mathbf{L}$ we get by direct calculation

$$\begin{aligned}(x^i, y_\alpha^i) &\xrightarrow{\mathcal{L}} (q^i, p_i^\alpha = \partial_i^\alpha \mathbf{L}) \xrightarrow{\mathcal{L}^{-1}} (x^i, y_\alpha^i = \partial_\alpha^i \mathbf{H}) = \\ &= (x^i, \partial(y_\beta^j \mathbf{L} - \mathbf{L})/\partial(\partial_i^\alpha \mathbf{L})) = (x^i, \delta_j^i \delta_\alpha^\beta y_\beta^j - 0) = (x^i, y_\alpha^i).\end{aligned} \tag{7.4}$$

Conversely, since $\mathbf{L} = p_j^\beta y_\beta^j - \mathbf{H}$ and the function $\mathbf{H}$ does not depend on $\partial_\alpha^i \mathbf{H}$ we directly obtain

$$\begin{aligned}(q^i, p_i^\alpha) &\xrightarrow{\mathcal{L}^{-1}} (x^i, y_\alpha^i = \partial_\alpha^i \mathbf{H}) \xrightarrow{\mathcal{L}} (q^i, \partial \mathbf{L} | \partial(\partial_\alpha^i \mathbf{H})) = \\ &= (q^i, \partial(p_j^\beta \partial_\beta^j \mathbf{H} - \mathbf{H})/\partial(\partial_\alpha^i \mathbf{H})) = (q^i, p_j^\beta \delta_i^j \delta_\beta^\alpha - 0) = (q^i, p_i^\alpha). \ \diamondsuit\end{aligned} \tag{7.5}$$

If $\mathcal{L}^T$ and $(\mathcal{L}^{-1})^T$ are the *tangent maps* to $\mathcal{L}$ and $\mathcal{L}^{-1}$, then we have

$$\begin{aligned}&\text{(a)}\ \ \mathcal{L}^T(\partial_i) = \overset{*}{\partial}_i + \partial_i \partial_k^\beta(\mathbf{L}) \partial_\beta^k \\ &\text{(b)}\ \ \mathcal{L}^T(\partial_i^\alpha) = g_{ij}^{\alpha\beta} \partial_\beta^j \\ &\text{(c)}\ \ (\mathcal{L}^{-1})^T \overset{*}{\partial}_i = \partial_i + \overset{*}{\partial}_i \partial_\alpha^k(\mathbf{H}) \partial_k^\alpha \\ &\text{(d)}\ \ (\mathcal{L}^{-1})^T(\partial_\alpha^i) = \partial_\alpha^i \partial_\beta^j(\mathbf{H}) \partial_j^\beta = g_{\alpha\beta}^{ij} \partial_j^\beta.\end{aligned} \qquad \begin{pmatrix}\overset{*}{\partial}_i := \partial | \partial q^i \\ \partial_i := \partial | \partial x^i\end{pmatrix} \tag{7.6}$$

By using these formulae we shall prove

Theorem 7.1. *If* $\mathbf{L}$ *is a hyperregular Lagrangian on* $E = \bigoplus_1^k TM$ *(i.e. the Legendre morphism associated to it is global diffeomorphism), then* $\mathcal{L}$ *carries a nonlinear connection* $\mathbf{N}$ *on* $E = \bigoplus_1^k TM$ *into a nonlinear connection* $\overset{*}{\mathbf{N}}$ *on* $\overset{*}{E} = \bigoplus_1^k T^*M$. *If* $N_{\alpha j}^i(x,y)$ *are the local coefficients of* $\mathbf{N}$ *and* $\overset{*}{N}{}_{ij}^\alpha(q,p)$ *are the local coefficients of* $\overset{*}{\mathbf{N}}$, *then we have*

$$\overset{*}{N}{}_{ij}^\alpha(q,p) = -(\partial_\beta^k \overset{*}{\partial}_j(\mathbf{H}) + N_{\beta j}^k) g_{ki}^{\beta\alpha}. \tag{7.7}$$

Proof. From (7.6) (a) and (b) we deduce that

$$\begin{aligned}\mathcal{L}^T(\delta_i) &= \mathcal{L}^T(\partial_i - N_{\alpha i}^j \partial_j^\alpha) = \overset{*}{\partial}_i + \partial_i \partial_k^\beta(\mathbf{L}) \partial_\beta^k - N_{\alpha j}^j g_{jk}^{\alpha\beta} \partial_\beta^k = \\ &= \overset{*}{\partial}_i + (\partial_i \partial_k^\beta(\mathbf{L}) - N_{\alpha i}^j g_{jk}^{\alpha\beta}) \partial_\beta^k.\end{aligned} \tag{7.8}$$

Moreover, from the defintion of $\mathbf{H}$ in (7.2) induced by $\mathbf{L}$ it follows

$$
\begin{array}{ll}
\text{(a)} & \overset{*}{\partial}{}_i\mathbf{H} = -\partial_i\mathbf{L} \\
\text{(7.9) (b)} & \partial_i\partial_k^\beta(\mathbf{L}) = \partial_k^\beta(\partial_i\mathbf{L}) = -\partial_k^\beta(\overset{*}{\partial}{}_i\mathbf{H}) = -\partial_\alpha^j\partial_i(\mathbf{H})\partial_k^\beta p_j^\alpha = \\
& = -\partial_\alpha^j\partial_i(\mathbf{H})\partial_k^\beta\partial_j^\alpha\mathbf{L} = -\partial_\alpha^j\partial_i(\mathbf{H})g_{kj}^{\beta\alpha} \qquad (x^i = q^i).
\end{array}
$$

Hence for the local basis adapted to the horizontal distribution $\mathbf{N}$ on E ([8]) we get

$$\mathcal{L}^T(\delta_i) = \overset{*}{\partial}{}_i + (-\partial_\alpha^j\overset{*}{\partial}{}_i(\mathbf{H}) - N_{\alpha i}^j)g_{j\ k}^{\alpha\beta}\partial_\beta^k. \tag{7.10}$$

Putting

$$\overset{*}{N}{}_{ki}^\beta = -(\partial_\alpha^j\overset{*}{\partial}{}_i(\mathbf{H}) + N_{\alpha i}^j)g_{jk}^{\alpha\beta} \tag{7.11}$$

we have obtained that $\mathcal{L}^T$ maps $\{\delta_i\}$ to the local basis $\{\overset{*}{\delta}\}$ adapted to $\overset{*}{\mathbf{N}}$ on $\overset{*}{E}$ and the formula (7.7) holds. $\diamondsuit$

Remark 7.1. If $\mathbf{L}$ is only regular then Theorem 7.1. is valid only locally, i.e. on an open set of $\bigoplus_1^k TM$ for which $\mathcal{L}$ is diffeomorphism.

Remark 7.2 Even though $\mathcal{L}$ is only a local diffeomorphism the coefficients $\overset{*}{N}{}_{ki}^\beta$ define a global nonlinear connection since they satisfy the usual transformation law as it can be seen by a long calculation.

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Mathematica Pannonica
2/1 (1990), 95 – 104

DREHZYKLIDEN DES GALILEISCHEN RAUMES G_3

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Received October 1990

AMS Subject Classification: 51 N 25

Keywords: Cyclides of revolution, Galileian space.

Abstract: A Galilean space G_3 is a three-dimensional affine space with an absolute $\{\omega, f, J\}$, where f is a line in the plane of infinity ω and J an elliptic involution on f. A cyclide in G_3 is an algebraic surface of order 4 that has f as double-line. In this paper we investigate all cyclides, generated by an euclidean rotation in G_3.

Zykliden des dreidimensionalen euklidischen Raumes E_3 sind nach G. Darboux [1] algebraische Flächen 4. Ordnung, die den absoluten Kegelschnitt als Doppelkurve enthalten. Analog definiert man Zykliden des einfach isotropen Raumes I_3 als algebraische Flächen 4. Ordnung, welche das absolute Geradenpaar dieses Raumes als Doppelgeraden besitzen (H. Sachs [11] – [14]; D. Palman [2] – [6]).

Im galileischen Raum G_3 können wir analog *Zykliden als jene algebraischen Flächen 4. Ordnung definieren, welche die absolute Gerade f als Doppelgerade enthalten und keine weiteren Fernpunkte besitzen.*

In dieser Arbeit werden wir jene Zykliden des galileischen Raumes G_3 betrachten, die durch eine euklidische Drehung erzeugt werden können.

1. Im reellen dreidimensionalen projektiven Raum $P_3(\mathrm{R})$, in dem wir die Punkte wie üblich durch reelle homogene Koordinaten

$$(1) \qquad (x_0 : x_1 : x_2 : x_3) \neq (0:0:0:0)$$

beschreiben, zeichnen wir eine reelle Ebene $\omega(x_0 = 0)$ und in dieser Ebene ω eine reelle Gerade $f(x_0 = x_1 = 0)$ aus. Außerdem definieren wir auf der Geraden f durch

$$(2) \qquad J: \quad (0:0:x_2:x_3) \to (0:0:x_3:-x_2)$$

eine elliptische Involution J.

Mit der so definierten Absolutfigur $[\omega, f, J]$ wird auf bekannte Weise ein galileischer Raum G_3 eingeführt (O. Röschel [9]). Die projektive Automorphismengruppe von $[\omega, f, J]$ ist achtparametrig und enthält eine ausgezeichnete sechsparametrige Untergruppe B_6, die man als *Bewegungsgruppe des galileischen Raumes* bezeichnet ([9,6]). Alle i.f. benützten Resultate aus der Geometrie des galileischen Raumes G_3 können in [9] nachgelesen werden.
Hier sei nun angemerkt, daß man in G_3 zwei Typen eigentlicher Ebenen unterscheidet: *Euklidische Ebenen* (dies sind Ebenen, die f enthalten) und *isotrope Ebenen* (dies sind Ebenen, die f nicht enthalten). Die von B_6 in einer euklidischen Ebene induzierte *Metrik* ist *euklidisch*, die in einer isotropen Ebene induzierte *Geometrie* ist *isotrop* (H. Sachs [11]). Als *Punktkugeln* des galileischen Raumes G_3 bezeichnen wir jene parabolischen Zylinder, welche die Fernebene längs der absoluten Geraden f berühren. Die Gleichung dieser Kugeln ist von der Form

$$(3) \qquad Ax^2 + Bx - 2Cy - 2Dz + E = 0, \qquad (A \neq 0,\ C^2 + D^2 \neq 0).$$

Die *Spitze* einer solchen Punktkugel liegt im Punkt $S(0:0:D:$ $:-C)$ auf der absoluten Geraden f.

Im galileischen Raum G_3 existieren zwei verschiedene Arten von *Kreisen*:

1. *Euklidische Kreise:* Das sind Kegelschnitte, die in euklidischen Ebenen liegen und die beiden konjugiert-komplexen Doppelpunkte $(0:$ $:0:1:\pm i)$ der absoluten Involution J enthalten.

2. *Isotrope Kreise:* Dies sind Parabeln in isotropen Ebenen. Die isotropen Kreise berühren die Fernebene in einem Punkt der absoluten Geraden f.

Die Gruppe B_6 enthält 2 Typen von Drehungsgruppen als Untergruppen:

1. *Euklidische Drehungen* mit der Normalform

$$\left\{\begin{aligned} x(t) &= x_0 \\ y(t) &= y_0 \cos t + z_0 \sin t \\ z(t) &= -y_0 \sin t + z_0 \cos t. \end{aligned}\right. \tag{4}$$

Die Drehachse ist hier die x-Achse (Fixpunktgerade). Die Bahnkurven sind euklidische Kreise in euklidischen Ebenen.

2. *Isotrope Drehungen* mit der Normalform

$$\left\{\begin{aligned} x(t) &= x_0 + bt \\ y(t) &= y_0 + x_0 t + b\tfrac{t}{2}^2 \\ z(t) &= z_0. \end{aligned}\right. \tag{5}$$

Die Bahnkurven dieser Gruppe sind isotrope Kreise vom Radius b in isotropen Ebenen.

2. In dieser Arbeit werden wir nur jene Flächen untersuchen, die sich durch euklidische Drehung (4) eines isotropen Kreises κ um die x-Achse erzeugen lassen. Je nach der Lage von κ können wir einige Fälle unterscheiden. Der einfachste Fall liegt vor, wenn κ ein Kreis in einer *Meridianebene*, d.h. einer Ebene durch die x-Achse ist. Man kann dann κ ohne Einschränkung der Allgemeinheit in $z = 0$ mittels

$$\kappa \ \ldots \qquad \begin{aligned} x &= v \\ y &= 2pv^2 + A \\ z &= 0 \end{aligned} \tag{6}$$

ansetzen. Unterwirft man ihn der euklidischen Drehung (4), so erhält man eine Fläche 4. Ordnung der Gestalt

$$y^2 + z^2 = (2px^2 + A)^2. \tag{7}$$

Dies ist offensichtlich eine Zyklide, deren Meridiankurven isotrope Kreise sind, also eine *Torusfläche* des galileischen Raumes G_3. Solche Torusflächen wurden von O. Röschel in [10] ausführlich untersucht.

3. Wir betrachten weiter einen isotropen Kreis mit der Parameterdarstellung

$$\kappa \ldots \quad \begin{aligned} x &= v \\ y &= 2pv^2 + A \\ z &= kv. \end{aligned} \tag{8}$$

Dieser isotrope Kreis liegt in der Ebene

$$\rho \ldots z = kx, \tag{9}$$

die ersichtlich keine Meridianebene ist.

Unterwirft man den Kreis κ (8) der euklidischen Drehung (4), so erhält man eine Fläche Φ mit der Gleichung

$$\Phi \ldots y^2 + z^2 = (2px^2 + A)^2 + k^2x^2. \tag{10}$$

Das ist offensichtlich eine Drehzyklide Φ. *Die Zyklide Φ (10) besitzt die absolute Gerade f als Doppelgerade und die konjugiert komplexen absoluten Punkte* $(0 : 0 : 1 : \pm i)$ *als uniplanare Knotenpunkte.*

Die Meridiankurve in der Meridianebenen $z = 0$ ist durch

$$\mu \ldots y^2 = (2px^2 + A)^2 + k^2x^2 \quad \text{bzw.} \tag{11}$$

$$4p^2x^4 + (k^2 + 4pA)x^2 - y^2 + A^2 = 0 \tag{12}$$

gegeben. Dies ist eine (bezüglich der x-Achse) axialsymmetrische, vollständig zirkuläre Kurve 4. Ordnung in der isotropen Ebenen $z = 0$ vom Typus (2,2) (vgl. [8]). Die Gleichung (12) läßt sich auch in der Form

$$\begin{aligned} &(2px^2 - y + \tfrac{k^2+4pA}{4p})(2px^2 + y + \tfrac{k^2+4pA}{4p}) - \\ &\quad - \tfrac{(k^2+4pA)^2}{16p^2} + A^2 = 0 \end{aligned} \tag{13}$$

schreiben. Die Meridiankurve (13) hat daher zwei isotrope asymptotische Kreise

$$\kappa_1, \kappa_2 \ \ldots 2px^2 \pm y + \tfrac{k^2+4pA}{4p} = 0, \tag{14}$$

die keinen eigentlichen Punkt mit der Kurve (13) gemeinsam haben. Die Radien der asymptotischen Kreise sind p bzw. $-p$. Die Kreise liegen symmetrisch bezüglich der x-Achse. Weiters hat die Meridiankurve (13) einen Selbstberührungspunkt im absoluten Punkt.

Da die Zyklide Φ (10) auch durch Drehung der Meridiankurve (13) erzeugt werden kann, folgt, daß die *Drehzyklide* Φ (10) *sich längs* *der absoluten Geraden f selbst berührt und gleichzeitig die Fernebene ω berührt.*

Nach (13) läßt sich die Gleichung der Zyklide (10) in der Form

$$y^2 + z^2 = \left(2px^2 + \tfrac{k^2+4pA}{4p}\right)^2 - \left(\tfrac{k^2+4pA}{4p}\right)^2 + A^2 \tag{15}$$

schreiben. Andererseits erhält man durch Drehung der asymptotischen Kreise (14) der Meridiankurve (13) eine galileische Torusfläche τ mit der Gleichung

$$\tau \ \dots \ y^2 + z^2 = \left(2px^2 + \tfrac{k^2+4pA}{4p}\right)^2. \tag{16}$$

Aus (15) ist ersichtlich, daß die Torusfläche τ (16) keinen eigentlichen Punkt mit der Drehzyklide Φ (10) gemeinsam hat, d.h. wir können die Torusfläche τ (16) als *asymptotischen Torus der Drehzyklide* Φ (10) bezeichnen.

4. Wir wollen nun untersuchen, welche Kreise auf der Drehzyklide Φ (10) liegen.

Betrachten wir zunächst den erzeugenden isotropen Kreis κ (8) in der Ebene ρ (9). Die Ebene ρ schneidet die Drehzyklide Φ (10) in noch einem weiteren isotropen Kreis $\bar{\kappa}$, der zum Kreis κ bezüglich der xz-Ebene symmetrisch liegt. Bei Drehung der Ebene ρ umhüllen diese Ebenen einen Drehkegel und die Kreise κ und $\bar{\kappa}$ beschreiben zwei Systeme von isotropen Kreisen der Drehzyklide Φ (10). *Diese beiden Systeme bezeichnen wir mit K_1 und K_2.* Die Kreise der beiden Systeme liegen in den Tangentialebenen des erwähnten Drehkegels.

Man sieht leicht, daß *durch jeden Punkt der Zyklide Φ je ein isotroper Kreis beider Systeme K_1 und K_2 hindurchgeht.*

Betrachten wir weiter die Ebene

$$\sigma \ \dots \ y = \sqrt{k^2 + 8pA}\ x \tag{17}$$

und schneiden wir die Drehzyklide Φ (10) mit dieser Ebene. Durch Einsetzen von (17) in (10) erhalten wir die Projektion der Schnittkurve auf die xz-Ebene in der Form

$$z^2 = (2px^2 + A)^2. \tag{18}$$

Das sind zwei isotrope Kreise, und die Ebene σ ist eine Doppeltangentialebene der Zyklide Φ. Durch Drehung der Ebene σ um die x-Achse erhalten wir zwei Systeme von isotropen Kreisen in Ebenen, die wieder einen Kegel umhüllen. Dieser Kegel berührt die Drehzyklide Φ (10) längs zweier euklidischer Parallelkreise. *Diese beiden Kreissysteme bezeichnen wir mit L_1 und L_2.* Daraus schließt man leicht, daß durch jeden Punkt der Zyklide je ein Kreis beider Systeme L_1 und L_2 hindurchgeht. Die isotropen Kreise der beiden Systeme L_1 und L_2 der Drehzyklide Φ (10) sind das Analogon zu den *Villarceauschen Kreisen* der Torusfläche (vgl. [10]). Zusammenfassend gilt der

Satz 1. *Durch einen allgemeinen Punkt P der Drehzyklide Φ (10) des galileischen Raumes G_3 geht ein euklidischer Kreis (Parallelkreis der Drehzyklide Φ) und je ein isotroper Kreis der vier Kreissysteme K_1, K_2, L_1* und *L_2.*

5. Die Kreise der betrachteten Kreissysteme können auch nicht reell sein. Wir betrachten im folgenden verschiedene Fälle betreffend die Realität der erzeugenden Kreise der Zyklide Φ (10). Dabei werden die Koeffizienten p, A und k stets als reell vorausgesetzt.

1. Bei der Drehzyklide Φ mit der Gleichung (10) handelt es sich um eine Fläche, bei der alle vier Kreissysteme K_1, K_2, L_1, L_2 reell sind. Die asymptotische Torusfläche ist ebenfalls reell.

2. Bei der durch

$$y^2 + z^2 = 4p^2x^2 - (k^2 - 4pA)x^2 \tag{19}$$

gegebenen Drehzyklide sind die Kreise der Systeme K_1 und K_2 imaginär. Diese Kreise haben reelle Projektionen in die xy-Ebene, liegen aber in imaginären Ebenen.

3. Bei der durch

$$4p^2x^4 - (k^2 - 4pA)x^2 + y^2 + A^2 = 0 \tag{20}$$

gegebenen Drehzyklide sind die Kreise der Systeme K_1 und K_2 konjugiert-komplex, liegen aber stets in reellen Ebenen.

6. Bei unseren bisherigen Betrachtungen war die Lage des erzeugenden Kreises κ (8) nicht ganz allgemein. Man kann leicht beweisen, daß ein isotroper Kreis des galileischen Raumes, der allgemeine Lage besitzt, sich stets in der Normalform

$$\kappa \ \ldots \quad \begin{aligned} x &= v \\ y &= 2pv + A \\ z &= kv + 1 \end{aligned} \tag{21}$$

darstellen läßt. Dies gelingt stets durch Anwendung einer geeigneten euklidischen Drehung (4). Unterwirft man den Kreis κ (21) der euklidischen Drehung (4), so erhält man wieder eine Drehzyklide, deren Gleichung

$$\Phi \ \ldots \ y^2 + z^2 = (2px^2 + A)^2 + (kx + 1)^2 \tag{22}$$

lautet. Die Gleichung einer allgemeinen Drehzyklide des galileischen Raumes, die durch euklidische Drehung eines isotropen Kreises entsteht, kann man somit immer auf die Form (22) transformieren. Was die Kreissysteme einer solchen Fläche betrifft, gilt der Satz 1 in analoger Weise.

7. Es liegt nun die Frage nahe, ob durch (22) alle euklidischen Drehzykliden des galileischen Raumes erfaßt werden, oder ob es noch andere gibt. Um diese Frage zu beantworten, wollen wir die Gleichung einer allgemeinen Zyklide Φ des galileischen Raumes G_3 betrachten. Dabei muß man, bezugnehmend auf die Definition einer Zyklide des G_3, die drei folgenden Punkte beachten:

1. Die Zyklide Φ besitzt außer der absoluten Ferngeraden f (Ferngerade der Ebene $x = 0$) keine weiteren Fernpunkt, d.h. die Gleichung der Zyklide Φ kann nur ein Glied vierten Grades nämlich x^4 enthalten.

2. Da die Ferngerade f Doppelgerade der Fläche Φ ist, schneidet jede Ebene $x = const.$ die Zyklide Φ (außer in f) nach einer Kurve 2. Ordnung, d.h., wenn man in der Gleichung der Zyklide $x = const.$ setzt, so muß eine Gleichung zweiten Grades in y und z übrig bleiben.

Zieht man dies alles in Betracht, so lautet die Gleichung einer allgemeinen Zyklide

$$x^4 + \alpha x^3 + x^2 p_1^1(y,z) + x p_2^2(y,z) + p_3^2(y,z) = 0, \tag{23}$$

wo die p_i^n Polynome n-ten Grades bezeichnen und $\alpha \in \mathrm{R}$ gilt. Sucht man nun die Gleichung einer allgemeinen *euklidischen Drehzyklide* Φ des galileischen Raumes G_3, d.h. einer Zyklide, die durch euklidische Drehung um die x-Achse entsteht, so muß man noch folgendes beachten:

3. Schneidet man eine solche Drehzyklide Φ mit einer Ebene σ mit der Gleichung $x = const.$, so ist der Schnitt ein euklidischer Kreis mit dem Mittelpunkt im Schnittpunkt der x-Achse mit der Ebene σ. Hieraus folgt durch eine einfache Rechnung, daß sich die euklidischen Drehzykliden des G_3 in der Gestalt

$$(24) \qquad \Phi \ \ldots \ (Ax + B)(y^2 + z^2) + q^4(x) = 0$$

schreiben lassen.

Man kann hier zwei Typen von euklidischen Drehzykliden unterscheiden:

Typ I: $A = 0$

Die Gleichung der euklidischen Drehzyklide lautet dann

$$(25) \qquad \Phi_I \ \ldots \ y^2 + z^2 + r^4(x) = 0.$$

Durch Vergleich mit der Gleichung (22) und nach den bisherigen Betrachtungen erkennt man, daß die Drehzykliden Φ_I (25) alle jene Drehzykliden sind, die durch euklidische Drehung eines isotropen Kreises um x-Achse erzeugbar sind.

Die Drehzykliden Φ_I besitzen eine Selbsberührung längs der absoluten Geraden f und haben in den absoluten Punkten I_1, I_2 zwei uniplanare Knotenpunkte.

Typ II: $A \neq 0$

Die Gleichung (24) kann man dann in der Form

$$(26) \qquad \Phi_{II} \ \ldots \ (Ax + B)(y^2 + z^2) + g^3(x) + S = 0$$

schreiben. Daraus entnimmt man unmittelbar, daß zur Drehzyklide Φ_{II} eine euklidische Ebene τ mit der Gleichung

$$(27) \qquad \tau \ \ldots \ Ax + B = 0,$$

existiert, die keinen eigentlichen Punkt mit der Drehzyklide Φ_{II} (26) gemeinsam hat. τ könnte man als *asymptotische Ebene* bezeichnen. Man erhält (27) auch dadurch, daß man die Flächengleichung (26) partiell nach z differenziert, und die entstehende Gleichung gleich Null setzt. Somit ist (27) die zu (26) gehörige Hauptachsenfläche Σ^*, wie sie von H. SACHS in [14] eingeführt wurde und bei der Klassifikation der Zykliden des Flaggenraumes benützt wurde.

Es existiert hier weiter eine Fläche 3. Ordnung mit der Gleichung

$$(28) \qquad \Sigma \ \ldots \ y^2 + z^2 + g^3(x) = 0,$$

welche die absolute Gerade f als einfache Gerade enthält, und außerdem keine weiteren uneigentlichen Punkte trägt. Man kann diese Fläche Σ als euklidische Drehzyklide 3. Ordnung bezeichnen. Die Fläche Σ berührt und schneidet gleichzeitig die Fernebene längs der absoluten Geraden f. Wir bezeichnen diese Fläche Σ (28) als *asymptotische*

Fläche der Drehzyklide Σ (26). Die Zykliden 3. Ordnung des Flaggenraumes wurden von H. SACHS in [15] vollständig klassifiziert. Gemäß dieser Klassifikation gehört (28) zum ersten Haupttyp, Unterklasse IA, elliptischer Fall.

Die Fläche Σ (28) ist eine Drehfläche (euklidische Drehung um x-Achse) mit der Meridiankurve

$$y^2 + g^3(x) = 0 \tag{29}$$

in der xy-Ebene. Dies ist eine divergente Parabel (vgl. [7]). Die asymptotische Drehfläche Σ entsteht somit durch euklidische Drehung einer divergenten Parabel um x-Achse.

Die Meridiankurve

$$(Ax + B)[y^2 + q^3(x)] + S = 0 \tag{30}$$

der Drehzyklide Φ_{II} (26) ist eine vollständig zirkuläre Kurve 4. Ordnung; sie besitzt eine asymptotische Gerade ($Ax + B = 0$) und eine asymptotische, divergente Parabel. Demnach ist sie vom Typus $(1,3)_1$ (vgl. [8]).

Die Drehzyklide Φ_{II} (26) berührt und schneidet die Fernebene ω längs der absoluten Geraden f, und schneidet die Fernebene ω nochmals längs f. Wir fassen zusammen im

SATZ 2. *Im galileischen Raum G_3 existieren 2 Typen von euklidischen Drehzykliden, die sich durch die Normalformen (25) bzw. (26) beschreiben lassen. Die Drehzykliden vom Typ II (26) besitzen eine euklidische Drehzyklide 3. Ordnung als asymptotische Fläche. Auf den Drehzykliden (26) existieren außer den euklidischen Parallelkreisen keine weiteren Kreise.*

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Mathematica Pannonica
2/1 (1991), 105 – 121

CHARACTERIZATION OF BINARY OPERATIONS ON THE UNIT INTERVAL SATISFYING THE GENERALIZED MODUS PONENS INFERENCE RULE

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Received May 1990

AMS Subject Classification: 03 B 52

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Keywords: Fuzzy implication, triangular norms and conorms, comositional rule of inference, modus ponens, many valued logic.

Abstract: The generalized modus ponens inference rule is examined in a formal way and a characterization of the $[0,1]^2$-$[0,1]$ mappings, especially fuzzy implication operators, is given according to their behaviour with respect to the sup-triangular norm inference rule. The analogies between triangular norms and their dual triangular conorms on the one hand, and fuzzy implication operators on the other hand are described. Finally it is proven that a fuzzy implication operator that is an extension of the classical formula (NOT P) OR Q in the sense that the negation is replaced by a strict complement operator and the disjunction by a triangular conorm, never satisfies the sup-minimum inference rule.

1. Introduction

The compositional rule of inference was introduced by L. A. Zadeh [17] as an extension of the classical reasoning scheme "modus ponens". Its main purpose is to infer a possibility distribution, given a relationship between two linguistic variables modelled as possibility distributions on their respective universes of discourse and a possibility distribution which represents the vague knowledge about the matching of the antecedent. This inference rule can be represented as

$$\begin{array}{l} x \text{ is } A \Rightarrow y \text{ is } B \\ \underline{x \text{ is } A' \qquad\qquad\quad} \\ \qquad\qquad\quad y \text{ is } B'. \end{array}$$

Here, A and A' are possibility distributions on a universe of discourse $\mathcal{U}$, B and the derived distribution B' are possibility distributions on a universe $\mathcal{V}$. The distribution B' is calculated as

$$B' : \mathcal{V} \to [0,1] : v \mapsto \sup_{u \in \mathcal{U}} \min(A'(u), \mathcal{F}(A(u), B(v))),$$

where $\mathcal{F}$ is a $[0,1]^2$–$[0,1]$ mapping representing the relationship between A and B. This generalized inference scheme is very powerful as it is able to deduce knowledge from incomplete and uncertain information [15,18]. As mentioned in [4] and [7] the formalism for deriving the distribution B' can be easily generalized by replacing the minimum operator by a general triangular norm T (Definition 1.2, [14]). When $\mathcal{F}$ is a fuzzy implication operator (Definition 1.1) the proposed inference scheme is an extension of the classical "modus ponens" inference scheme. In [7,8,9] extensive case studies are presented by Martin-Clouaire and Mizumoto where the minimum operator is replaced by a general t-norm and $\mathcal{F}$ is a fuzzy implication operator. The results of these studies and the idea of modus ponens generating functions by Trillas and Valverde [15] suggest a strong connection between the choice of the fuzzy implication operator used to model the linguistic rule and fuzzy relation "IF x is A THEN y is B" and the triangular norm in the generalized modus ponens inference rule or compositional rule of inference. With regard to other inference schemes like "modus tollens" and "syllogism" the same remark can be made. In the sequel a formal treatment of the properties of $[0,1]^2$–$[0,1]$ mappings and especially fuzzy implication operators, w.r.t. their relationship with the triangular norm in the sup-triangular norm generalized inference scheme is presented. The family of equations

$$(\forall y \in [0,1])(y = \sup_{x\in[0,1]} T(x, \mathcal{F}(x,y)))$$

is examined and the mapping $\mathcal{F}$ is characterized w.r.t. the triangular norm in the sup-triangular norm inference rule. In every example $\mathcal{F}$ is a fuzzy implication operator. This section is mainly concerned with definitions and notations. In section 2 some properties of $[0,1]^2$–$[0,1]$ mappings, based on properties of fuzzy implication operators are presented. In section 3, the restrictions for the modus ponens inference rule yield three possible classes of mappings. Section 4 deals with the properties of one of these classes. Section 5 deals with the similarities of triangular conorms and the properties of the fuzzy implication operator, and in section 6 it is proven that the classical formula (NOT P) OR Q cannot be generalized without loss of the sup-min modus ponens inference rule.

Definition 1.1. A $[0,1]^2$–$[0,1]$ mapping $\mathcal{I}$ satisfying the *boundary conditions*

$$\mathcal{I}(0,0) = \mathcal{I}(0,1) = \mathcal{I}(1,1) = 1 \text{ and } \mathcal{I}(1,0) = 0$$

is a *fuzzy implication operator.*

Definition 1.1 of a fuzzy implication operator is weaker than Weber's definition [16]. These conditions are the weakest that can be imposed: a fuzzy implication operator is a $[0,1]^2$–$[0,1]$ mapping that is an extension of the material, binary implication operator.

Definition 1.2. [14]. A $[0,1]^2$–$[0,1]$ mapping T satisfying

(T1) *boundary conditions:*

$$(\forall x \in [0,1])(T(1,x) = x),$$

(T2) *symmetry property:*

$$(\forall (x,y) \in [0,1]^2)(T(x,y) = T(y,x)),$$

(T3) *associative property:*

$$(\forall (x,y,z) \in [0,1]^3)(T(x,T(y,z)) = T(T(x,y),z)),$$

(T4) *monotonicity:*

$$\begin{aligned}&(\forall (x,y) \in [0,1]^2)(\forall (x',y') \in [0,1]^2)\\ &\qquad ((x \leq x') \wedge (y \leq y') \Rightarrow T(x,y) \leq T(x',y')),\end{aligned}$$

is a *triangular norm* (shortly *t-norm*).

Definition 1.3. A $[0,1]^2$–$[0,1]$ mapping S satisfying

(T1') *boundary conditions:*

$$(\forall x \in [0,1])(S(0,x) = x)$$

and (T2)–(T4) is a *triangular conorm* (shortly *t-conorm*).

Definition 1.4. A $[0,1]$–$[0,1]$ mapping C that is strictly decreasing and involutive and satisfies $C(0) = 1$ and $C(1) = 0$ is a *strict complement operator.*

Remark. The following two well-known properties can be easily proved [3]:

1. If T is a t-norm and C a strict complement operator then S_T^C is a t-conorm, where
$$S_T^C : [0,1]^2 \to [0,1] : (x,y) \mapsto C(T(C(x),C(y))).$$
2. If S is a t-conorm and C is a strict complement operator then T_S^C is a t-norm, where
$$T_S^C : [0,1]^2 \to [0,1] : (x,y) \mapsto C(S(C(x),C(y))).$$

Definition 1.5. Let T be a t-norm and $\mathcal{F}$ a $[0,1]^2$–$[0,1]$ mapping then $\mathcal{F}$ satisfies the *sup-T modus ponens inference rule* iff

$$(\forall y \in [0,1])(y = \sup_{x\in[0,1]} T(x, \mathcal{F}(x,y)))$$

This definition is a natural extension of the sup-min modus ponens inference rule.

Definition 1.6. Let f and g be two $[0,1]^2 - [0,1]$ mappings then $f \leq g$ iff $(\forall (x,y) \in [0,1]^2)(f(x,y) \leq g(x,y))$.

2. Potential properties of the fuzzy implication operator

Let $\mathcal{I}$ be a fuzzy implication operator and C a strict complement operator. The following potential properties for $\mathcal{I}$ are defined [6].

Definition 2.1. $\mathcal{I}$ satisfies the *contrapositive symmetry* iff

$$(\forall (x,y) \in [0,1]^2)(\mathcal{I}(x,y) = \mathcal{I}(C(y), C(x))).$$

Definition 2.2. $\mathcal{I}$ satisfies the *exchange principle* iff

$$(\forall (x,y,z) \in [0,1]^3)(\mathcal{I}(x, \mathcal{I}(y,z)) = \mathcal{I}(y, \mathcal{I}(x,z)))).$$

Definition 2.3. $\mathcal{I}$ is *hybrid monotonous* [2,6] iff

$$\begin{aligned}&(\forall (x,y) \in [0,1]^2)\ (\forall (x',y') \in [0,1]^2)\\ &\qquad\qquad ((x \leq x') \wedge (y \geq y') \Rightarrow \mathcal{I}(x,y) \geq \mathcal{I}(x',y')).\end{aligned}$$

Although the definition of the hybrid monotonicity of $\mathcal{I}$ seems rather strange, it satisfies the intuitive idea that the less the antecedent is true and the more the consequence is true, the more the implication should be true. The following property is easily verified.

Property 2.1. *If $\mathcal{I}$ is a hybrid monotonous fuzzy implication operator then*

$$(\forall x \in [0,1])(\mathcal{I}(0,x) = 1).$$

Definition 2.4. $\mathcal{I}$ satisfies the *neutrality principle* iff

$$(\forall x \in [0,1])(\mathcal{I}(1,x) = x).$$

Remark. Obviously these definitions can be extended to general $[0,1]^2$–$[0,1]$ mappings.

3. Natural restrictions for the generalized modus ponens inference rule

In this section some natural restrictions on the mapping

$$\sup T : [0,1] \to [0,1] : y \mapsto \sup_{x\in[0,1]} T(x, \mathcal{F}(x,y))$$

are introduced, where T, $\mathcal{F}$ respectively, is a t-norm, a $[0,1]^2$–$[0,1]$ mapping respectively. These restrictions generate three disjoint classes of mappings that are examined to determine whether or not these mappings satisfy the generalized modus ponens.

Property 3.1. *If $\mathcal{F}$ is a $[0,1]^2$–$[0,1]$ mapping and T a t-norm then*

$$(\forall y \in [0,1])(\mathcal{F}(1,y) \leq \sup_{x\in[0,1]} T(x,\mathcal{F}(x,y)) \leq \sup_{x\in[0,1]} \min(x,\mathcal{F}(x,y))).$$

Proof. As $(\forall y \in [0,1])(\mathcal{F}(1,y) = T(1,\mathcal{F}(1,y)))$ and $T \leq \min$ for every t-norm T the result is immediately obtained. $\diamondsuit$

Considering Property 3.1 three disjoint classes of mappings can be defined:

Class I: $(\forall y \in [0,1])(\mathcal{F}(1,y) = y)$, i.e. $\mathcal{F}$ satisfies the neutrality principle,

Class II: $(\forall y \in [0,1])(\mathcal{F}(1,y) \leq y)$ and $(\exists y_0 \in [0,1])(\mathcal{F}(1,y_0) < y_0)$,

Class III: $(\exists y_0 \in [0,1])(\mathcal{F}(1,y_0) > y_0)$.

Remarks.

1. The three disjoint classes are a partition of the set of the $[0,1]^2$–$[0,1]$ mappings.
2. For the mappings of classes I and II the following property is easily proven:

Property 3.2. *Let T, T_1 and T_2 be t-norms. If $\mathcal{F}$ satisfies the sup-T_1 and the sup-T_2 modus ponens inference rule and if $T_1 \leq T \leq T_2$ then $\mathcal{F}$ satisfies the sup-T modus ponens inference rule.*

Proof.

$$(\forall (x,y) \in [0,1]^2)(T_1(x,\mathcal{F}(x,y)) \leq T(x,\mathcal{F}(x,y)) \leq T_2(x,\mathcal{F}(x,y)))$$

and thus

$$(\forall y \in [0,1])(\sup_{x\in[0,1]} T_1(x,\mathcal{F}(x,y)) \leq \sup_{x\in[0,1]} T(x,\mathcal{F}(x,y)) \leq$$

$$\leq \sup_{x\in[0,1]} T_2(x, \mathcal{F}(x,y))).\diamondsuit$$

3. From Property 3.1 it follows that the mappings of class III never satisfy the generalized modus ponens.

In section 4 some results on mappings satisfying the neutrality principle are presented.

4. $[0,1]^2$–$[0,1]$ mappings satisfying the neutrality principle

In this section T is an arbitrary t-norm and $\mathcal{F}$ a $[0,1]^2$–$[0,1]$ mapping that satisfies the neutrality principle.

Definition 4.1 ([4,16]):

$$\triangleright_T : [0,1]^2 \to [0,1] : (x,y) \mapsto \sup\{z \mid T(x,z) \leq y\}.$$

The following theorem is based on some theorems proved by Dubois and Prade [3,4] and deals with the existence of a maximal solution $\mathcal{F}$ of the family of equations

$$(\forall y \in [0,1])(y = \sup_{x\in[0,1]} T(x, \mathcal{F}(x,y)));$$

it gives a sufficient and necessary condition to determine whether or not $\mathcal{F}$, satisfying the neutrality principle, is a solution of the above family of equations.

Theorem 4.1. *Let T be a t-norm such that every partial mapping of T is infra-semicontinuous and $\mathcal{F}$ a $[0,1]^2$–$[0,1]$ mapping satisfying the neutrality principle. $\mathcal{F}$ satisfies the sup-T modus ponens inference rule iff $\mathcal{F} \leq \triangleright_T$.*

Proof.1. If $\mathcal{F}$ satisfies the sup-T modus ponens then $\mathcal{F} \leq \triangleright_T$ [4].

2. $\triangleright_T$ is a solution of the family of equations

$$(\forall y \in [0,1])(y = \sup_{x\in[0,1]} T(x, \mathcal{F}(x,y))).$$

Although this has already been proven for a continuous t-norm T this property holds for every t-norm which has infra-semicontinuous partial mappings. The proof is entirely based on the exchange of supremum and T. This property is proven in the Appendix.

3. It has already been proven that if $\mathcal{F}$ satisfies the generalized sup-T modus ponens then $\mathcal{F} \leq \triangleright_T$. The reverse implication is established now. Let $\mathcal{F}$ be a mapping that satisfies the condition $\mathcal{F} \leq \triangleright_T$ and T a t-norm with infra-semicontinuous partial mappings then

$$(\forall y \in [0,1])(\mathcal{F}(1,y) \leq \sup_{x\in[0,1]} T(x, \mathcal{F}(x,y)) \leq \sup_{x\in[0,1]} T(x, x \triangleright_T y))$$

or

$$(\forall y \in [0,1])(y \leq \sup_{x\in[0,1]} T(x, \mathcal{F}(x,y)) \leq y). \diamondsuit$$

Remarks.

1. If the partial mappings of T are not infra-semicontinuous Theorem 4.1 cannot be generalized. Counterexample: let T be $\mathbb{Z}$ then

$$\triangleright_{\mathbb{Z}} : [0,1]^2 \to [0,1] : (x,y) \mapsto \begin{cases} 1 & ; \quad \forall (x,y) \in [0,1[\times[0,1] \\ y & ; \quad x = 1. \end{cases}$$

It is easily verified that $\sup_{x\in[0,1]} \mathbb{Z}(x, x \triangleright_{\mathbb{Z}} y_0) = 1$ or the sup-$\mathbb{Z}$ modus ponens inference rule does not hold when $\mathcal{F} = \triangleright_{\mathbb{Z}}$.

2. Consider for $a \in]0,1[$ the mapping

$$T_a : [0,1]^2 \to [0,1] : (x,y) \mapsto \begin{cases} 0 & ; \quad \text{if } \max(x,y) \leq a \\ \min(x,y) & ; \quad \text{elsewhere.} \end{cases}$$

Then every partial mapping of T_a is infra-semicontinuous and T_a is a t-norm. Hence, there exists t-norms that are not continuous and that have infra-semicontinuous partial mappings.

Corollary 4.1. *If T is an arbitrary t-norm and $\mathcal{F}$ satisfies the sup-T modus ponens then $\mathcal{F} \leq \triangleright_T$.*

Proof. This is an immediate consequence of the first part of the proof of Theorem 4.1. $\diamondsuit$

Corollary 4.2. *If $\mathcal{F}$ is a $[0,1]^2$–$[0,1]$ mapping satisfying the neutrality principle and the sup-min modus ponens then $\mathcal{F}$ satisfies the sup-T modus ponens inference rule, where T is an arbitrary t-norm.*

Proof. Obvious considering Property 3.1. $\diamondsuit$

Corollary 4.3. *Let T_1 and T_2 be arbitrary t-norms then T_2 satisfies the sup-T_1 modus ponens inference rule.*

Proof. For every t-norm T_2 the inequality $T_2 \leq \min \leq \triangleright_{\min}$ holds. As $(\forall x \in [0,1])(T_2(1,x) = x)$ considering Theorem 4.1 and Corollary 4.2, T_2 satisfies the sup-T_1 modus ponens. $\diamondsuit$

Example 4.1. In the examples only $[0,1]^2$–$[0,1]$ mappings that are implication operators are considered. Let T be W (Lucasiewicz t-norm [14]) or explicitly

$$W : [0,1]^2 \to [0,1] : (x,y) \mapsto \max(0, x+y-1)$$

and $\mathcal{F}$ be the Kleene-Dienes implication operator[13]

$$\mathcal{F} : [0,1]^2 \to [0,1] : (x,y) \mapsto \max(1-x, y).$$

Considering Theorem 4.1 and the inequality $\mathcal{F} \leq \triangleright_W$, $\mathcal{F}$ satisfies the sup-W modus ponens inference rule.
Example 4.2. Let T be the well-known algebraic product $\times$, then $\triangleright_\times$ is the operator $G43$ of [1,5,13]:

$$\triangleright_\times : [0,1]^2 \to [0,1] : (x,y) \mapsto \begin{cases} 1 & ; \quad \text{if} \quad x \leq y \\ y/x & ; \quad \text{elsewhere.} \end{cases}$$

Consider as defined in $\mathcal{F}$ [10]

$$\mathcal{F} : [0,1]^2 \to [0,1] : (x,y) \mapsto \max(\min(x,y), 1-x).$$

Let $x_0 = 1/3$ and $y_0 = 1/8$ then $\mathcal{F}(x_0, y_0) = 2/3$ and $\triangleright_\times(x_0, y_0) = 3/8$. Considering Theorem 4.1 $\mathcal{F}$ does not satisfy the sup-$\times$ modus ponens inference rule.
Example 4.3. Let T be any t-norm, then define

$$T_\mathcal{I} : [0,1]^2 \to [0,1] : \begin{cases} (0,0) \mapsto 1 \\ (0,1) \mapsto 1 \\ (x,y) \mapsto T(x,y) & ; \quad \text{elsewhere.} \end{cases}$$

Obviously $T_\mathcal{I}$ is a fuzzy implication operator that satisfies every sup-T modus ponens inference rule, whatever T is (Theorem 4.1 and Corollary 4.2).

5. On the extension of the classical formula (NOT P) OR Q

The properties of the mappings that are extensions of the classical formula (NOT P) OR Q are examined. The negation is fuzzified by a strict complement operator C and the disjunction by a t-conorm S.

These extensions are the implication operators of type I of Dubois and Prade [3].

Definition 5.1. Let S be a t-conorm and C a strict complement operator. The mapping $\mathcal{I}_S^C$ is defined as

$$\mathcal{I}_S^C : [0,1]^2 \to [0,1] : (x,y) \mapsto S(C(x),y).$$

Obviously the mapping $\mathcal{I}_S^C$ is a fuzzy implication.

Properties 5.1.

1. *$\mathcal{I}_S^C$ satisfies the contrapositive symmetry,*
2. *$\mathcal{I}_S^C$ satisfies the exchange principle,*
3. *$\mathcal{I}_S^C$ is hybrid monotonous,*
4. *$\mathcal{I}_S^C$ satisfies the neutrality principle,*
5. *If S is continuous then $\mathcal{I}_S^C$ is continuous.*

Proof. Immediate from the symmetric and associative properties, the monotonicity and the boundary conditions of S and the involutive property of C. Property 5. is proven by the chain rule for continuous functions. $\Diamond$

Definition 5.2. Let $\mathcal{I}$ be a fuzzy implication operator and C a strict complement operator. The mapping $S_{\mathcal{I}}^C$ is defined by

$$S_{\mathcal{I}}^C : [0,1]^2 \to [0,1] : (x,y) \mapsto \mathcal{I}(C(x),y).$$

It is easily proven that $S_{\mathcal{I}}^C$ is an extension of the classical union operator; i.e. $S_{\mathcal{I}}^C \mid_{\{0,1\}^2}$ is the classical union operator $\cup$.

Properties 5.2.

1. *if $\mathcal{I}$ satisfies the contrapositive symmetry then $S_{\mathcal{I}}^C$ is symmetric,*
2. *if $\mathcal{I}$ satisfies the exchange principle and the contrapositive symmetry then $S_{\mathcal{I}}^C$ is associative,*
3. *if $\mathcal{I}$ is monotonous then $S_{\mathcal{I}}^C$ is increasing,*
4. *if $\mathcal{I}$ satisfies the neutrality principle then $S_{\mathcal{I}}^C$ satisfies the condition*

$$(\forall y \in [0,1])(S_{\mathcal{I}}^C(0,y) = y),$$

5. *if $\mathcal{I}$ is continuous then $S_{\mathcal{I}}^C$ is continuous.*

Proof. As an example 1. is proved. Let $(x,y) \in [0,1]^2$ then

$$S_{\mathcal{I}}^C(x,y) = \mathcal{I}(C(x),y)$$

and as C is involutive and $\mathcal{I}$ satisfies the contrapositive symmetry

$$\mathcal{I}(C(x),y) = \mathcal{I}(C(y),C(C(x))) = \mathcal{I}(C(y),x) = S_{\mathcal{I}}^C(y,x)\Diamond$$

Corollary 5.1. *Let S be a t-conorm and $\mathcal{I}$ a fuzzy implication operator then*

$$S^C_{\mathcal{I}^C_S} = S \quad ; \quad \mathcal{I}^C_{S^C_{\mathcal{I}}} = \mathcal{I}.$$

Corollary 5.2.

1. *$S^C_{\mathcal{I}}$ is a t-conorm iff $\mathcal{I}$ satisfies the contrapositive symmetry, the exchange principle and neutrality principle and $\mathcal{I}$ is hybrid monotonous;*
2. *$S^C_{\mathcal{I}}$ is continuous iff $\mathcal{I}$ is continuous.*

Proof. Straightforward taking into account Properties 5.1, 5.2 and Corollary 5.1. ◊

6. A special case : the sup-min modus ponens inference rule

In this section the special case of the sup-min modus ponens inference rule is considered. As minimum is a continuous mapping Theorem 4.1 assures us that if $\mathcal{F}$ satisfies the neutrality principle then $\mathcal{F}$ satisfies the sup-min modus ponens inference rule iff $\mathcal{F} \leq \triangleright_{\min}$ where $\triangleright_{\min}$ is the Gödel implication [4,5,7,13,16]]. The condition $\mathcal{F} \leq \triangleright_{\min}$ can be easily transformed into the formula of Theorem 6.1.

Theorem 6.1. *If $\mathcal{F}$ is $[0,1]^2$–$[0,1]$ mapping satisfying the neutrality principle then $\mathcal{F}$ satisfies the sup-min modus ponens inference rule iff*

$$(\forall(x,y) \in [0,1]^2)(x > y \Rightarrow \mathcal{F}(x,y) \leq y)$$

Proof. Obvious considering Theorem 4.1 and the definition of the Gödel implication. ◊

Example 6.1. Let $\mathcal{F}$ be $\triangleright_W$ then

$$\mathcal{F} : [0,1]^2 \to [0,1] : (x,y) \mapsto \min(1, 1 - x + y).$$

Let $x_0 = 0.9$ and $y_0 = 0.8$ then $\mathcal{F}(x_0, y_0) > y_0$. Considering Theorem 6.1 the sup-min modus ponens inference rule does not hold for $\triangleright_W$.

Example 6.2. Let $\mathcal{F}$ be the Kleene-Dienes implication operator [13]:

$$\mathcal{F} : [0,1]^2 \to [0,1] : (x,y) \mapsto \max(1 - x, y).$$

Let $x_0 = 0.5$ and $y_0 = 0.3$ then $\mathcal{F}(x_0, y_0) > y_0$. Considering Theorem

6.1 the sup-min modus ponens inference rule does not hold for this implication operator.

Theorem 6.2. *If $\mathcal{F}$ is a $[0,1]^2$–$[0,1]$ mapping satisfying*

$$(\forall y \in [0,1])(\mathcal{F}(1,y) \leq y) \text{ and } (\exists y_0 \in [0,1])(\mathcal{F}(1,y_0) < y_0)$$

then the sup-min modus ponens inference rule holds iff

$$(\forall (x,y) \in [0,1]^2)((x > y \Rightarrow \mathcal{F}(x,y) \leq y) \textit{ and}$$
$$(\sup_{0 \leq x \leq y} \min(x, \mathcal{F}(x,y)) = y \textit{ or } \sup_{y < x \leq 1} \min(x, \mathcal{F}(x,y)) = y)).$$

Proof. Considering Corollary 4.1, the inequality

$$(\forall (x,y) \in [0,1]^2)(x > y \Rightarrow \mathcal{F}(x,y) \leq y)$$

is immediately obtained as $\mathcal{F} \leq \triangleright_{\min}$ should be satisfied. The second part of the conjunction is proven as follows. Suppose

$$(\exists y_0 \in [0,1])(\sup_{0 \leq x \leq y_0} \min(x, \mathcal{F}(x,y_0)) \neq y_0 \qquad \text{and}$$
$$\sup_{y_0 < x \leq 1} \min(x, \mathcal{F}(x,y_0)) \neq y_0)$$

then

$$\sup_{0 \leq x \leq 1} \min(x, \mathcal{F}(x,y_0)) = \max(\sup_{0 \leq x \leq y_0} \min(x, \mathcal{F}(x,y_0)),$$
$$\sup_{y_0 < x \leq 1} \min(x, \mathcal{F}(x,y_0))) \neq y_0$$

or the sup-min modus ponens inference rule does not hold, which is clearly a contradiction.

The reverse implication is obtained in a similar way. $\diamondsuit$

Example 6.3. Let $\mathcal{F}$ be $\mathcal{I}_{\nabla}$ of [9] or explicitly

$$\mathcal{I}_{\nabla} : [0,1]^2 \to [0,1[: (x,y) \mapsto \begin{cases} \min\left(1, \frac{y}{x}, \frac{1-x}{1-y}\right) & ; \forall (x,y) \in]0,1] \times [0,1] \\ 1 & ; \text{elsewhere.} \end{cases}$$

If $x_0 = 0.5$ and $y_0 = 0.5$ then $\mathcal{I}_{\nabla}(x_0, y_0) = 1 > y_0$ so sup-min modus ponens inference rule does not hold for $\mathcal{I}_{\nabla}$.

Example 6.4. Let

$$\mathcal{F} : [0,1]^2 \to [0,1] : (x,y) \mapsto \begin{cases} 1 & ; \quad x \leq y \\ \min(1-x, y) & ; \quad \text{elsewhere.} \end{cases}$$

By definition

$$(\forall(x,y)\in[0,1]^2)(x>y\Rightarrow \mathcal{F}(x,y)\leq y)$$

holds and $\sup\limits_{0\leq x\leq y}\min(x,\mathcal{F}(x,y))=y$ and thus the sup-min modus ponens inference rule holds although $\mathcal{F}$ does not satisfy the neutrality principle.

Theorem 6.3. *Let S be a t-conorm and C a strict complement operator then $\mathcal{I}_S^C$ never satisfies the sup-min modus ponens inference rule.*

Proof.

1. As C is a continuous, strictly decreasing $[0,1]$–$[0,1]$ mapping and $C(0)=1$, $C(1)=0$,
$$(\exists p_0\in]0,1[)(C(p_0)=p_0).$$
2. Let $y_0<x_0<p_0$, then $p_0<C(x_0)<C(y_0)$ and hence
$$(\exists(x_0,y_0)\in[0,1]^2)(x_0>y_0\wedge C(x_0)>y_0).$$
3. S is a t-conorm, hence
$$S(C(x_0),y_0)\geq\max(C(x_0),y_0)=C(x_0).$$
4. Combining (2) and (3) yields
$$(\exists(x_0,y_0)\in[0,1]^2)(x_0>y_0\text{ and }\mathcal{I}_S^C(x_0,y_0)>y_0).$$

By Theorem 6.1, $\mathcal{I}_S^C$ does not satisfy the sup-min modus ponens inference rule. ◇

Corollary 6.1. *If $\mathcal{I}$ is a fuzzy implication operator satisfying the exchange principle, neutrality principle, contrapositive symmetry and which is hybrid monotonous then $\mathcal{I}$ does not satisfy the sup-min modus ponens inference rule.*

Proof. Immediate from Theorem 6.3 and Corollary 5.1. ◇

7. Conclusion

The interaction between $[0,1]^2$–$[0,1]$ mappings $\mathcal{F}$ and, as special cases the fuzzy implication operators, and the generalized sup-T modus ponens inference rule have been investigated. As indicated in the introduction there is a very strong connection between $\mathcal{F}$, in practice a fuzzy implication operator, and the t-norm T of the inference rule. Several authors already suggested this connection [2,3,7,15].

Weber [16] considers a $[0,1]^2$–$[0,1]$ mapping as a fuzzy implication operator iff there is some relationship with a t-norm or conorm and a strict complement operator and if the mapping is an extension of the binary, material implication. Theorem 6.3 proves that this relationship cannot be the fuzzification of the classical formula (NOT P) OR Q without loss of the sup-min modus ponens inference rule when negation and disjunction are fuzzified by a strict complement operator respectively a t-conorm.

Whenever T has infra-semicontinuous partial mappings, a mapping $\mathcal{F}$ that satisfies the neutrality principle and the inequality $\mathcal{F} \leq \triangleright_T$ satisfies the sup-T modus ponens inference rule : this is a remarkable fact since $\mathcal{F}$ should not even be an extension of the binary implication.

8. Appendix

Let F be a $[0,1]-[0,1]$ mapping then F is *infra-semicontinuous* iff

$$(\forall x_0 \in [0,1])(\forall \epsilon > 0)(\exists \delta > 0)(\forall x \in [0,1])(\mid x - x_0 \mid < \delta \Rightarrow$$
$$\Rightarrow F(x_0) - \epsilon < F(x)).$$

Theorem A1. *Let T be a t-norm, then T is completely distributive w.r.t. supremum iff every partial mapping of T is infra-semicontinuous.*

Proof.

1. The "if" part is established now for t-norms with infra-semicontinuous partial mappings. Let T be a t-norm with infra-semicontinuous partial mappings. First the inequality

$$\sup_{j \in J} T(x_j, y) \leq T(\sup_{j \in J} x_j, y) \quad ; \quad \forall y \in [0,1]$$

is proved, $(x_j)_{j \in J}$ being a family in $[0,1]$ and J an arbitrary index set. From $x_i \leq \sup_{j \in J} x_j$ and the non-decreasingness of $T(\cdot, y)$ it follows that

$$(\forall i \in J)(T(x_i, y) \leq T(\sup_{j \in J} x_j, y))$$

and hence

$$\sup_{i\in J} T(x_i, y) \leq T(\sup_{j\in J} x_j, y).$$

To prove the equality, suppose $T(\sup_{j\in J} x_j, y) > \sup_{j\in J} T(x_j, y)$ and let

$$\varepsilon_0 = T(\sup_{j\in J} x_j, y) - \sup_{j\in J} T(x_j, y) > 0.$$

The infra-semicontinuity of $T(\cdot, y)$ in $\sup_{j\in J} x_j$ implies for $\varepsilon = \varepsilon_0$

$$(\exists\delta_0 > 0)(\forall x \in [0,1])(\sup_{j\in J} x_j - \delta_0 < x < \sup_{j\in J} x_j + \delta_0 \Rightarrow$$
$$\Rightarrow \sup_{j\in J} T(x_j, y) < T(x, y)).$$

From the characterization of supremum it is deduced that $\sup_{j\in J} x_j - \delta_0$ is no lower an upper bound for $(x_j)_{j\in J}$ and hence

$$(\exists i_0 \in J)(\sup_{j\in J} x_j - \delta_0 < x_{i_0} < \sup_{j\in J} x_j + \delta_0)$$

and so

$$\sup_{j\in J} T(x_j, y) < T(x_{i_0}, y),$$

a contradiction.

2. The reverse implication is proven now. Suppose the partial mapping $T(\cdot, y_0)$ is not infra-semicontinuous in x_0.

$$(\exists\varepsilon_0 > 0)(\forall\delta > 0)(\exists x \in [0,1])(\mid x - x_0 \mid < \delta \text{ and}$$
$$T(x, y_0) \leq T(x_0, y_0) - \varepsilon_0).$$

Choose $\varepsilon_0 > 0$ and let $\delta_n = 1/n$; $\forall n \in \mathbb{N}^*$. From the previous formula it follows that for each δ_n there exists an x_n satisfying the condition $\mid x_n - x_0 \mid < 1/n$ and $T(x_n, y_0) \leq T(x_0, y_0) - \varepsilon_0$. Obviously $\lim_{n\to\infty} x_n = x_0$. From the monotonicity of T it is deduced that

$$(\forall n \in \mathbb{N}^*)(x_n \leq x_0)$$

and hence

$$\lim_{n\to\infty} x_n = \sup_{n\in\mathbb{N}^*} x_n = x_0$$

and

$$(\forall n \in \mathbb{N}^*)(T(x_n, y_0) \leq T(x_0, y_0) - \varepsilon_0)$$

or

$$\sup_{n \in \mathbb{N}^*} T(x_n, y_0) \leq T(x_0, y_0) - \varepsilon_0 < T(x_0, y_0).$$

Thus $\sup_{n \in \mathbb{N}^*} T(x_n, y_0) \neq T(\sup_{n \in \mathbb{N}^*} x_n, y_0)$. $\diamondsuit$

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