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ON A PROBLEM OF R. SCHILLING II

Wolfgang **Förg-Rob**

Institut für Mathematik, Universität Innsbruck, Technikerstraße 25/7, A-6020 Innsbruck, Austria

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Abstract: Studies of a physical problem led to the functional equation

(1) $f(qx)=\frac{1}{4q}(f(x+1)+f(x-1)+2f(x))$ for all $x\in\mathbb{R}$

with the boundary condition

(2) $f(x)=0$ for all x with $|x|>Q:=\frac{q}{1-q}$

where $q\in]0,1[$ is a fixed real number. It turns out that the behaviour of solutions of (1) which fulfill the boundary condition (2) is quite different, depending heavily on the value of q. An — in some sense "complete" — answer on the general solution of (1) under the condition (2) (including investigations on continuity, differentiability, measurability, integrability) can be given in the following cases: $0<q<\frac{1}{3}$, $q=\frac{1}{3}$, and $q=\frac{1}{2}$.

Studies of a physical problem (cf. [4]) led Prof. R. Schilling to the functional equation given below. It was known that in the case $q=\frac{1}{2}$ there is a continuous solution with bounded support. Now the question arose to find all the solutions of this equation.

Let the functional equation

$$(1)\qquad f(qx)=\frac{1}{4q}\big(f(x+1)+f(x-1)+2f(x)\big)\quad\text{for all } x\in\mathbb{R}$$

and the boundary condition

$$(2)\qquad f(x)=0\quad\text{for all } x \text{ with }\quad |x|>Q:=\frac{q}{1-q}$$

be given, where $q\in]0,1[$ is a fixed real number. In the previous paper [2] we dealt with the problem of finding solutions of equ. (1) with unbounded support. Now we turn over to some results on solutions with bounded support. As we will see, the problem (and its solutions) have a quite different behaviour depending on the values of q. Though the problem in general is far from being solved, some special cases can be treated in the sequel:

III. Solutions with bounded support

a) The case $0 < q < \frac{1}{3}$

In this case we have $0 < Q < \frac{1}{2} < 1 - Q < 1$. The following theorem will give 5 conditions (α)-(ε) equivalent to (1)-(2). Though they are more than two (as originally given) their structure is much more easier than (1)+(2):

Theorem 20. *Let $q < \frac{1}{3}$. Then the system (1) and (2) is equivalent to the system*

(α)	$f(x) = 2qf(qx)$	*for all* $x \in [-Q, Q]$
(β)	$f(x) = 4qf(q(x+1))$	*for all* $x \in [-Q, Q]$
(γ)	$f(x) = 4qf(q(x-1))$	*for all* $x \in [-Q, Q]$
(δ)	$f(x) = 0$	*for all* x *with* $qQ < \lvert x\rvert < q(1-Q)$
(ε)	$f(x) = 0$	*for* $\lvert x\rvert > Q$.

Proof. (a) Let f fulfill (1) and (2). We show that f is a solution of (α)-(ε):

(ε) is trivial, as $(2)=(\varepsilon)$.

(α) Let $x \in [-Q, Q]$. Then $x + 1 > Q$, $x - 1 < -Q$. Thus $f(x+1) = f(x-1) = 0$ and $f(qx) = \frac{1}{4q}2f(x)$, i.e. (α) holds.

(β) Let $x \in [-Q, Q]$. Then $y := x + 1 \in [1 - Q, 1 + Q]$ and therefore $y > Q$, $y + 1 > Q$, $y - 1 = x \in [-Q, Q]$, $qy \in [q(1 - Q), q(1 + Q)]$. Remembering that $q(1+Q) = Q$ we have $qy \in [-Q, Q]$ and $f(qy) = \frac{1}{4q}f(y-1)$, thus $f(q(x+1)) = \frac{1}{4q}f(x)$.

(γ) like (β).

(δ) Let $qQ < \lvert x\rvert < q(1-Q)$ and put $y := \frac{x}{q}$. Then $Q < \lvert y\rvert < 1-Q$ and therefore $\lvert y\rvert > Q$, $\lvert y+1\rvert > Q$, $\lvert y-1\rvert > Q$, thus $f(y) = f(y+1) = = f(y-1) = 0$. That implies $f(x) = f(qy) = 0$.

(b) On the other hand, suppose that f is a function which fulfills (α)-(ε). We show that f is a solution of (1)-(2):

(2) is trivial, as $(2)=(\varepsilon)$.

(1) If $x \in [-Q, Q]$ then $x-1 < -Q$, $x+1 > Q$, and therefore $f(x+1) = f(x-1) = 0$ by (ε). Thus (α) implies that $f(qx) = \frac{1}{2q} f(x) = \frac{1}{4q}\big(f(x+1) + f(x-1) + 2f(x)\big)$. If $Q < |x| < 1-Q$ then $|x| > Q$, $|x-1| > Q$, $|x+1| > Q$. By (ε) $f(x) = f(x+1) = f(x-1) = 0$. By (δ) $f(qx) = 0$, and thus $f(qx) = \frac{1}{4q}\big(f(x+1) + f(x-1) + 2f(x)\big)$. If $x \in [1-Q, 1+Q]$, then $x > Q$, $x+1 > Q$, $x-1 \in [-Q, Q]$ and $qx \in [-Q, Q]$. (β) and (ε) imply $f(x-1) = 4qf(qx)$ and therefore $f(qx) = \frac{1}{4q}\big(f(x+1) + f(x-1) + 2f(x)\big)$. The case $x \in [-(1+Q), -(1-Q)]$ is treated like $[1-Q, 1+Q]$ by use of (γ) and (ε). If $|x| > 1+Q$, then $|x| > Q$, $|x+1| > Q$, $|x-1| > Q$, $|qx| > Q$, and therefore (ε) implies that (1) is fulfilled. $\Diamond$

As the next theorem shows, the conditions (α)–(ε) give rise to a detailed description of all the solutions:

Theorem 21. *Suppose that f is a solution of (α)–(ε). Let*

$$A_1 := \{x \mid qQ < |x| < q(1-Q)\}$$

and $\varphi_0, \varphi_1, \varphi_{-1} : [-Q, Q] \to [-Q, Q]$ be the functions

$$\varphi_0(x) := qx, \quad \varphi_1(x) := q(x+1), \quad \varphi_{-1}(x) := q(x-1).$$

Define the sets A_n recursively by

$$A_{n+1} := \varphi_0(A_n) \cup \varphi_1(A_n) \cup \varphi_{-1}(A_n).$$

Then the following holds:

(a) *Each A_n and the set $A := \bigcup_{n \in \mathbb{N}} A_n$ are open;*

(b) *$\lambda(A) = 2Q$, where λ is the Lebesgue measure on $\mathbb{R}$;*

(c) *$f(x) = 0$ for each $x \in A$ (i.e. $f = 0$ a.e.);*

(d) *$[-Q, Q] \setminus A = \left\{x \mid x = \sum_{n=1}^{\infty} a_i q^i, \text{ where } a_i \in \{0, 1, -1\}\right\}$, and this set is uncountable.*

Proof. $\varphi_0, \varphi_1, \varphi_{-1}$ are linear-affine, order-preserving homeomorphisms.

(a) As A_1 is open, by induction each A_n is open and, therefore, the set A, too.

(b) First we show that $(A_n)_{n \in \mathbb{N}}$ is a family of pairwise disjoint sets. We compute a detailed description of A_n. Let $J :=]Q, 1-Q[$. Then $A_1 = qJ \cup (-q)J$, and by induction one can easily see that A_n is the union of all the sets

$$\sum_{i=1}^{n-1} a_i q^i + q^n J \quad \text{and} \quad \sum_{i=1}^{n-1} a_i q^i - q^n J,$$

where $a_i \in \{0, 1, -1\}$. We show that all these sets are disjoint: Let

$m, n \in \mathbb{N}$ and $a_i, b_j \in \{0, 1, -1\}$, and

$$x \in \Big(\sum_{i=1}^{m-1} a_i q^i \pm q^m J\Big) \cap \Big(\sum_{j=1}^{n-1} b_j q^j \pm q^n J\Big).$$

Suppose that $a_1 > b_1$ and $m, n \geq 2$: As $J \subseteq]0,1[$ we have

$$x \geq a_1 q - \sum_{i=1}^{\infty} 1 q^i = a_1 q - q\frac{q}{1-q} > a_1 q - \frac{1}{2}q;$$

on the other hand

$$x \leq b_1 q + \sum_{i=2}^{\infty} 1 q^i = b_1 q + q\frac{q}{1-q} < b_1 q + \frac{1}{2}q,$$

a contradiction. Thus, if the intersection of two such sets is nonvoid then necessarily $a_1 = b_1$, and by a simple induction argument we are reduced to the case

$$x \in \Big(\sum_{i=1}^{m-1} a_i q^i \pm q^m J\Big) \cap (qJ).$$

Suppose $m \geq 2$: If $a_1 = 0$ or $a_1 = -1$ then $x < \sum_{i=2}^{\infty} 1q^i = q\frac{q}{1-q} = qQ$, and $x \in qJ$ implies $x > qQ$, a contradiction. If $a_1 = 1$, then $x < q(1-Q)$, and $x > q - \sum_{i=2}^{\infty} 1q^i = q - q\frac{q}{1-q} = q(1-Q)$, a contradiction. Therefore, the only possible case is $m = 1$, and we have shown that all these sets are disjoint. Now the Lebesgue measure of an interval $\sum_{i=1}^{n-1} a_i q^i \pm q^n J$ is equal to $q^n \lambda(J) = q^n(1-2Q)$, and therefore we get $\lambda(A) = 2\lambda(J) \sum_{n=1}^{\infty} q^n 3^{n-1}$, because there are 3^{n-1} polynomials $\sum_{i=1}^{n-1} a_i q^i$ with $a_i \in \{0, 1, -1\}$. Thus

$$\lambda(A) = 2q\frac{1}{1-3q}(1-2Q) = 2q\frac{1}{1-3q}\frac{1-3q}{1-q} = 2Q.$$

(c) Using $f(x) = 0$ for $x \in A_1$ (by (δ)), by induction and (α), (β), (γ) we can show that $f(x) = 0$ for $x \in A_n$, where $n \in \mathbb{N}$. Thus $f(x) = 0$ for each $x \in A$.

(d) As A is an open set $[-Q, Q] \setminus A$ is a closed set which contains all the border points of the intervals $\sum_{i=1}^{n-1} a_i q^i \pm q^n J$, that is, all the points

$$\sum_{i=1}^{n-1} a_i q^i \pm q^n Q = \sum_{i=1}^{n-1} a_i q^i \pm \sum_{i=n+1}^{\infty} 1 q^i$$

and

$$\sum_{i=1}^{n-1} a_i q^i \pm q^n (1-Q) = \sum_{i=1}^{n-1} a_i q^i \pm \Big(q^n + \sum_{i=n+1}^{\infty} (-1) q^i\Big).$$

Furthermore, all the limit points of these border points belong to the set $[-Q, Q] \setminus A$.

On the other hand, $\lambda(A) = 2Q$, and therefore $[-Q, Q]$ contains no proper interval belonging to $[-Q, Q] \setminus A$. Thus every element of $[-Q, Q] \setminus A$ is a limit point of the points given above. Now the set $\Big\{x \,|\, x = \sum_{n=1}^{\infty} a_i q^i$, where $a_i \in \{0, 1, -1\}\Big\}$ is homeomorphic to the product space $\{0, 1, -1\}^{\mathbb{N}}$ via the bijection $(a_i)_{i \in \mathbb{N}} \to \sum_{i=1}^{\infty} a_i q^i$, because $q < \frac{1}{3}$. It is easy to see that the set of border points of the intervals is dense in the set

$$\Big\{x \mid x = \sum_{n=1}^{\infty} a_i q^i, \quad \text{where} \quad a_i \in \{0, 1, -1\}\Big\},$$

and this set is closed. ◊

Corollary 4. *Let* $q < \frac{1}{3}$. *Then any solution of* (1) *and* (2) *is equal to* 0 *almost everywhere (and therefore measurable). Thus any continuous solution is identically zero.*

The next theorem gives an idea how to find all the solutions of (1) and (2) in the case $q < \frac{1}{3}$. But before we have to give a definition.

Definition 2. Let $q < \frac{1}{3}$ and use the notations of Th. 21. Let

$$B := [-Q, Q] \setminus A = \Big\{x \mid x = \sum_{n=1}^{\infty} a_i q^i, \text{ where } a_i \in \{0, 1, -1\}\Big\}.$$

Define a relation $\sim$ on B by

$$\sum_{i=1}^{\infty} a_i q^i \sim \sum_{i=1}^{\infty} b_i q^i : \Leftrightarrow \exists m, n \in \mathbb{N} : a_{m+i} = b_{n+i} \text{ for all } i \in \mathbb{N}.$$

It is easy to see that $\sim$ is an equivalence relation on B. Furthermore, $\sim$ has the following property:

Lemma 4. *If $x, y \in B$, then $x \sim y$ iff there is a $z \in B$ and there are $\psi_1, \dots, \psi_k$, $\omega_1, \dots, \omega_p \in \{\varphi_1, \varphi_0, \varphi_{-1}\}$ such that $x = \psi_1(\psi_2(\dots \psi_k(z)\dots))$ and $y = \omega_1(\omega_2(\dots \omega_p(z)\dots))$.*

Proof. For $j \in \{-1, 0, 1\}$ we have $\varphi_j\left(\sum_{i=1}^{\infty} c_i q^i\right) = jq + \sum_{i=2}^{\infty} c_{i-1} q^i$. Now let $x = \sum_{i=1}^{\infty} a_i q^i$, $y = \sum_{i=1}^{\infty} b_i q^i$. First suppose that $x \sim y$ and let m, n be as in the definition. Let $z = \sum_{i=1}^{\infty} a_{m+i} q^i = \sum_{i=1}^{\infty} b_{n+i} q^i$. Then

$$x = \varphi_{a_1}\big(\varphi_{a_2}(\dots \varphi_{a_m}(z)\dots)\big) \quad \text{and} \quad y = \varphi_{b_1}\big(\varphi_{b_2}(\dots \varphi_{b_n}(z)\dots)\big).$$

On the other hand, if $z = \sum_{i=1}^{\infty} c_i q^i$ and

$$x = \varphi_{\alpha_1}\big(\varphi_{\alpha_2}(\dots \varphi_{\alpha_m}(z)\dots)\big) \quad \text{and} \quad y = \varphi_{\beta_1}\big(\varphi_{\beta_2}(\dots \varphi_{\beta_n}(z)\dots)\big)$$

for $\alpha_1, \alpha_2, \dots, \alpha_m, \beta_1, \beta_2, \dots, \beta_n \in \{-1, 0, 1\}$, then $a_{m+i} = c_i = b_{n+i}$. ◊

Theorem 22. *Let f be a solution of (α)–(ε). Then the following holds:*

(a) *$f(x) = 0$ for $x \notin B$;*

(b) *for $x \in B$, the value $f(x)$ determines the values $f(y)$ on the equivalence class $[x] = \{y \mid y \sim x\}$;*

(c) *if $x = \sum_{i=1}^{\infty} a_i q^i$ is periodic, i.e. there are positive integers m, p such that $a_{m+i} = a_{m+p+i}$ for all $i \in \mathbb{N}$, and if s is the number of zeroes in the period, i.e. $s = \#\{i \mid m < i \le m+p, a_i = 0\}$, then $f(x) = 0$ whenever $(4q)^p \neq 2^s$.*

Proof. (a) was shown in Th. 21.

(b) is a trivial consequence of Lemma 4 and equations (α), (β), (γ).

(c) We have

$$x \sim \sum_{i=1}^{\infty} a_{m+i} q^i = \sum_{i=1}^{\infty} a_{m+p+i} q^i =: y.$$

Then $y = \varphi_{a_{m+1}}\big(\varphi_{a_{m+2}}(\dots \varphi_{a_{m+p}}(y)\dots)\big)$ and therefore by (α), (β), (γ): $f(y) = (2q)^s (4q)^{p-s} f(y)$. Thus, $f(y) = 0$ whenever $(2q)^s (4q)^{p-s} \neq 1$, and in this case we have $f(x) = 0$ by (α)–(γ). ◊

Let us denote by B_p the set $B_p := \{x \mid x \in B \text{ and } x \text{ periodic}\}$, and by B_{np} the set $B \setminus B_p$ (those x which are not periodic), then we can state the following

Lemma 5.

(a) *If* $x \in B_p$ *and* $x \sim y$*, then* $y \in B_p$.

(b) *The set of periods of minimal length form a system of representatives for* $\sim$ *on* B_p.

The proof is easy and omitted. ◊

Now we can give the general solution of (α)–(ε) in the case $q < \frac{1}{3}$. We do this in the next 3 theorems, distinguishing between different cases.

Theorem 23. *Suppose that* $0 < q < \frac{1}{4}$*. Then the general solution can be given in the following way:*

(a) $f(x) = 0$ *for* $x \in A$ *and* $|x| > Q$.

(b) *Let* $(x_k)_{k \in K}$ *be a system of representatives for* $\sim$ *on* B_{np}*, choose* $(f(x_k))_{k \in K}$ *arbitrarily and extend* f *onto the equivalence class* $[x_k]$ *of* x_k *as described in* Th. 22(b).

(c) $f(x) = 0$ *for* $x \in B_p$.

Theorem 24. *Suppose that* $\frac{1}{4} \le q < \frac{1}{3}$ *and that* $q = 2^r$ *for some rational* r*. Then the general solution can be given in the following way;*

(a) $f(x) = 0$ *for* $x \in A$ *and* $|x| > Q$.

(b) *Let* $(x_k)_{k \in K}$ *be a system of representatives for* $\sim$ *on* B_{np}*, choose* $(f(x_k))_{k \in K}$ *arbitrarily and extend* f *onto the equivalence class* $[x_k]$ *of* x_k *as described in* Th. 22(b).

(c) *Let* $(\beta_p)_{p \in P}$ *be the system of periods of minimal length (which is a system of representatives for* $\sim$ *on* B_p*) and denote by* $\ell(\beta_p)$ *the length and by* $z(\beta_p)$ *the number of zeroes of* β_p*. Choose values* $g(\beta_p)$ *in the following way:*

$$g(\beta_p) = \begin{cases} \text{arbitrary} & \text{if } \dfrac{z(\beta_p)}{\ell(\beta_p)} = 2 + r \\ 0 & \text{otherwise.} \end{cases}$$

For any $p \in P$ *choose an element* $x_p \in B_p$ *with period* β_p*, define* $f(x_p) := g(\beta_p)$ *and extend* f *onto* B_p *as in* (b).

Theorem 25. *Suppose that* $\frac{1}{4} \le q < \frac{1}{3}$ *and that* $q \neq 2^r$ *for any rational* r*. Then the general solution can be given in the following way:*

(a) $f(x) = 0$ *for* $x \in A$ *and* $|x| > Q$.

(b) *Let* $(x_k)_{k \in K}$ *be a system of representatives for* $\sim$ *on* B_{np}*, choose* $(f(x_k))_{k \in K}$ *arbitrarily and extend* f *onto the equivalence class* $[x_k]$ *of* x_k *as described in* Th. 22(b).

(c) $f(x) = 0$ *for* $x \in B_p$.

Proof. (a) was shown in Th. 21.

(b) If x is nonperiodic, $x = \sum\limits_{i=1}^{\infty} a_i q^i$, for no $m \neq n$ the equality $a_{m+i} = a_{n+i}$ can hold for all $i \in \mathbb{N}$. Therefore it is easy to see that the computation of $f(y)$ for $y \sim x$ cannot give "different results" via the choice of a z as in Lemma 4.

(c) If $x \sim y$ and x is periodic, then y has the same period as x. By Th. 22 $f(x) \neq 0$ is possible only in the case when $(4q)^\ell = 2^z$, that is, $q = 2^r$, where $r = \frac{z}{\ell} - 2$. ◊

Corollary 5. *Let* $q = \frac{1}{4}$. *Then* $Q = \frac{1}{3} = \sum\limits_{i=1}^{\infty} 1\left(\frac{1}{4}\right)^i$, *which is periodic with period* 1. *Also* $-Q = -\frac{1}{3} = \sum\limits_{i=1}^{\infty}(-1)\left(\frac{1}{4}\right)^i$ *is periodic with period* 1. *Furthermore,* $-Q$ *is not equivalent to* Q. *Thus* $\ell = 1$, $z = 0$, *and as* $\frac{1}{4} = 2^{-2}$ *and* $2+(-2) = \frac{0}{1}$ *there exist solutions* f *of* (1)-(2) *in this case such that*

$$\begin{array}{llll} f(Q) \neq 0 & \textit{and} & f(-Q) \neq 0 & \textit{or} \\ f(Q) = 0 & \textit{and} & f(-Q) \neq 0 & \textit{or} \\ f(Q) \neq 0 & \textit{and} & f(-Q) = 0 & \textit{or} \\ f(Q) = 0 & \textit{and} & f(-Q) = 0. & \end{array}$$

In other words, we have seen that any of the cases $-Q, Q \in S(f) \subseteq [-Q, Q]$, $-Q \in S(f) \subseteq [-Q, Q[$, $Q \in S(f) \subseteq]-Q, Q]$, $S(f) \subseteq]-Q, Q[$ *really can occur.*

After this investigation into the case $q < \frac{1}{3}$ we turn over to the next value for q:

b) The case $\mathbf{q = \frac{1}{3}}$

The methods used in this case to give the solutions of (1)-(2) are very similar to those used in the case $q < \frac{1}{3}$. First we give a system equivalent to (1)-(2):

Theorem 26. *Let* $q = \frac{1}{3}$. *Then* $Q = \frac{1}{2}$. *and the system* (1)-(2) *is equivalent to the system*

$$(\alpha) \qquad f(x) = \frac{2}{3} f\Big(\frac{1}{3}x\Big) \qquad \textit{for all } x \in \Big[-\frac{1}{2}, \frac{1}{2}\Big]$$

$$(\beta) \qquad f(x) = \frac{4}{3} f\Big(\frac{1}{3}(x+1)\Big) \qquad \textit{for all } x \in \Big[-\frac{1}{2}, \frac{1}{2}\Big]$$

$$(\gamma) \qquad f(x) = \frac{4}{3} f\Big(\frac{1}{3}(x+1)\Big) \qquad \textit{for all } x \in \Big[-\frac{1}{2}, \frac{1}{2}\Big]$$

$$(\delta) \qquad f(x) = 0 \qquad \textit{for all } |x| \geq \frac{1}{2}.$$

Proof. This proof is nearly the same as the proof of Th. 20, but we have to take care that $x \in [-Q, Q]$ does not imply that $x+1, x-1 \notin [-Q, Q]$, because $-Q + 1 = Q$.

(a) Suppose that f is a solution of (1) and (2). We show that f fulfills (α)–(δ):

(δ) follows directly from (2).

(α)–(γ) is shown in the same manner as in Th. 20 for all values $x \in \left]-\frac{1}{2}, \frac{1}{2}\right[$ (for these values the same arguments as in Th. 20 hold). $x = \pm\frac{1}{2}$: As $f\big(\frac{1}{2}\big) = f\big(-\frac{1}{2}\big) = 0$, (α)–(γ) are trivial consequences of (1) and (2).

(b) On the other hand, let f be a solution of (α)–(δ).

(2) is a trivial consequence of (δ), and (1) can be derived from (α)–(δ) as in the case $q < \frac{1}{3}$. ◊

As in the case $q < \frac{1}{3}$, we give an equivalence relation $\sim$ in order to describe the solutions. Therefore, let $I := \left[-\frac{1}{2}, \frac{1}{2}\right]$ and denote by $\varphi_0, \varphi_1, \varphi_{-1} \colon I \to I$ the functions $\varphi_j(x) = \frac{1}{3}(x + j)$. We denote by A the set

$$A := \left\{ \psi_1\big(\psi_2(\ldots\psi_k(z)\ldots)\big) \;\middle|\; \begin{array}{c} k \in \mathbb{Z}, k \geq 0, z \in \left\{0, \frac{1}{2}, -\frac{1}{2}\right\}, \\ \psi_1 \ldots \psi_k \in \{\varphi_1, \varphi_0, \varphi_{-1}\} \end{array} \right\},$$

and let $B := I \setminus A$. For two numbers

$$x = \sum_{i=1}^{\infty} a_i \Big(\frac{1}{3}\Big)^i \quad \text{and} \quad y = \sum_{i=1}^{\infty} b_i \Big(\frac{1}{3}\Big)^i,$$

where $a_i, b_j \in \{0, 1, -1\}$ and $x, y \in B$, we define

$$x \sim y \colon \;\Leftrightarrow\; \exists m, n \colon\; a_{m+i} = b_{n+i} \;\text{ for all } i \in \mathbb{N}.$$

Lemma 6. *With the notation above the following holds:*

(a) $A = \left\{ x = \sum\limits_{i=1}^{n} a_i \big(\frac{1}{3}\big)^i + \alpha \big(\frac{1}{3}\big)^n \;\middle|\; \begin{array}{c} n \in \mathbb{N},\; a_i \in \{0, 1, -1\}, \\ \alpha \in \left\{0, \frac{1}{2}, -\frac{1}{2}\right\} \end{array} \right\};$

(b) *Each* $x \in I$ *has a representation* $x = \sum_{i=1}^{\infty} a_i\left(\frac{1}{3}\right)^i$, *where* $a_i \in \{0,1,-1\}$. *This representation is unique whenever* $x \in B$. *Furthermore, the real number* $\sum_{i=1}^{\infty} a_i\left(\frac{1}{3}\right)^i$ *(where* $a_i \in \{(0,1,-1\}$*) is an element of* B *iff this representation has no period of length* 1 *(i.e. for all* $m \in \mathbb{N}$ *there is an* $n \in \mathbb{N}$ *such that* $n > m$ *and* $a_n \neq a_m$*).*

(c) *The relation* $\sim$ *is well-defined and an equivalence relation on the set* B.

(d) *Let* $\mathbb{Q}$ *denote the set of rational numbers. Then*

$$I \cap \mathbb{Q} = A \cup \{x \in B \mid x \text{ has a periodic representation}\}.$$

(e) *If* $x, y \in B$, x *rational and* $x \sim y$, *then* y *is rational, too.*

Proof. (a) Let

$$A' := \left\{x = \sum_{i=1}^{n} a_i\left(\frac{1}{3}\right)^i + \alpha\left(\frac{1}{3}\right)^n \;\middle|\; \begin{matrix} n \in \mathbb{N},\ a_i \in \{0,1,-1\}, \\ \alpha \in \{0, \frac{1}{2}, -\frac{1}{2}\} \end{matrix}\right\}.$$

It is easy to see that $A \subseteq A'$ and $A' \subseteq A$. (b) and (d) are well known from elementary analysis, (c) and (e) are immediate consequences. ◊

Now let $B_r := B \cap \mathbb{Q}$ be the rational points of B, and $B_{np} := B \setminus B_r$ the set of those $x \in B$ which have a nonperiodic representation. Furthermore, let $(x_k)_{k \in K}$ be a system of representatives for $\sim$ on the set B_{np}. The preceding lemma gives the technical details for proving the following theorems on the structure of the solutions.

Theorem 27. *Let* f *be a solution of* (α)–(δ). *Then:*

(a) *If* $x, y \in B$, $x \sim y$, $x = \sum_{i=1}^{\infty} a_i\left(\frac{1}{3}\right)^i$, $y = \sum_{i=1}^{\infty} b_i\left(\frac{1}{3}\right)^i$ *and* $m, n \in \mathbb{N}$ *such that* $a_{m+i} = b_{n+i}$ *for all* $i \in \mathbb{N}$, *then* $\alpha_{a_1}\alpha_{a_2}\dots\alpha_{a_m} f(x) = \alpha_{b_1}\alpha_{b_2}\dots\alpha_{b_n} f(y)$, *where* $\alpha_0 = \frac{2}{3}$ *and* $\alpha_1 = \alpha_{-1} = \frac{4}{3}$;

(b) $f(x) = 0$ *for all* $x \in \mathbb{Q}$.

Proof. (a) Let

$$z := \sum_{i=1}^{\infty} a_{m+i}\left(\frac{1}{3}\right)^i = \sum_{i=1}^{\infty} b_{n+i}\left(\frac{1}{3}\right)^i.$$

Then

$$x = \varphi_{a_1}\left(\varphi_{a_2}(\dots(\varphi_{a_m}(z)\dots)\right) \text{ and } y = \varphi_{b_1}\left(\varphi_{b_2}(\dots\varphi_{b_n}(z)\dots)\right).$$

By (α)–(γ) we have $\alpha_j f(\varphi_j(u)) = f(u)$ for all $u \in I$. Thus

$$f(z) = \alpha_{a_1} \cdot \alpha_{a_2} \cdot \ldots \cdot \alpha_{a_m} \cdot f(x) = \alpha_{b_1} \cdot \alpha_{b_2} \cdot \ldots \cdot \alpha_{b_n} \cdot f(y).$$

(b) If $x \in \mathbb{Q}$, then x has a (not necessarily unique) periodic representation $x = \sum_{i=1}^{\infty} a_i \left(\frac{1}{3}\right)^i$. Let $m, p \in \mathbb{N}$, $p \geq 1$, such that $a_{m+i} = a_{m+p+i}$ for all $i \in \mathbb{N}$. Because of (α)–(γ) we have

$$\alpha_{a_i} \cdot \alpha_{a_2} \cdot \ldots \cdot \alpha_{a_m} \cdot f(x) = \alpha_{a_1} \cdot \alpha_{a_2} \cdot \ldots \cdot \alpha_{a_{m+p}} \cdot f(x),$$

i.e.,

$$\left(\alpha_{a_{m+1}} \cdot \alpha_{a_{m+2}} \cdot \ldots \cdot \alpha_{a_{m+p}} - 1\right) \cdot f(x) = 0.$$

Now $\alpha_{a_{m+1}} \cdot \alpha_{a_{m+2}} \cdot \ldots \cdot \alpha_{a_{m+p}} = \left(\frac{2}{3}\right)^k \left(\frac{4}{3}\right)^{p-k}$ for some $k \in \mathbb{N}$. As this product cannot be equal to 1 we must conclude that $f(x) = 0$. $\Diamond$

Corollary 6. *In the case $q = \frac{1}{3}$, the only continuous solution of* (1) *and* (2) *is identically* 0.

Proof. By Th. 27, $f(x) = 0$ for all $x \in \mathbb{R} \setminus I$ and for $x \in I \cap \mathbb{Q}$. $\Diamond$

On the other hand, we can give the general solution of equations (α)–(δ):

Theorem 28. *Let $g\colon \{x_k \mid k \in K\} \to \mathbb{R}$ be given arbitrarily and define $f\colon I \to \mathbb{R}$ by*

$$f(x) := \begin{cases} g(x_k) & \text{if } x = x_k \\ \text{defined by the formula given in Th. 26(a)} & \text{if } x \sim x_k \\ 0 & \text{if } x \in I \cap \mathbb{Q} \\ 0 & \text{if } |x| \geq \frac{1}{2}. \end{cases}$$

Then f is a solution of (α)–(δ).

Proof. As any x which is equivalent to some x_k is an element of B_{np}, the representation $x = \sum_{i=1}^{\infty} a_i \left(\frac{1}{3}\right)^i$ is unique. Therefore, f is well-defined. It is easy to see that f fulfills (α)–(δ). By Th. 27, the function f given above is the only possible extension of the given function g. $\Diamond$

Next we will show that any measurable solution vanishes almost everywhere.

Theorem 29. *Let $q = \frac{1}{3}$. Then any (Lebesgue-)measurable solution of* (1) *and* (2) *vanishes almost everywhere.*

Proof. Let f be a measurable solution of (α)–(δ), and denote by $A_r := \{x \mid x \in I \text{ and } |f(x)| > r\}$ for any real $r > 0$. As $A_r \subseteq B \subseteq I$ and $\bigcap_{r>0} A_r = \emptyset$, the function $\mu\colon r \to \mu(r) := \lambda(A_r)$ (Lebesgue measure of A_r) is nonincreasing, bounded by 1, and $\lim\limits_{r\to\infty}\mu(r)=0$. Now let $t(s):=\frac{3}{2}s$ and $u(s):=\frac{3}{4}s$. Then one can immediately see that

$$\begin{aligned}\varphi_0(A_s)&=\left]-\frac{1}{6},\frac{1}{6}\right[\cap A_{t(s)} &&\text{by } (\alpha)\\ \varphi_1(A_s)&=\left]\frac{1}{6},\frac{1}{2}\right[\cap A_{u(s)} &&\text{by } (\beta)\\ \varphi_{-1}(A_s)&=\left]-\frac{1}{2},-\frac{1}{6}\right[\cap A_{u(s)} &&\text{by } (\gamma)\end{aligned}$$

for each $s>0$. Using the numbers α_j of Th. 27 we have $A_r = \varphi_0(A_{\alpha_0 r})\cup\varphi_1(A_{\alpha_1 r})\cup\varphi_{-1}(A_{\alpha_{-1} r})$, and the sets $\varphi_j(A_{\alpha_j r})$ are pairwise disjoint. Therefore, for each $r>0$ the equation

$$\mu(r)=\frac{1}{3}\mu\Big(\frac{2}{3}r\Big)+\frac{2}{3}\mu\Big(\frac{4}{3}r\Big)$$

holds. By induction, we get the equation

$$\mu(r)=\sum_{k=0}^{n}\binom{n}{k}\Big(\frac{2}{3}\Big)^k\Big(\frac{1}{3}\Big)^{n-k}\mu\Big(2^k\Big(\frac{2}{3}\Big)^n r\Big)$$

for each $r>0$, $n\in\mathbb{N}$. Now $2^2 2^3=32>27=3^3$, thus $2^{2/3}(\frac{2}{3})>1$. As the function $x\to\frac{2}{3}2^x$ is continuous, there is a $v=\frac{p}{q}\in\mathbb{Q}$ such that $p,q\in\mathbb{N}$, $v<\frac{2}{3}$ and $1<2^p(\frac{2}{3})^q<\Big(2^{2/3}(\frac{2}{3})\Big)^q$. Thus for any $n\in\mathbb{N}$ we get

$$\begin{aligned}\mu(r)=&\sum_{k=0}^{pn-1}\binom{qn}{k}\Big(\frac{2}{3}\Big)^k\Big(\frac{1}{3}\Big)^{qn-k}\mu\Big(2^k\Big(\frac{2}{3}\Big)^{qn}r\Big)+\\ &+\sum_{k=pn}^{qn}\binom{qn}{k}\Big(\frac{2}{3}\Big)^k\Big(\frac{1}{3}\Big)^{qn-k}\mu\Big(2^k\Big(\frac{2}{3}\Big)^{qn}r\Big)\le\\ &\le\sum_{k=0}^{pn-1}\binom{qn}{k}\Big(\frac{2}{3}\Big)^k\Big(\frac{1}{3}\Big)^{qn-k}1+\\ &+\sum_{k=pn}^{qn}\binom{qn}{k}\Big(\frac{2}{3}\Big)^k\Big(\frac{1}{3}\Big)^{qn-k}\mu\Big(2^{pn}\Big(\frac{2}{3}\Big)^{qn}r\Big)\le\\ &\le\sum_{k=0}^{pn-1}\binom{qn}{k}\Big(\frac{2}{3}\Big)^k\Big(\frac{1}{3}\Big)^{qn-k}1+\\ &+\mu\Big(2^{pn}\Big(\frac{2}{3}\Big)^{qn}r\Big)\sum_{k=0}^{qn}\binom{qn}{k}\Big(\frac{2}{3}\Big)^k\Big(\frac{1}{3}\Big)^{qn-k}=\end{aligned}$$

$$= \sum_{k=0}^{pn-1} \binom{qn}{k} \left(\frac{2}{3}\right)^k \left(\frac{1}{3}\right)^{qn-k} 1 + \mu\left(\left(2^p \left(\frac{2}{3}\right)^q\right)^n r\right).$$

Now choose a real number w such that $v < w < \frac{2}{3}$ and define the continuous function g on the interval $[0,1]$ by

$$g(x) := \begin{cases} 1 & \text{for } 0 \le x \le v \\ \frac{w-x}{w-v} & \text{for } v < x < w \\ 0 & \text{for } w \le x \le 1. \end{cases}$$

Then

$$\sum_{k=0}^{pn-1} \binom{qn}{k} \left(\frac{2}{3}\right)^k \left(\frac{1}{3}\right)^{qn-k} 1 =$$

$$= \sum_{k=0}^{pn-1} \binom{qn}{k} \left(\frac{2}{3}\right)^k \left(\frac{1}{3}\right)^{qn-k} g\left(\frac{k}{qn}\right) \le$$

$$\le \sum_{k=0}^{qn} \binom{qn}{k} \left(\frac{2}{3}\right)^k \left(\frac{1}{3}\right)^{qn-k} g\left(\frac{k}{qn}\right).$$

The last expression is the approximation of g by Bernstein polynomials at $x = \frac{2}{3}$, and, therefore,

$$\lim_{n\to\infty} \sum_{k=0}^{qn} \binom{qn}{k} \left(\frac{2}{3}\right)^k \left(\frac{1}{3}\right)^{qn-k} g\left(\frac{k}{qn}\right) = g\left(\frac{2}{3}\right) = 0.$$

Furthermore, $2^p\left(\frac{2}{3}\right)^q > 1$, and therefore $\lim \mu\left(\left(2^p\left(\frac{2}{3}\right)^q\right)^n r\right) = 0$. Thus we see that $\mu(r) \le 0$ for each $r > 0$, and this fact implies that the solution f has to vanish almost everywhere. ◊

Corollary 7. *As any continuous function is measurable, the preceding theorem gives another proof for the fact that in the case $q = \frac{1}{3}$ the zero function is the only continuous solution of the system* (1)-(2).

Finally, the last case which can be said to have been completely solved is

c) The case $\mathbf{q} = \frac{1}{2}$

Theorem 30. *Let $q = \frac{1}{2}$. Then the system* (1) *and* (2) *is equivalent to the system*

$$(\alpha) \qquad f(x) = 0 \qquad\qquad \text{for } |x| \geq 1$$

$$(\beta) \qquad f(x) = 2f\Big(\frac{x+1}{2}\Big) - 2f(x+1) \qquad\qquad \text{for } x \in [-1, 0]$$

$$(\gamma) \qquad f\Big(\frac{x}{4}\Big) = \frac{3}{2}f\Big(\frac{x}{2}\Big) - \frac{1}{2}f(x) \qquad\qquad \text{for } x \in [0, 1]$$

$$(\delta) \qquad f\Big(\frac{x+1}{2}\Big) = \frac{1}{2}f(x) \qquad\qquad \text{for } x \in [0, 1]$$

$$(\varepsilon) \qquad f\Big(\frac{x+1}{4}\Big) = f\Big(\frac{x}{2}\Big) - \frac{1}{4}f(x) \qquad\qquad \text{for } x \in [0, 1].$$

Proof. (a) Let f be a solution of (1) and (2). We show that f fulfills (α)–(ε):

(α) Let $x = 2$. By (2), $f(2) = f(3) = 0$, and therefore by (1) $f(1) = f\big(\frac{1}{2}2\big) = \frac{1}{2}\big(f(3) + f(1) + 2f(2)\big) = \frac{1}{2}f(1)$. Thus $f(1) = 0$, and $f(-1) = 0$ is shown in the same way.

(β) Let $x \in [-1, 0]$ and let $y := x + 1$. Then $f(y+1) = 0$ by (α) and $f\big(\frac{y}{2}\big) = \frac{1}{2}\big(f(y+1) + f(y-1) + 2f(y)\big)$, thus $f\big(\frac{x+1}{2}\big) = \frac{1}{2}\big(f(x) + {} + 2f(x+1)\big)$ and $f(x) = 2f\big(\frac{x+1}{2}\big) - 2f(x+1)$.

(δ) Let $x \in [0, 1]$. Then $x + 2 > x + 1 \geq 1$ and $\frac{x+1}{2} \in [0, 1]$. By (1) and (α) — used for the real number $x + 1$ — we have $f\big(\frac{x+1}{2}\big) = {} = \frac{1}{2}\big(f(x) + f(x+2) + 2f(x+1)\big) = \frac{1}{2}f(x)$.

(ε) Let $x \in [0, 1]$ and $y := x - 1 \in [-1, 0]$. By (1) and (α) we have $f(y-1) = 0$ and therefore $f\big(\frac{y}{2}\big) = \frac{1}{2}\big(f(y+1) + 2f(y)\big)$. By (β) we get $2f\Big(\frac{\frac{y}{2}+1}{2}\Big) - 2f\big(\frac{y}{2} + 1\big) = \frac{1}{2}\big(f(y+1) + 4f\big(\frac{y+1}{2}\big) - 4f(y+1)\big)$, or $2f\big(\frac{x+1}{4}\big) - 2f\big(\frac{x+1}{2}\big) = \frac{1}{2}f(x) + 2f\big(\frac{x}{2}\big) - 2f(x)$, which implies (ε) by use of (δ).

(γ) Let $x \in [0, 1]$ and $y := x - 1$, then $y - 2 < y - 1 \leq -1$, and $\frac{y-1}{2} \in [-1, 0]$. By (1) and (2) (used for the number $y - 1$) we have $f\big(\frac{y-1}{2}\big) = \frac{1}{2}\big(f(y-2) + f(y) + 2f(y-1)\big) = \frac{1}{2}f(y)$. By (β),

$$2f\Big(\frac{\frac{y-1}{2}+1}{2}\Big) - 2f\Big(\frac{y-1}{2} + 1\Big) =$$
$$= f\Big(\frac{y-1}{2}\Big) = \frac{1}{2}f(y) = f\Big(\frac{y+1}{2}\Big) - f(y+1).$$

Thus $2f\big(\frac{x}{4}\big) - 2f\big(\frac{x}{2}\big) = f\big(\frac{x}{2}\big) - f(x)$, which implies (γ).

(b) On the other hand, let f be a solution of (α)–(ε). We show that f fulfills (1) and (2):

(2) is an immediate consequence of (α).

(1): We show that (1) is fulfilled for any $x \in \mathbb{R}$:

(1.1) Let $|x| \geq 2$. Then $f(x) = f(\frac{x}{2}) = f(x+1) = f(x-1) = 0$, and (1) is fulfilled.

(1.2) Let $x \in [1,2]$ and $y := x - 1$. Then

$$(1) \Leftrightarrow f\left(\frac{x}{2}\right) = \frac{1}{2}f(x-1) \text{ by } (\alpha) \Leftrightarrow f\left(\frac{y+1}{2}\right) = \frac{1}{2}f(y),$$

which is fulfilled by (δ).

(1.3) Let $x \in [0,1]$, $y := x - 1$. Then

$$(1) \Leftrightarrow f\left(\frac{x}{2}\right) = \frac{1}{2}(f(x-1) + 2f(x)) \text{ by } (\alpha) \Leftrightarrow$$
$$\Leftrightarrow f(y) = 2f\left(\frac{y+1}{2}\right) - 2f(y+1) \Leftrightarrow (\beta).$$

(1.4) Let $x \in [-1,0]$ and $y := x + 1$. Then

$$(1) \Leftrightarrow f\left(\frac{x}{2}\right) = \frac{1}{2}(f(x+1) + 2f(x)) \text{ by } (\alpha) \Leftrightarrow$$
$$\Leftrightarrow 2f\left(\frac{x+2}{4}\right) - 2f\left(\frac{x+2}{2}\right) =$$
$$= \frac{1}{2}f(x+1) + 2f\left(\frac{x+1}{2}\right) - 2f(x+1) \text{ by } (\beta) \Leftrightarrow$$
$$\Leftrightarrow 4f\left(\frac{y+1}{4}\right) - 4f\left(\frac{y+1}{2}\right) = 4f\left(\frac{y}{2}\right) - 3f(y) \Leftrightarrow$$
$$\Leftrightarrow 4f\left(\frac{y+1}{4}\right) = 4f\left(\frac{y}{2}\right) - f(y) \text{ by } (\delta) \Leftrightarrow (\varepsilon).$$

(1.5) Let $x \in [-2,-1]$, $y := x + 1$, $w := x + 2$. Then

$$(1) \Leftrightarrow f\left(\frac{x}{2}\right) = \frac{1}{2}f(x+1) \text{ by } (\alpha) \Leftrightarrow$$
$$\Leftrightarrow 2f\left(\frac{x+2}{4}\right) - 2f\left(\frac{x+2}{2}\right) = f\left(\frac{x+1+1}{2}\right) - f(x+1+1) \text{ by } (\beta) \Leftrightarrow$$
$$\Leftrightarrow f\left(\frac{w}{4}\right) = \frac{3}{2}f\left(\frac{w}{2}\right) - \frac{1}{2}f(w) \Leftrightarrow (\gamma). \ \Diamond$$

Corollary 8. *Let $q = \frac{1}{2}$, and (α)-(ε) as in the preceding theorem. Then any function $g\colon [0,1] \to \mathbb{R}$ which fulfills (γ)-(ε) has a unique extension to a solution $f\colon \mathbb{R} \to \mathbb{R}$ of (1) and (2).*

Proof. As (1)-(2) is equivalent to (α)–(ε), the statement is trivial because such an f is given on $[-1,0]$ by (β), and for $|x| \geq 1$ by (α). The only problem might arise at the points $0, 1, -1$ because of some fact of "confusion". But:

(δ) implies that $g(1) = g\left(\frac{1+1}{2}\right) = \frac{1}{2}g(1)$, therefore $g(1) = 0$;

(β) gives $f(0) = 2f\left(\frac{1}{2}\right)$, the same as ($\delta$), and $f(-1) = 2f(0) - 2f(0) = 0$. ◊

Remark 4. As the proofs of Th. 30 and Cor. 8 show, the other case — restriction to the interval $[-1,0]$ instead of $[0,1]$ — can be dealt in the same way. Thus for describing solutions of (1)-(2) we always may restrict ourselves to solutions of (γ)–(ε) on the interval $[0,1]$.

Theorem 31. *Let* $q = \frac{1}{2}$, *and* (α)–(ε) *as in* Th. 30. *A function* $f\colon [0,1] \to \mathbb{R}$ *is a solution of* (γ)–(ε) *iff*

$$(*) \qquad f\left(\frac{x+m}{2^k}\right) = \left(2 - \frac{m+1}{2^{k-1}}\right) f\left(\frac{x}{2}\right) + \left(\frac{m+2}{2^k} - 1\right) f(x)$$

holds for any $x \in [0,1]$ *and any nonnegative integers* k, m *such that* $0 \leq m < 2^k$.

Proof. (a) Suppose that $(*)$ is fulfilled. We show that (γ)–(ε) hold. (γ): Put $m = 0$, $k = 2$. (δ): Put $m = 1$, $k = 1$. (ε): Put $m = 1$, $k = 2$.

(b) On the other hand, suppose that (γ)–(ε) are fulfilled. We show that $(*)$ holds.

$k = 0$: Then $m = 0$, and $(*)$ is nothing else but $f(x) = f(x)$.

$m = 0$: $f\left(\frac{x}{2}\right) = f\left(\frac{x}{2}\right)$, a trivial statement.

$m = 1$: nothing else but (δ).

$k > 1$: We do the proof by induction on k:

$m = 2n$: Then $\frac{x+m}{2^k} = \frac{x+2n}{2^k} = \frac{\frac{x}{2}+n}{2^{k-1}}$, and $0 \leq m < 2^k$ implies that $0 \leq n < 2^{k-1}$, thus

$$f\left(\frac{x+m}{2^k}\right) = f\left(\frac{\frac{x}{2}+n}{2^{k-1}}\right) = \text{(by induction)}$$

$$= \left(2 - \frac{n+1}{2^{k-2}}\right) f\left(\frac{x}{4}\right) + \left(\frac{n+2}{2^{k-1}} - 1\right) f\left(\frac{x}{2}\right) =$$

$$= \left(2 - \frac{n+1}{2^{k-2}}\right)\left(\frac{3}{2} f\left(\frac{x}{2}\right) - \frac{1}{2} f(x)\right) + \left(\frac{n+2}{2^{k-1}} - 1\right) f\left(\frac{x}{2}\right) =$$

$$= \left(2 - \frac{2n+1}{2^{k-1}}\right) f\left(\frac{x}{2}\right) + \left(\frac{2n+2}{2^k} - 1\right) f(x) \Leftrightarrow (*).$$

$m = 2n+1$: Then $\frac{x+m}{2^k} = \frac{x+2n+1}{2^k} = \frac{\frac{x+1}{2}+n}{2^{k-1}}$, and $0 \leq m < 2^k$ implies that $0 \leq n < 2^{k-1}$, thus

$$f\left(\frac{x+m}{2^k}\right) = f\left(\frac{\frac{x+1}{2}+n}{2^{k-1}}\right) = \text{(by induction)}$$

$$= \left(2 - \frac{n+1}{2^{k-2}}\right) f\left(\frac{x+1}{4}\right) + \left(\frac{n+2}{2^{k-1}} - 1\right) f\left(\frac{x+1}{2}\right) =$$

$$= \left(2 - \frac{n+1}{2^{k-2}}\right)\left(f\left(\frac{x}{2}\right) - \frac{1}{4}f(x)\right) + \left(\frac{n+2}{2^{k-1}} - 1\right)\frac{1}{2}f(x) =$$

$$= \left(2 - \frac{2n+2}{2^{k-1}}\right) f\left(\frac{x}{2}\right) + \left(\frac{2n+3}{2^k} - 1\right) f(x) \Leftrightarrow (*). \ \diamond$$

Remark 5. The equation $(*)$ can be written in the following way: let $z = \frac{x+m}{2^k}$, then we have

$$f(z) = \left(1 - z - \frac{1-x}{2^k}\right)\left(2f\left(\frac{x}{2}\right) - f(x)\right) + \frac{1}{2^k}f(x).$$

Proof. We have $2^k z - x = m$. Using this equation in $(*)$, we get

$$f(z) = f\left(\frac{x+m}{2^k}\right) = \frac{2^k - 2^k z + x - 1}{2^{k-1}} f\left(\frac{x}{2}\right) + \frac{2^k z - x + 2 - 2^k}{2^k} f(x) =$$

$$= \left(1 - z - \frac{1-x}{2^k}\right)\left(2f\left(\frac{x}{2}\right) - f(x)\right) + \frac{1}{2^k}f(x). \ \diamond$$

In order to describe the solutions of (α)–(ε) we introduce some equivalence relations on the interval $[0,1[$. Let $\mathbf{M}$ be the set

$$\mathbf{M} := \left\{\frac{p}{2^k} \mid p,k \in \mathbb{Z}\right\}.$$

It is easy to see that $\mathbf{M}$ is dense in $\mathbb{R}$ and a group with respect to addition. Furthermore, $\mathbf{M}$ is invariant under multiplication with powers of 2 resp. $\frac{1}{2}$.

Definition. Let $x, y \in [0,1[$. We define

$$x \sim y : \Leftrightarrow x - y \in \mathbf{M}$$

and

$$x \approx y : \Leftrightarrow \exists k \in \mathbb{Z}: \ 2^k x - y \in \mathbf{M}.$$

Lemma 7. *The relations $\sim$ and $\approx$ on $[0,1[$ are equivalence relations, and for $x, y \in [0,1[$ we have*
$x \approx y \Leftrightarrow$ *there is a $z \in [0,1[$ and there are nonnegative integers k, m, n, p such that $0 \le m < 2^k$, $0 \le n < 2^p$ and $x = \frac{z+m}{2^k}$ and $y = \frac{z+n}{2^p}$.*

Proof. $\sim$ is an equivalence relation because $\mathbf{M}$ is a group.

$\approx$: Reflexivity is evident, for symmetry we have $2^{-k}y - x = -2^{-k}(2^k x - y) \in -2^{-k}\mathbf{M} = \mathbf{M}$. Transitivity: $2^k x - y = m \in \mathbf{M}$, $2^p y - z = n \in \mathbf{M} \Rightarrow 2^{k+p}x - z = 2^p m + n \in \mathbf{M}$.

As to the last statement of the lemma:

a) Suppose that there are z, k, m, n, p for x and y. Then $x = \frac{z+m}{2^k}$, $y = \frac{z+n}{2^p}$, thus $2^k x - 2^p y = (z+m)-(z+n) = m-n \in \mathbb{Z}$. Furthermore, $2^{k-p}x - y = \frac{m-n}{2^p} \in \mathbf{M} \Rightarrow x \approx y$.

b) Suppose that $x \approx y$ and $2^k x - y = \frac{m}{2^p} \in \mathbf{M}$. Because of symmetry we may assume that $k \geq 0$ and, of course, $p \geq 0$. Then $2^{k+p}x = 2^p y + m$. We put $z := 2^{k+p}x (\operatorname{mod} 1)$. Then $z \in [0,1[$, and $r := 2^{k+p}x - z \in \mathbb{Z}$, $0 \leq r < 2^{k+p}$, $x = \frac{z+r}{2^{k+p}}$. Furthermore, $2^p y = = 2^{k+p}x - m = z + r - m$. Thus $0 \leq r - m < 2^p$ and $y = \frac{z+r-m}{2^p}$. ◊

In the following $[x]$ will denote the equivalence class of $x \in [0,1[$ with respect to the relation $\approx$. We distinguish two types of these classes:
"type 1": there is an integer k, $k > 0$ and a $y \in [x]$ such that $2^k y - y \in \mathbf{M}$,
"type 2": for any $k \in \mathbb{N}$ and any $y \in [x]$ we have $2^k y - y \notin \mathbf{M}$.

The following remark shows that these types exclude one another.

Remark 6. Suppose that $[x]$ is of type 1 and $2^k y - y \in \mathbf{M}$ for some $k \in \mathbb{N}$, $y \in [x]$. Then:

α) For any $z \in [x]$ $2^k z - z \in \mathbf{M}$ holds;

β) $x \approx z = \frac{m}{2^k - 1}$ for some integer m, $0 \leq m < 2^k - 1$. m can be chosen in such a way that $x = \frac{z+n}{2^p}$ for some nonnegative integers n and p.

Proof. α) $2^k y - y = m \in \mathbf{M}$, $y \approx z \Rightarrow 2^p y - z = n \in \mathbf{M}$ for some $p \in \mathbb{Z}$. Thus $z = 2^p y - n$ and $2^k z - z = 2^{k+p}y - 2^k n - 2^p y + n = 2^p(2^k y - y) - - 2^k n + n = 2^p m - 2^k n + n \in \mathbf{M}$.

β) By α), $2^k x - x = \frac{m}{2^p}$ for some integers m, p. Let $z := 2^p x (\operatorname{mod} 1)$, $n = 2^p x - z$. Then $n \in \mathbb{Z}$ and therefore $z \approx x$. Furthermore, $2^k z - z = = 2^{k+p}x - 2^k n - 2^p x + n = 2^p(2^k x - x) - 2^k n + n = m - 2^k n + n \in \mathbb{Z}$. Furthermore, $x = \frac{z+n}{2^p}$. ◊

Remark 7. Let us denote by $\langle x \rangle$ the equivalence class of x with respect to $\sim$. Then $\langle x \rangle \subseteq [x]$ for any $x \in [0,1[$, and all the sets $\langle x \rangle$, $[x]$ are countable and dense subsets of $[0,1[$. Furthermore, if $[x]$ is of type 2, for two numbers $y, z \in [x]$ for which $2^k y - x \in \mathbf{M}$ and $2^p z - x \in \mathbf{M}$ we have $y \sim z$ if and only if $k = p$.

After these remarks we are able to describe the structure of the general solution of $(*)$:

Theorem 32. *Let f be a solution of $(*)$ on the interval $[0,1]$, and let $x \in [0,1[$. Then the following holds:*

(a) *The function $y \to 2f\left(\frac{y}{2}\right) - f(y) =: c_y$ is constant on $[x]$.*

(b) *If* $m \in \mathbf{M}$ *and* $x+m \in [0,1[$ *then in any case*

$$(**) \qquad f(x) - f(x+m) = mc_x.$$

(c) f *is given on* $\langle x \rangle$ *by the formula*

$$f(y) = c(1-y) + d, \text{ for constants } c \text{ and } d.$$

(d) *If* $[x]$ *is of* type 1, *then* f *is uniquely determined on* $[x]$ *by the value* $f(x)$, *and each value of* $f(x)$ *gives rise to a solution* f *on the set* $[x]$. *The value* $f(y)$ *for* $y \approx x$ *can be computed by the formula*

$$f(y) = f(x)\frac{1-y}{1-x}.$$

(e) *If* $[x]$ *is of* type 2, *then* f *is uniquely determined on* $[x]$ *by the values* $f(x)$ *and* $f\left(\frac{x}{2}\right)$, *and any choice of values of* $f(x)$ *and* $f\left(\frac{x}{2}\right)$ *gives rise to a solution* f *on the set* $[x]$.

Proof. (a) We show that for each $y \approx x$ the value $f(y)$ can be computed from the values $f(x)$ and $f\left(\frac{x}{2}\right)$: As $y \approx x$, there is a $z \in [0,1[$ and there are nonnegative integers k, m, n, p such that $x = \frac{z+m}{2^k}$ and $y = \frac{z+n}{2^p}$. Now $(*)$ implies that

$$f(x) = f\left(\frac{z+m}{2^k}\right) = \left(2 - \frac{m+1}{2^{k-1}}\right) f\left(\frac{z}{2}\right) + \left(\frac{m+2}{2^k} - 1\right) f(z)$$

and

$$f\left(\frac{x}{2}\right) = f\left(\frac{z+m}{2^{k+1}}\right) = \left(2 - \frac{m+1}{2^k}\right) f\left(\frac{z}{2}\right) + \left(\frac{m+2}{2^{k+1}} - 1\right) f(z).$$

This system of two linear equations in the unknowns $f(z), f\left(\frac{z}{2}\right)$ has a unique solution for any given values of $f(x)$, $f\left(\frac{x}{2}\right)$ because the determinant of the coefficients is nonzero. Computation of the determinant of the coefficients:

$$\det = \left(2 - \frac{m+1}{2^{k-1}}\right)\left(\frac{m+2}{2^{k+1}} - 1\right) - \left(2 - \frac{m+1}{2^k}\right)\left(\frac{m+2}{2^k} - 1\right) =$$

$$= \frac{1}{2^{2k}}\left((2^k - m - 1)(m + 2 - 2^{k+1}) - (2^{k+1} - m - 1)(m + 2 - 2^k)\right) =$$

$$= \frac{1}{2^{2k}}\left(-2^{2k+1} + 2^k(m+2+2m+2) - (m+1)(m+2) + \right.$$

$$\left. + 2^{2k+1} - 2^k(m+1+2m+4) + (m+1)(m+2)\right) = -\frac{1}{2^k}.$$

By $(*)$, the value $f(y)$ can be computed from the values $f(z)$ and $f\left(\frac{z}{2}\right)$. Furthermore, we see that $2f\left(\frac{x}{2}\right) - f(x) = 2f\left(\frac{z}{2}\right) - f(z)$. Thus the function $y \to 2f\left(\frac{y}{2}\right) - f(y)$ is constant on each equivalence class $[x]$.

(b) Let $m \in \mathbf{M}$ and suppose that $x+m \in [0,1[$. As $x \approx x+m$, we have $c_x = c_{x+m}$. Thus we may assume for proving (b) that $m = \frac{p}{2^k} > 0$. Let $z := 2^k x \pmod 1$, $n := 2^k x - z \in \mathbb{Z}$. Then $x = \frac{z+n}{2^k}$, $x+m = \frac{z+n+p}{2^k}$. By (*), we have

$$f(x) = f\Big(\frac{z+n}{2^k}\Big) = \Big(2 - \frac{n+1}{2^{k-1}}\Big) f\Big(\frac{z}{2}\Big) + \Big(\frac{n+2}{2^k} - 1\Big) f(z)$$

and

$$f(x+m) = f\Big(\frac{z+n+p}{2^k}\Big) = \Big(2 - \frac{n+p+1}{2^{k-1}}\Big) f\Big(\frac{z}{2}\Big) + \Big(\frac{n+p+2}{2^k} - 1\Big) f(z).$$

Thus

$$f(x) - f(x+m) = \frac{p}{2^k}\Big(2f\Big(\frac{z}{2}\Big) - f(z)\Big) = mc_z = mc_x.$$

(c) Let $c := c_x = 2f\big(\frac{x}{2}\big) - f(x)$, $d = f(x) - c(1-x)$. Then $f(x) = c(1-x) + d$, and for $y \in \langle x \rangle$ we have (by (b)) $f(y) = f(x) + (x-y)c = d + c(1-x) + c(x-y) = d + c(1-y)$.

(d) Suppose that $[x]$ is of type 1. Then $x = \frac{z+n}{2^p}$, where $z = \frac{m}{2^k - 1}$, for some integers k, m, n, p, i.e. $[x] = [z]$. Now $z = \frac{z+m}{2^k}$ and (*) imply that

$$f(z) = f\Big(\frac{z+m}{2^k}\Big) = \Big(1 - z - \frac{1-z}{2^k}\Big)\Big(2f\Big(\frac{z}{2}\Big) - f(z)\Big) + \frac{1}{2^k} f(z).$$

Thus

$$\Big(1 - \frac{1}{2^k}\Big) f(z) = (1-z)\Big(1 - \frac{1}{2^k}\Big)\Big(2f\Big(\frac{z}{2}\Big) - f(z)\Big),$$

and $2f\big(\frac{z}{2}\big) - f(z) = \frac{f(z)}{1-z}$. Now we use this result and (*) for x:

$$f(x) = f\Big(\frac{z+n}{2^p}\Big) = \Big(1 - x - \frac{1-z}{2^p}\Big)\Big(2f\Big(\frac{z}{2}\Big) - f(z)\Big) + \frac{1}{2^p} f(z) =$$

$$= \Big(1 - x - \frac{1-z}{2^p}\Big)\frac{f(z)}{1-z} + \frac{1}{2^p} f(z)$$

$$f\Big(\frac{x}{2}\Big) = f\Big(\frac{z+n}{2^{p+1}}\Big) = \Big(1 - \frac{x}{2} - \frac{1-z}{2^{p+1}}\Big)\Big(2f\Big(\frac{z}{2}\Big) - f(z)\Big) + \frac{1}{2^{p+1}} f(z) =$$

$$= \Big(1 - \frac{x}{2} - \frac{1-z}{2^{p+1}}\Big)\frac{f(z)}{1-z} + \frac{1}{2^{p+1}} f(z).$$

Thus $(1-x) f\big(\frac{x}{2}\big) - \big(1 - \frac{x}{2}\big) f(x) = 0$. Therefore, the value $f\big(\frac{x}{2}\big)$ is uniquely determined by $f(x)$. In order to show that there is a solution on $[x]$ for any choice of the value $f(x)$, one has to check that $y \to \frac{1-y}{1-x} f(x)$ is a solution of (*) on $[x]$. This is easy and omitted.

(e) Suppose that $[x]$ is of type 2. We only have to show that any choice of values of $f(x)$ and $f\left(\frac{x}{2}\right)$ gives rise to a solution on the set $[x]$. As it has been shown in (1), for any $y \in [x]$ the value $f(y)$ can be computed from $f(x)$ and $f\left(\frac{x}{2}\right)$. Thus it only has to be shown that $f(x)$ and $f\left(\frac{x}{2}\right)$ can be chosen arbitrarily. Here we indicate an explicit construction of the solution: Let $f(x)$ and $f\left(\frac{x}{2}\right)$ be given and define $c := 2f\left(\frac{x}{2}\right) - f(x)$, $d := f(x) - c(1-x)$ and $g\colon [x] \to \mathbb{R}$ by $g(y) := := c(1-y) + d2^k$, where $2^k x - y \in \mathbf{M}$. Then:

$$g(x) = c(1-x) + d1 = f(x)$$

$$g\left(\frac{x}{2}\right) = c\left(1 - \frac{x}{2}\right) + d\frac{1}{2} =$$

$$= c\frac{2-x}{2} + \frac{1}{2}f(x) - c\frac{1-x}{2} = \frac{1}{2}c + \frac{1}{2}f(x) = f\left(\frac{x}{2}\right).$$

g is a solution of (γ)–(ε). In fact, let $y \in [x]$, $2^k x - y \in \mathbf{M}$:

$$(\gamma) \qquad \begin{aligned} &\frac{3}{2}g\left(\frac{y}{2}\right) - \frac{1}{2}g(y) = \\ &= \frac{3}{2}c\left(1 - \frac{y}{2}\right) + \frac{3}{2}d2^{k-1} - \frac{1}{2}c(1-y) - \frac{1}{2}d2^k = \\ &= c\left(1 - \frac{y}{4}\right) + d2^{k-2} = g\left(\frac{y}{4}\right) \end{aligned}$$

$$(\delta) \qquad \begin{aligned} g\left(\frac{y+1}{2}\right) &= c\left(1 - \frac{y+1}{2}\right) + d2^{k-1} = \\ &= \frac{1}{2}\left(c(1-y) + d2^k\right) = \frac{1}{2}g(y) \end{aligned}$$

$$(\varepsilon) \qquad \begin{aligned} g\left(\frac{y}{2}\right) - \frac{1}{4}g(y) &= c\left(1 - \frac{y}{2}\right) + d2^{k-1} - \frac{1}{4}c(1-y) - \frac{1}{4}d2^k = \\ &= c\left(1 - \frac{y+1}{4}\right) + d2^{k-2} = g\left(\frac{y+1}{4}\right). \end{aligned}$$

Thus g is a solution and, therefore, $g = f$.

The only fact to show is that g is well defined (i.e. the integer k is unique). Suppose that $2^k x - y = m \in \mathbf{M}$, $2^p x - y = n \in \mathbf{M}$, where $k > p$. Then $2^k x - 2^p x = m - n \in \mathbf{M}$ and, therefore, $2^{k-p}x - x \in \mathbf{M}$ – a contradiction to the assumption that $[x]$ is of type 2. ◊

Theorem 33. *The general solution of* $(*)$ *on the interval* $[0,1[$ *is given in the following way: Let* $\{x_i\}_{i \in T}$ *be a system of representatives for the relation* $\approx$. *For each* $i \in T$ *choose* $c_i \in \mathbb{R}$ *arbitrarily and*

$$d_i \begin{cases} \textit{arbitrarily} & \textit{if } [x_i] \textit{ is of type } 2 \\ = 0 & \textit{if } [x_i] \textit{ is of type } 1. \end{cases}$$

Define $f\colon [0,1[\to \mathbb{R}$ *by* $f(y)=c_i(1-y)+d_i 2^k$ *whenever* $2^k x_i - y \in \mathbf{M}$.

The **proof** has been given in the preceding theorem. ◊

From this result one can easily deduce the general structure of continuous solutions.

Corollary 9. *Let* f *be a solution of* $(*)$ *on* $[0,1[$ *which is continuous on a nondegenerate interval* J. *Then* $f(x) = c(1-x)$ *for some real constant* c.

Proof. As $20 - 0 = 0 \in \mathbf{M}$, the set $[0]$ is of type 1. Thus f is given on $[0]$ by $f(x) = c(1-x)$ for some constant c. As $[0]$ is dense in $[0,1[$, $f(x) = c(1-x)$ on J. Now let $y \in [0,1[$ be arbitrary. On $\langle y\rangle$ f is given by $f(z) = c'(1-z) + d'$ for some constants c', d'. As $\langle y\rangle$ is dense in $[0,1[$, there are at least two elements $u, u' \in \langle y\rangle \cap J$, $u \neq u'$. Thus we have

$$c'(1-u) + d' = f(u) = c(1-u)$$

$$c'(1-u') + d' = f(u') = c(1-u')$$

and therefore $c' = c$, $d' = 0$. As y was chosen arbitrarily we have $f(x) = c(1-x)$ on $[0,1[$. ◊

We also can use the result of Th. 33 to give the structure of solutions of $(*)$ which are continuous at one point:

Theorem 34. *Let* $f\colon [0,1[\to \mathbb{R}$ *be a solution of* $(*)$ *which is continuous at a point* $x_0 \in [0,1[$. *Then* $f(x) = c(1-x)$ *for some constant* c.

Proof. Let $y \in [0,1[$ and $z = \frac{y}{2}$. Then $[z] = [y]$, and we have

$$f(t) = c_y(1-t) + d_y \quad \text{for} \quad t \in \langle y\rangle$$

and

$$f(t) = c_y(1-t) + d_y\frac{1}{2} \quad \text{for} \quad t \in \langle z\rangle.$$

As $\langle y\rangle$ and $\langle z\rangle$ are dense in $[0,1[$ and as f is continuous at x_0 we have

$$\begin{aligned} c_y(1-x_0) + d_y\frac{1}{2} &= \lim_{\substack{t\to x \\ t\in\langle z\rangle}} f(t) = f(x_0) = \\ &= \lim_{\substack{t\to x \\ t\in\langle y\rangle}} f(t) = c_y(1-x_0) + d_y. \end{aligned}$$

Therefore, $d_y = 0$ and $c_y = \frac{f(x_0)}{1-x_0}$. As y was arbitrary, $f(x) = c(1-x)$, where $c = \frac{f(x_0)}{1-x_0}$. ◊

Remark 8. For getting the result of Th. 34 it is essential that the point of continuity is an element of the open interval $]-1,1[$. The following example shows that continuity at the point $x = 1$ is not sufficient to guarantee the continuity of the solution f: Let $\{x_i\}_{i\in T}$ be a system of representatives for the relation $\approx$, choose $c_i := x_i$, $d_i := 0$ for each $i \in T$ and define $f\colon [0,1[\to \mathbb{R}$ by $f(x) = c_i(1-x)$ whenever $x \approx x_i$. By Th. 33, f is a solution of $(*)$ and, by Th. 34, f is not continuous at any point $x \in [0,1[$. (As Th. 35 will show, f is not measurable, too.) As $c_i \in [0,1[$ for all $i \in T$ we have $0 \le f(x) < 1-x$ on the interval $[0,1[$, which implies that f is continuous at $x = 1$.

Next we deal with measurable solutions. A heavy instrument for treating this question is Šmital's lemma, which can be written in the following way:

Lemma 8. *Let $A, B \subseteq \mathbb{R}$ be such that A has positive Lebesgue measure and B is dense in $\mathbb{R}$. Then the set $A + B$ has full Lebesgue measure, i.e. the complement of $A + B$ has measure 0.*

A proof can be found in [3].

Before we give the theorem on measurable solutions we need a lemma which bases on Šmital's lemma:

Lemma 9. *Let $J \subseteq [0,1[$ be a nondegenerate interval, $g\colon J \to \mathbb{R}$ a measurable function which is constant on the equivalence classes $\langle x\rangle$ for any $x \in [0,1[\cap J$. Then g is constant a.e.*

Proof. Suppose that g is not constant a.e. As g is real there must be a number $c \in \mathbb{R}$ such that both of the sets $A := \{x \mid f(x) \le c\}$ and $B := \{x \mid f(x) > c\}$ have positive Lebesgue measure. As g is constant on each equivalence class $\langle x\rangle$ we have $A = J \cap (A + \mathbf{M})$ and $B = J \cap$ $\cap (B + \mathbf{M})$. By Šmital's lemma $\lambda(A) = \lambda(B) = \lambda(J)$, a contradiction to the fact that $A \cap B = \emptyset$. Thus g must be constant a.e. $\Diamond$

Theorem 35. *Let $f\colon [0,1[\to \mathbb{R}$ be a solution of $(*)$ which is measurable on a measurable set $S \subseteq [0,1[$ of positive Lebesgue measure. Then there is a constant $c \in \mathbb{R}$ such that $f(x) = c(1-x)$ almost everywhere.*

Proof. We give the proof in several steps:

(a) By Šmital's lemma the set $S + (\mathbf{M}\setminus\{0\})$ has full Lebesgue measure, thus $\lambda(S \cap (S + (\mathbf{M}\setminus\{0\}))) = \lambda(S) > 0$. Now $S + (\mathbf{M}\setminus\{0\}) =$ $= \bigcup_{m\in\mathbf{M}\setminus\{0\}} (S + m)$, and $\mathbf{M}$ is countable. Therefore, there is an $m \in \mathbf{M}$, $m \neq 0$, such that $\lambda\big(S \cap (S+m)\big) > 0$. Now choose such an m and let $A := \big(S \cap (S+m)\big) - m$. Then $A \subseteq S$ and $A + m \subseteq S$, thus the functions $x \to f(x)$, $x \to f(x+m)$ are both measurable on the set A.

(b) As it was shown, on A the function

$$x \to c_x = 2f\left(\frac{x}{2}\right) - f(x) \text{ is given by } c_x = \frac{f(x) - f(x+m)}{m}.$$

Now let $n \in \mathbf{M}$ be arbitrary. Then we have

$$f(x+n) = f(x) - nc_x = f(x) + \frac{n}{m}\big(f(x+m) - f(x)\big).$$

Thus f is measurable on the set $\big((A+n) \cap [0,1[\big)$, for arbitrary $n \in \mathbf{M}$, and therefore measurable on $B := [0,1[\cap \bigcup_{n \in \mathbf{M}} (A+n)$. By Šmital's lemma, $\lambda(B) = 1$, and therefore f is measurable on the whole interval $[0,1[$.

(c) By Th. 33, f is given by $f(x) = c(x)(1-x) + d(x)$, where c is constant on the set $[x]$ and d is constant on the set $\langle x \rangle$ for any $x \in [0,1[$. As f is measurable the function $c(x) = 2f\left(\frac{x}{2}\right) - f(x)$ is measurable, too, which implies that d is also measurable. By Lemma 9, c and d are constant functions a.e. Keeping in mind the structure of the function d as it is given in Th. 33 ($d(x) = d_i 2^k$), the only possible case for d to be constant a.e. is that d vanishes a.e. Thus $f(x) = c(1-x) + 0$ a.e. $\Diamond$

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ÜBER DERIVATIONEN AUF GEWISSEN FUNKTIONENRÄUMEN UND DEREN BEZIEHUNG ZUM DIFFERENTIATIONSOPERATOR

Zbigniew **POWĄZKA**

Poland, PL-30-612 Kraków, ul. W. Witosa 33/99

Michael **ROSE**

Germany, 38678 Clausthal-Zellerfeld, Am Schlagbaum 24

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Abstract: In this paper we consider derivations on special sets of functions, mainly on the set of infinitely differentiable functions. We will be able to characterize the derivations on this set in such a way that the problem is reduced to that of determining the derivations on the real field, which is already done in the literature. Finally we characterize the differentiation operator as a derivation with some additional properties.

Der erste Autor hat auf der vierten „International Conference on Functional Equations and Inequalities" im Februar 1993 einen Vortrag mit dem Titel „Über ein Funktionalgleichungssystem des Differentiationsoperators" gehalten. Der zweite Autor hat dort darauf hingewiesen, daß das vorgestellte Funktionalgleichungssystem, welches Endomorphismen auf der Menge der stetigen Funktionen beschreibt, die zusätzlich noch die Produktregel und Kettenregel erfüllen, nur die triviale Lösung (den Nulloperator) besitzt. Durch gewisse Modifikationen konnte er

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jedoch sämtliche Derivationen auf der Menge der auf einem Intervall beliebig oft differenzierbaren Funktionen beschreiben.

1. Vorbemerkungen

$I \subset \mathbb{R}$ sei ein fest gewähltes Intervall. Gehören Randpunkte zum Intervall dazu, so sind unter Ableitungen in diesen Punkten im folgenden stets Einseitige zu verstehen. Die uns interessierenden Funktionenräume sind

$\mathcal{F} := \{f\colon I \to \mathbb{R}\}$ die Menge der reellwertigen Funktionen auf I,

$\mathcal{F}_k := \{f \in F;\ f$ konstant$\}$,

$\mathcal{P} := \{p \in F;\ p$ Polynom$\}$,

$\mathcal{C} := \{f \in F;\ f$ stetig$\}$,

$\mathcal{A}^n := \{f \in F;\ f$ n-mal differenzierbar$\}(n \in \mathbb{N} \cup \{\infty\})$.

$\mathcal{F}$ ist bzgl. der punktweise erklärten Addition und (Skalar-)Multiplikation eine kommutative Algebra mit Einselement τ_0; $\mathcal{F}_k$, $\mathcal{P}$, $\mathcal{C}$ und $\mathcal{A}^n$ sind Teilalgebren. Dabei wird das n-te Monom für $n \in \mathbb{N}_0$ mit $\tau_n\colon I \ni t \to t^n \in \mathbb{R}$ bezeichnet. Alle Ergebnisse gelten sinngemäß auch dann, wenn man komplexwertige Funktionen betrachtet.

1.1. Definition. Wir sagen die Transformation $T\colon \mathcal{D}(T) \to \mathcal{F}$ erfülle (D) dann und nur dann, wenn gilt:

(1) $\quad \mathcal{P} \subset \mathcal{D}(T) \subset \mathcal{F}$ und $\mathcal{D}(T)$ ist Algebra, $T\colon \mathcal{D}(T) \to \mathcal{F}$,

$$\left.\begin{aligned} &(2) & T(f+g) &= Tf + tg \\ &(3) & T(f \cdot g) &= fTg + gTf \end{aligned}\right\} \text{ für alle } f, g \in \mathcal{D}(T).$$

Die Transformation $T\colon \mathcal{D}(T) \to \mathcal{F}$ ist also eine Derivation, an deren Definitionsbereich $\mathcal{D}(T)$ gewisse Anforderungen gestellt werden.

Derivationen sind schon von vielen Autoren ausgiebig untersucht worden. Das Buch [4] gibt einen guten Überblick über die gängigen algebraischen Strukturen; Derivationen werden dort auf Seite 120 eingeführt. Die hier zugrundeliegende Menge, auf der die Derivationen erklärt werden, hat jedoch eine vergleichsweise reiche Struktur; es werden algebraische und analytische Eigenschaften zusammen betrachtet.

1.2. Definition. Die Transformation $T\colon \mathcal{D}(T) \to \mathcal{F}$ erfülle (D′) dann und nur dann, wenn sie (D) erfüllt und zusätzlich gilt:

(4) $$T_{\tau_1} = \tau_0, \quad T\phi = \Theta \text{ für } \phi \in \mathcal{F}_k,$$

wobei $\Theta \in \mathcal{F}$ die Nullfunktion bezeichnet.

Die letzte Bedingung dient als Normierung; sie liefert im Satz (2.5) die Übereinstimmung der betrachteten Derivation mit dem Differentiationsoperator, der natürlich (D′) erfüllt.

1.3. Bemerkung. *T erfülle* (D) *und zusätzlich*

(4a) *$\mathcal{D}(T)$ ist ∘-abgeschlossen, d.h.: $f, g \in \mathcal{D}(T) \Rightarrow f \circ g \in \mathcal{D}(T)$,*

(4b) $$T(f \circ g) = (Tf \circ g) \cdot Tg, \text{ für alle } f, g \in \mathcal{D}(T),$$

(4c) *zu jedem $t \in I$ existiert ein $f \in \mathcal{D}(T)$ mit $(Tf)(t) \neq 0$.*

Dann erfüllt T such (D′); *es gibt also* (4). Dies sieht man so:

$$T\phi = T(\phi \circ \Theta) \overset{(4b)}{=} (T\phi \circ \Theta) \cdot T\Theta \overset{(2)}{=} \Theta \qquad \text{für } \phi \in \mathcal{F}_k,$$

$$Tf = T(f \circ \tau_1) \overset{(4b)}{=} (Tf \circ \tau_1) \cdot T\tau_1 = Tf \cdot T\tau_1 \text{ für } f \in \mathcal{D}(T),$$

und weiter

$$Tf \cdot (T\tau_1 - \tau_0) = \Theta \qquad \text{für } f \in \mathcal{D}(T).$$

Wegen (4c) erhalten wir daraus $T\tau_1 = \tau_0$, also insgesamt die Bedingung (4). ◊

Die Normierungsbedingung (4) kann also durch die Kettenregel (4a, 4b) und eine weitere relativ schwache Bedingung (4c) ersetzt werden:

$$\text{(D)},\text{(4a-c)} \Rightarrow \text{(D}'\text{)}.$$

Der Sinn der Bemerkung (1.3) liegt einzig und allein in dem Nachweis, daß die im weiteren Verlauf ausschließlich benutzten Bedingungen (D′) wirklich weniger fordern, als die im anfangs erwähnten Vortrag von Z. Powązka.

Vorab wird nun noch der für die weiteren Untersuchungen wesentliche stetig fortgesetzte Differenzenquotient einer differenzierbaren Funktion betrachtet:

1.4. Definition und Lemma. Zu $f \in \mathcal{F}$, f differenzierbar im Punkt $t_0 \in I$ sei

(5) $$r(t) := r_{f,t_0}(t) := \begin{cases} \frac{f(t)-f(t_0)}{t-t_0} & \text{für } t \neq t_0, \\ f'(t_0) & \text{für } t = t_0 \end{cases}$$

der an der Stelle t_0 stetig fortgesetzte Differenzquotient von f. Damit gilt:

(6) *Für* $n \in \mathbb{N}_0$, $f \in \mathcal{A}^{n+1}$ *und* $t_0 \in I$ *ist* $r_{f,t_0} \in \mathcal{A}^n$.

Beweis. Seien $f \in \mathcal{A}^{n+1}$ und $t_0 \in I$ gewählt. Definiere $R \in \mathcal{A}^{n+1}$ durch

$$f(t) =: \sum_{k=0}^{n+1} \frac{f^{(k)}(t_0)}{k!} \cdot (t-t_0)^k + R(t), \quad t \in \mathbb{R}.$$

Aus dieser Gleichheit folgt durch sukzessive Differentiation für $m \in \{0, \dots, n+1\}$

$$f^{(m)}(t) = \sum_{k=0}^{n+1-m} \frac{f^{(m+k)}(t_0)}{k!} \cdot (t-t_0)^k + R^{(m)}(t).$$

Weil $R^{(m)}(t)$ das Restglied zum $(n+1-m)$-ten Taylorpolynom von $f^{(m)}(t)$ ist, gilt für $0 \leq m \leq n$ (vergleiche etwa [3], Seite 239):

(7) $$R^{(m)}(t) = o\big((t-t_0)^{n+1-m}\big) \qquad (t \to t_0).$$

Definiere $S \in \mathcal{F}$ durch

$$r(t) =: \sum_{k=0}^{n} \frac{f^{(k+1)}(t_0)}{(k+1)!} \cdot (t-t_0)^k + S(t), \qquad t \in I.$$

Daraus ergibt sich:

$$S(t) = \begin{cases} \frac{R(t)}{t-t_0}, & t \neq t_0 \\ 0, & t = t_0 \end{cases}$$

Zu zeigen bleibt, daß S n-mal auf I differenzierbar ist, denn dann gilt dies auch für r. Für $t \notin t_0$ ist $S(t)$ sogar $(n+1)$-mal differenzierbar. Mit Hilfe der Leibnizschen Regel für die mehrfache Differentiation von Produkten erhält man für $0 \leq m \leq n+1$, $t \neq t_0$:

$$S^{(m)}(t) = \sum_{k=0}^{m} \binom{m}{k} R^{(k)}(t) \cdot \frac{(-1)^{m-k}(m-k)!}{(t-t_0)^{m-k+1}} =$$

$$= \sum_{k=0}^{m} (-1)^{m-k} \frac{m!}{k!} \cdot (t-t_0)^{n-m} \cdot \frac{R^{(k)}(t)}{(t-t_0)^{n+1-k}}.$$

Mit (7) bekommt man hiermit für $0 \leq m \leq n-1$ schließlich $\frac{S^{(m)}(t)}{t-t_0} \xrightarrow{t \to t_0} 0$, woraus weiter $S'(t_0) = \dots = S^{(n)}(t_0) = 0$, also insgesamt die n-malige Differenzierbarkeit von S im Punkt t_0 folgt. ◊

Das Beispiel $f(t) := \operatorname{sgn} t \cdot t^2$, $r_{f,0}(t) = |t|$ zeigt übrigens schnell, daß es Fälle gibt, in denen r in $\mathcal{A}^n \setminus \mathcal{A}^{n+1}$ liegt ($n \in \mathbb{N}_0$).

2. Folgerungen

Ohne die Bedingung (4) wird man erwarten, daß es auf den Funktionenräumen $\mathcal{A}^n$ eine Fülle von Derivationen geben wird. Das folgende Lemma gibt eine Schar davon an, die sich im Falle des Raumes $\mathcal{A}^\infty$ als erschöpfend erweisen wird:

2.1. Definition und Lemma. Es sei $\alpha \in \mathcal{F}$ und B: $\mathcal{F}_k \mapsto \mathcal{F}$ eine Derivation. Definiere für $f \in \mathcal{A}^1$

$$(K_B f)(t) := \big(B(f(t)\tau_0) - f' \cdot B(t\tau_0)\big)(t), \quad t \in I,$$
$$Tf := \alpha \cdot f' + K_B f.$$

Dann sind K_B und T zwei Derivationen (D) *auf $\mathcal{A}^1$.* Dies kann durch direktes Nachrechnen der Bedingungen (2) und (3) eingesehen werden.

Das nächste Lemma zeigt nun, daß sich jede Derivation nach (D) wenigstens auf einem Teilbereich in dieser Form darstellen läßt:

2.2. Lemma. *T erfülle* (D), *und es sei $\alpha := T\tau_1$ und $B := T|_{\mathcal{F}_k}$. Dann gilt:*

Für $f \in \mathcal{A}^1 \cap \mathcal{D}(T)$ und $t_0 \in I$ mit $r_{f,t_0} \in \mathcal{D}(T)$ ist $(Tf)(t_0) = (\alpha \cdot f' + K_B f)(t_0)$.

Beweis. Unter den Voraussetzungen an f und $r = r_{f,t_0}$ folgt:

$$\begin{aligned} f &= f(t_0)\tau_0 + (\tau_1 - t_0\tau_0) \cdot r, \\ Tf &= B(f(t_0)\tau_0) + (\alpha - B(t_0\tau_0)) \cdot r + (\tau_1 - t_0\tau_0) \cdot Tr, \\ (Tf)(t_0) &= (\alpha f')(t_0) + (B(f(t_0)\tau_0) - f' \cdot B(t_0\tau_0))(t_0) \\ &= (\alpha f' + K_B f)(t_0). \ \Diamond \end{aligned}$$

Es sei angemerkt, daß die Forderung $r_{f,t_0} \in \mathcal{D}(T)$ für $f \in \mathcal{A}^\infty \subset \mathcal{D}(T)$ immer erfüllt ist.

Wir kommen nun zu den abschließenden Sätzen, die das Wesentliche noch einmal zusammenfassen:

2.3. Satz. *T erfülle* (D), *es sei $\alpha := T\tau_1$, $B := T|_{\mathcal{F}_k}$, und für $f \in \mathcal{A}^1$ sei $\tilde{T}f := \alpha f' + K_B f$. Dann ist $\tilde{T}$ eine Derivation, und es gilt:*

a) $$\mathcal{C} \subset \mathcal{D}(T) \Rightarrow T|_{\mathcal{A}^1} = \tilde{T},\ T(\mathcal{C}) \overset{\neq}{\supset} \mathcal{C},$$

b) $$n \in \mathbb{N} \text{ und } \mathcal{A}^n \subset \mathcal{D}(T) \Rightarrow T|_{\mathcal{A}^{n+1}} = \tilde{T}|_{\mathcal{A}^{n+1}},$$

c) $$\mathcal{A}^\infty \subset \mathcal{D}(T) \Rightarrow T|_{\mathcal{A}^\infty} = \tilde{T}|_{\mathcal{A}^\infty}.$$

Zu beweisen bleibt nur Punkt a): $\mathcal{C}$ ist eine echte Teilmenge der Menge aller f' mit $f \in \mathcal{A}^1$; die Letztere ist wiederum wegen $Tf = f'$ in $T(\mathcal{C})$ enthalten. ◊

Ist T zusätzlich linear, so ist B wegen $B\tau_0 = B(\tau_0 \cdot \tau_0) = 2\tau_0 \cdot \cdot B\tau_0 = 2B\tau_0$ notwendig das Nullfunktional, und es ergibt sich:

2.4. Satz. *T erfülle* (D), *sei linear, und es sei $\alpha := T\tau_1$. Dann gilt:*

a) $$\mathcal{C} \subset \mathcal{D}(T) \Rightarrow \text{für } f \in \mathcal{A}^1 \text{ ist } Tf = \alpha f',$$

b) $$n \in \mathbb{N} \text{ und } \mathcal{A}^n \subset \mathcal{D}(T) \Rightarrow \text{für } f \in \mathcal{A}^{n+1} \text{ ist } Tf = \alpha f',$$

c) $$\mathcal{A}^\infty \subset \mathcal{D}(T) \Rightarrow \text{für } f \in \mathcal{A}^\infty \text{ ist } Tf = \alpha f',$$

d) $$\text{für } f \in \mathcal{P} \text{ ist } Tf = \alpha f'.$$

Zu beweisen bleibt nur Punkt d): Wegen der Linearität reicht es, für $n \in \mathbb{N}$ nachzuweisen, daß $T\tau_n = n\alpha\tau_{n-1}$ gilt. Letzteres zeigt man induktiv. ◊

2.5. Satz. *T erfülle* (D′). *Dann gilt:*

a) $$\mathcal{C} \subset \mathcal{D}(T) \Rightarrow \text{für } f \in \mathcal{A}^1 \text{ ist } Tf = f',$$

b) $$n \in \mathbb{N} \text{ und } \mathcal{A}^n \subset \mathcal{D}(T) \Rightarrow \text{für } f \in \mathcal{A}^{n+1} \text{ ist } Tf = f',$$

c) $$\mathcal{A}^\infty \subset \mathcal{D}(T) \Rightarrow \text{für } f \in \mathcal{A}^\infty \text{ ist } Tf = f',$$

d) $$\text{für } f \in \mathcal{P} \text{ ist } Tf = f'.$$

3. Schlußbemerkung

Die Frage, ob es überhaupt Derivationen gibt, die auf ganz $\mathcal{C}$ definiert sind, konnte nicht beantwortet werden. Es hat sich hier jedoch gezeigt, daß sämtliche Derivationen von $\mathcal{A}^\infty$ Restriktionen der in Lemma 2.1 definierten Derivationen auf $\mathcal{A}^1$ sind und der Differentiationsoperator die einzige Derivation auf $\mathcal{A}^\infty$ ist, die zusätzlich noch die Normierungsbedingung (4), also insgesamt (D′) erfüllt.

Die Derivationen auf $\mathcal{A}^\infty$ hängen von einer Derivation $B\colon \mathcal{F}_k \mapsto \mathcal{F}$ und einer „Steigungsfunktion" α ab. B ist jedoch genau dann eine Derivation, wenn für jedes $t \in I$ die Funktion $\mathbb{R} \ni x \mapsto B(x\tau_0)(t) \in \mathbb{R}$ eine Derivation auf $\mathbb{R}$ ist; da es eine Fülle unstetiger Derivationen in $\mathbb{R}$ gibt, erkennt man, was für eine Vielzahl von Derivationen auf $\mathcal{A}^\infty$ existieren. Speziell die Derivationen auf dem Körper $\mathbb{R}$ werden relativ ausführlich in [1], Kapitel XIV, Seite 346–355 abgehandelt. Dort wird auch deutlich, daß sämtliche von Θ verschiedenen Derivationen auf $\mathbb{R}$ nicht meßbar, also ziemlich exotisch sind.

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SUBSTRUCTURES AND RADICALS OF MORITA CONTEXTS FOR NEAR-RINGS AND MORITA NEAR-RINGS

Stefan **VELDSMAN**

Department of Mathematics, University of Port Elizabeth, PO Box 1600, 6000 Port Elizabeth, South Africa

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Abstract: The relationships between certain substructures of a morita context for near-rings and the associated morita near-ring is determined. This is then used to determine the relationship between the radicals of the two near-rings in the morita context and the radical of the associated morita near-ring.

1. Introduction and preliminaries

Morita contexts have proved to be a useful tool in ring theory in determining the transfer of structural properties between two rings, especially as far as the radicals are concerned, see for example Amitsur [1] and Sands [6]. In [3] we have defined morita contexts for near-rings and in [4] we showed that two much studied cases from the theory of near-rings can be accommodated in this setting and how the tools provided by the morita context facilitates their investigation. The two cases referred to are, firstly, the transfer of structural properties between a (right) ring module and the associated near-ring of homogeneous maps on the group and secondly, that of a near-ring and the associated matrix near-ring. Here, in Section 3, we study the relationships between the radicals of the two near-rings L and R from a morita context for near-rings $\Gamma = (L, G, H, R)$ and the radical of the associated morita near-ring

$M_2(\Gamma)$. We give explicit conditions which ensures that the radical of $M_2(\Gamma)$ can be expressed in terms of the radicals L and R – initially having to first determine the relationship between the radicals of L and R. In doing this, we had to establish various relationships between some substructures of the morita context and the morita near-ring (Section 2). But firstly we have to recall some relevant definitions and earlier results.

All near-rings will be right distributive and 0-symmetric. Let R and L be near-rings and let G be a group. G is a *L-R-bimodule* if there are functions

$$L \times G \to G, (x, g) \longmapsto xy \quad \text{and} \quad G \times R \to G,\ (g, r) \longmapsto gr$$

such that $(x_1 + x_2)g = x_1g + x_2g$, $(g_1 + g_2)r = g_1r + g_2r$, $(x_1x_2)g = = x_1(x_2g)$, $(gr_1)r_2 = g(r_1r_2)$ and $(xg)r = x(gr)$ for all $x, x_1, x_2 \in L$, $g, g_1, g_2 \in G$, $r, r_1, r_2 \in R$. (Strictly speaking we should call G a near-ring L-R-bimodule, for even if both L and R are rings, G need not be a ring bimodule.) A normal subgroup K of G, G a L-R-bimodule, is an *ideal* of G if

$$KR := \{kr \mid k \in K, r \in R\} \subseteq K \quad \text{and}$$
$$L * K := \{x(g + k) - xg \mid x \in L, g \in G, k \in K\} \subseteq K.$$

Let $N_2 := \{1, 2\}$. For $i \in N_2$, we use i_c to denote the complement of i in N_2. For each $i, j \in N_2$, let Γ_{ij} be a group. The quadruple $\Gamma = (\Gamma_{11}, \Gamma_{12}, \Gamma_{21}, \Gamma_{22})$ is a *morita context* (for near-rings) if for every $i, j, k \in N_2$ there is a function

$$\Gamma_{jk} \times \Gamma_{ki} \to \Gamma_{ji},\ \ (x, y) \longmapsto xy$$

which satisfies $(a + b)c = ac + bc$ and $(db)e = d(be)$ for all $a, b \in \Gamma_{jk}$, $c \in \Gamma_{ki}$, $d \in \Gamma_{ij}$ and $e \in \Gamma_{km}$. It is clear that for each $i, j \in N_2$, Γ_{ij} is a Γ_{ii}-Γ_{jj}-bimodule and, in particular, for $i = j$, Γ_{ii} is a near-ring. As agreed earlier on, we only consider 0-symmetric near-rings. Hence we should add the requirement that $a0 = 0$ for each $a \in \Gamma_{ii}$, $i = 1, 2$. Then $x0_{jk} = 0_{ik}$ for all $x \in \Gamma_{ij}$, $i, j, k \in N_2$. We will usually not write the subscripts in 0_{ij}. It is clear that if $\Gamma = (\Gamma_{11}, \Gamma_{12}, \Gamma_{21}, \Gamma_{22})$ is a morita context, then so is $(\Gamma_{22}, \Gamma_{21}, \Gamma_{12}, \Gamma_{11})$, the one being called the dual of the other. Often, for a fixed $i \in N_2$, we will thus talk about the morita context $(\Gamma_{ii}, \Gamma_{ii_c}, \Gamma_{i_ci}, \Gamma_{i_ci_c})$. For each $i, j \in N_2$, let $\Delta_{ij} \subseteq \Gamma_{ij}$. The quadruple $\Delta = (\Delta_{11}, \Delta_{12}, \Delta_{21}, \Delta_{22})$ is an *ideal of the morita context* $\Gamma = (\Gamma_{11}, \Gamma_{12}, \Gamma_{21}, \Gamma_{22})$ if each Δ_{ij} is a normal subgroup of Γ_{ij}, $\Delta_{ij}\Gamma_{jk} \subseteq \Delta_{ik}$ and $\Gamma_{ki} * \Delta_{ij} := \{a(b+c)-ab \mid a \in \Gamma_{ki}, b \in \Gamma_{ij}, c \in \in \Delta_{ij}\} \subseteq \Delta_{kj}$ for all $i,j,k \in N_2$. In this case we get the quotient morita context

$$\frac{\Gamma}{\Delta} := \left[\frac{\Gamma_{11}}{\Delta_{11}}, \frac{\Gamma_{12}}{\Delta_{12}}, \frac{\Gamma_{21}}{\Delta_{21}}, \frac{\Gamma_{22}}{\Delta_{22}}\right]$$

where the relevant maps

$$\frac{\Gamma_{ij}}{\Delta_{ij}} \times \frac{\Gamma_{jk}}{\Delta_{jk}} \to \frac{\Gamma_{ik}}{\Delta_{ik}}$$

are defined by

$$(x + \Delta_{ij}, y + \Delta_{jk}) \longmapsto xy + \Delta_{ik}.$$

For the morita context $\Gamma = (\Gamma_{11}, \Gamma_{12}, \Gamma_{21}, \Gamma_{22})$, let $\Gamma^+ = \begin{bmatrix} \Gamma_{11} & \Gamma_{12} \\ \Gamma_{21} & \Gamma_{22} \end{bmatrix}$ be the associated matrix group. Let $\pi_{ij}\colon \Gamma^+ \to \Gamma_{ij}$ and $\tau_{ij}\colon \Gamma_{ij} \to \Gamma^+$ be the (i,j)-th projection and (i,j)-th injection respectively. For $i \in N_2$, let $\Gamma_{i1} \oplus \Gamma_{i2}$ be the direct sum of the two groups Γ_{i1} and Γ_{i2} and let $\pi_i\colon \Gamma^+ \to \Gamma_{i1} \oplus \Gamma_{i2}$ and $\tau_i\colon \Gamma_{i1} \oplus \Gamma_{i2} \to \Gamma^+$ be the obvious projection and injection respectively. Let $u_{ij}\colon \Gamma_{ij} \to \mathrm{Map}\,(\Gamma_{j1} \oplus \Gamma_{j2}, \Gamma_{i1} \oplus \Gamma_{i2})$ be defined by

$$u_{ij}(x) = u^x_{ij}\colon \Gamma_{ji} \oplus \Gamma_{j2} \to \Gamma_{i1} \oplus \Gamma_{i2}, \quad u^x_{ij}(a_1, a_2) := (xa_1, xa_2).$$

Finally, for each $i,j \in N_2$, $x \in \Gamma_{ij}$, let $s^x_{ij} := \tau_i \circ u^x_{ij} \circ \pi_j$.

The *morita near-ring determined by* Γ, denoted by $M_2(\Gamma)$, is the subnear-ring of

$$\mathrm{Map}\,(\Gamma^+, \Gamma^+) := \{f\colon \Gamma^+ \to \Gamma^+ \mid f \text{ a function with } f(0) = 0\}$$

generated by $\{s^x_{ij} \mid x \in \Gamma_{ij},\, i,j \in N_2\}$. $M_2(\Gamma)$ is a 0-symmetric near-ring which has an identity $s^1_{11} + s^1_{22}$ if both Γ_{11} and Γ_{22} have identities (here 1 denotes both the identity of Γ_{11} and Γ_{22}). A proof technique which is quite useful when dealing with elements of $M_2(\Gamma)$ is "induction on the weight $w(u)$ of $U \in M_2(\Gamma)$". The weight of $U \in M_2(\Gamma)$, written as $w(u)$, is the smallest number of s^x_{ij} needed to represent U. If Γ_d denotes the dual of the morita context Γ, then $M_2(\Gamma) \cong M_2(\Gamma_d)$. Some useful facilities for doing calculations in $M_2(\Gamma)$ are (cf. [3]):

1.1 $$s^x_{ij} + s^y_{ij} = s^{x+y}_{ij};$$

1.2 $$s^x_{ij} + s^y_{km} = s^y_{km} + s^x_{ij} \quad \text{if} \quad i \neq k;$$

1.3 $$s_{ij}^{x} s_{km}^{y} = \begin{cases} s_{im}^{xy} & \text{if } j = k \\ 0 & \text{if } j \neq k \end{cases};$$

1.4 $$s_{ij}^{x}(s_{1k_1}^{y_1} + s_{2k_2}^{y_2}) = s_{ik_j}^{xy_j};$$

1.5 for any $U \in M_2(\Gamma)$, $U \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = U \begin{bmatrix} a_{11} & 0 \\ a_{21} & 0 \end{bmatrix} + U \begin{bmatrix} 0 & a_{12} \\ 0 & a_{22} \end{bmatrix}$;

1.6
for any $U, V \in M_2(\Gamma)$, $U \begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix} + V \begin{bmatrix} 0 & c \\ 0 & d \end{bmatrix} = V \begin{bmatrix} 0 & c \\ 0 & d \end{bmatrix} + V \begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix}$;

1.7 for $k \in N_2$, $C_k := \{s_{1k}^{x_1} + s_{2k}^{x_2} \mid x_1 \in \Gamma_{1k}, x_2 \in \Gamma_{2k}\}$ is a left invariant subgroup of $M_2(\Gamma)$;

1.8 for $U \in M_2(\Gamma), U \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$ if and only if $U(s_{1i}^{a_{1i}} + s_{2i}^{a_{2i}}) = s_{1i}^{b_{1i}} + s_{2i}^{b_{2i}}$ for all $i \in N_2$.

All our consideration to follow, will be in what we call a standard morita context. A morita context $\Gamma = (\Gamma_{11}, \Gamma_{12}, \Gamma_{21}, \Gamma_{22})$ is a *standard morita context* if both Γ_{11} and Γ_{22} have identities, all bimodules in Γ are unital (both left and right) and $\overline{\Gamma_{jj_c}\Gamma_{j_cj}} = \Gamma_{jj}$ for all $j \in N_2$ where $\overline{\Gamma_{jj_c}\Gamma_{j_cj}}$ denotes the subgroup of Γ_{jj} generated by $\Gamma_{jj_c}\Gamma_{j_cj}$. Some useful consequences are:

1.9 For all $j, k \in N_2$ and $x \in \Gamma_{jk}$, $x \in (\overline{\Gamma_{jj_c}\Gamma_{j_cj}})x$;

1.10 $\Gamma_{jn} = \overline{\Gamma_{jk}\Gamma_{kn}}$ for all $j, k, n \in N_2$;

1.11 if $x \in \Gamma_{jk}$ and A is a subgroup of Γ_{jk}, then $\Gamma_{jj_c}\Gamma_{j_cj}x \subseteq A$ implies $x \in A$;

1.12 for $x \in \Gamma_{jk}$, if $\Gamma_{nj}x \subseteq \Delta_{nk}$, then $x \in \Delta_{nk}$ where $(\Delta_{11}, \Delta_{12}, \Delta_{21}, \Delta_{22})$ is an ideal of Γ.

Finally, it is clear that if Γ is a standard morita context, then so is its dual as well as $\frac{\Gamma}{\Delta}$ for any ideal Δ of Γ.

In all that follows, we assume that the morita contexts under discussion are all standard morita contexts.

2. Substructures of morita contexts and morita near-rings

Let $\Delta = (\Delta_{11}, \Delta_{12}, \Delta_{21}, \Delta_{22})$ be an ideal of the morita context $\Gamma = (\Gamma_{11}, \Gamma_{12}, \Gamma_{21}, \Gamma_{22})$. Then $\Delta^* := \{U \in M_2(\Gamma) \mid U\Gamma^+ \subseteq \Delta^+ +\}$ is an ideal of the near-ring $M_2(\Gamma)$ where Δ^+ is the matrix group $\Delta^+ = \begin{bmatrix} \Delta_{11} & \Delta_{12} \\ \Delta_{21} & \Delta_{22} \end{bmatrix}$. Let $\mathcal{T}$ be an ideal of the morita near-ring $M_2(\Gamma)$. For each $i, j \in N_2$, let $\mathcal{T}_{ij} = \{x \in \Gamma_{ij} \mid S^x_{ij} \in \mathcal{T}\}$. Then $\mathcal{T}_* := := (\mathcal{T}_{11}, \mathcal{T}_{12}, \mathcal{T}_{21}, \mathcal{T}_{22})$ is an ideal of the morita context Γ. If $U \in \mathcal{T}$, and $U \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$, then $s^{b_{ij}}_{ij} \in \mathcal{T}$ for all $i, j \in N_2$. It is clear that if $\mathcal{T}^1$ and $\mathcal{T}^2$ are ideals of $M_2(\Gamma)$ with $\mathcal{T}^1 \subseteq \mathcal{T}^2$, then $(\mathcal{T}^1)_* \subseteq (\mathcal{T}^2)_*$. Moreover, if $\mathcal{T}^\alpha$ is an ideal of $M_2(\Gamma)$ where each α is from some index set, then $(\bigcap_\alpha \mathcal{T}^\alpha)_* = \bigcap_\alpha ((\mathcal{T}^\alpha)_*)$. For any ideal Δ of Γ, $(\Delta^*)_* = \Delta$ and if $\mathcal{T}$ is an ideal of $M_2(\Gamma)$, then (in general) only $\mathcal{T} \subseteq (\mathcal{T}_*)^*$ holds. If $\mathcal{T} = (\mathcal{T}_*)^*$, then $\mathcal{T}$ is called a *full ideal* of $M_2(\Gamma)$. From the above and [3], we have

2.1 Proposition.

(1) *An ideal $\mathcal{T}$ of $M_2(\Gamma)$ is full if and only if it satisfies:*

$$UM_2(\Gamma)s^1_{kk} \subseteq \mathcal{T} \quad \textit{for all} \quad k \in N_2 \quad \textit{implies} \quad U \in \mathcal{T}.$$

(2) *There is a one-to-one correspondence, which preserves inclusions and intersections, between the ideals of the morita context Γ and all the full ideals of the associated morita near-ring $M_2(\Gamma)$ given by $\Delta \longmapsto \Delta \longmapsto \Delta^* \longmapsto (\Delta^*)_* = \Delta$.* $\Diamond$

Let Δ_{ij} be an ideal of the Γ_{ii}-Γ_{jj}-bimodule Γ_{ij}. For $k \in N_2$, let $\Delta_{ij}\Gamma^{-1}_{kj} := \{x \in \Gamma_{ik} \mid x\Gamma_{kj} \subseteq \Delta_{ij}\}$; it is an ideal of the Γ_{ii}-Γ_{kk}-bimodule Γ_{ik}. Let $\Gamma^{-1}_{ik}\Delta_{ij}$ be the ideal of the Γ_{kk}-Γ_{jj}-bimodule Γ_{kj} generated by $\Gamma_{ki} * \Delta_{ij}$. Part of the next result follows from [4]:

2.2 Proposition. *Let $i \in N_2$ be fixed. Let $\Gamma = (\Gamma_{ii}, \Gamma_{ii_c}, \Gamma_{i_ci}, \Gamma_{i_ci_c})$ be a* (as usual) *standard morita context. For each $j \in N_2$, let Δ_{jj} be an ideal of the near-ring Γ_{jj}. Let*

$$\Delta_i = (\Delta_{ii}, \Delta_{ii}\Gamma^{-1}_{i_ci}, \Gamma^{-1}_{ii_c}\Delta_{ii}, (\Gamma^{-1}_{ii_c}\Delta_{ii})\Gamma^{-1}_{i_ci})$$

and let

$$\Delta_{i_c} = ((\Gamma^{-1}_{i_c i}\Delta_{i_c i_c})\Gamma^{-1}_{i i_c}, \Gamma^{-1}_{i_c i}\Delta_{i_c i_c}, \Delta_{i_c i_c}\Gamma^{-1}_{i i_c}, \Delta_{i_c i_c}).$$

Then:

(1) *For every $j \in N_2$, Δ_j is an ideal of Γ if and only if $\Gamma_{jj_c} * \Gamma^{-1}_{jj_c}\Delta_{jj} \subseteq \subseteq \Delta_{jj}$. In case $\Gamma_{jj_c} * \Gamma^{-1}_{jj_c}\Delta_{jj} \subseteq \Delta_{jj}$, there is no need to insert brackets in $(\Gamma^{-1}_{jj_c}\Delta_{jj})\Gamma^{-1}_{j_c j}$, since $(\Gamma^{-1}_{jj_c}\Delta_{jj})\Gamma^{-1}_{j_c j} = \Gamma^{-1}_{jj_c}(\Delta_{jj}\Gamma^{-1}_{j_c j})$.*

(2) *If $\Delta_i = \Delta_{i_c}$, then Δ_i is an ideal of Γ.*

Proof. (1) follows from [4].

(2): From (1) above, we need $\Gamma_{ii_c} * \Gamma^{-1}_{ii_c}\Delta_{ii} \subseteq \Delta_{ii}$. But our assumption reduces our need to $\Gamma_{ii_c} * \Delta_{i_c i_c}\Gamma^{-1}_{ii_c} \subseteq (\Gamma^{-1}_{i_c i}\Delta_{i_c i_c})\Gamma^{-1}_{ii_c}$ i.e., we need

$$(\Gamma_{ii_c} * \Delta_{i_c i_c}\Gamma^{-1}_{ii_c})\Gamma_{ii_c} \subseteq \Gamma^{-1}_{i_c i}\Delta_{i_c i_c}$$

where the latter is the ideal of the Γ_{ii}-$\Gamma_{i_c i_c}$-bimodule generated by $\Gamma_{ii_c} * \Delta_{i_c i_c}$. Now $(\Gamma_{ii_c} * \Delta_{i_c i_c}\Gamma^{-1}_{ii_c})\Gamma_{ii_c} \subseteq \Gamma_{ii_c} * ((\Delta_{i_c i_c}\Gamma^{-1}_{ii_c})\Gamma_{ii_c})\Gamma_{ii_c}) \subseteq \subseteq \Gamma_{ii_c} * \Delta_{i_c i_c} \subseteq \Gamma^{-1}_{i_c i}\Delta_{i_c i_c}$. ◊

For any $j \in N_2$, let $\mathcal{S}^*(\Gamma_{jj}) := \{\Delta_{jj} \subseteq \Gamma_{jj} \mid \Delta_{jj}$ is an ideal of Γ_{jj} for which $\Gamma_{jj_c} * \Gamma^{-1}_{jj_c}\Delta_{jj} \subseteq \Delta_{jj}$ and for $x \in \Gamma_{jj}$, $x\Gamma_{jj_c}\Gamma_{j_c j} \subseteq \Delta_{jj}$ implies $x \in \Delta_{jj}\}$. In [4] it was shown that this class of ideals is closed under intersections and

2.3 Proposition [4]. *There is a one-to-one correspondence, which preserves inclusions and intersections, between $\mathcal{S}^*(\Gamma_{ii})$ and $\mathcal{S}^*(\Gamma_{i_c i_c})$ given by*

$$\Delta_{ii} \longmapsto \Gamma^{-1}_{ii_c}\Delta_{ii}\Gamma^{-1}_{i_c i} \longmapsto \Gamma^{-1}_{i_c i}(\Gamma^{-1}_{ii_c}\Delta_{ii}\Gamma^{-1}_{i_c i})\Gamma^{-1}_{ii_c} = \Delta_{ii}.$$

Recall, an ideal I of a near-ring N is a *2-semiprime ideal* if for any left invariant subgroup A of N, $A^2 \subseteq I$ implies $A \subseteq I$. I is a *3-semiprime ideal* if $xNx \subseteq I$ implies $x \in I$. The near-ring N is 2-semiprime (resp. 3-semiprime) if 0 is a 2-semiprime (resp. 3-semiprime) ideal of N. Any intersection of 2-semiprime (3-semiprime) ideals is 2-semiprime (3-semiprime). It is clear that any 3-semiprime near-ring is 2-semiprime; our interest here in 2-semiprime near-rings is mainly because of the following which is easy to verify:

2.4 Proposition. *Let N be a near-ring with identity. An ideal I of N is 3-semiprime if and only if it is 2-semiprime.* ◊

2.5 Proposition. *Any 3-semiprime (= 2-semiprime) ideal $\mathcal{T}$ of $M_2(\Gamma)$ is full.*

Proof. We use 2.1(1) above. Let $U \in M_2(\Gamma)$ and suppose $UM_2(\Gamma)s^1_{kk} \subseteq \mathcal{T}$ for all $k \in N_2$. Then $(s^1_{kk}U)M_2(\Gamma)(s^1_{kk}U) \subseteq \mathcal{T}$ and hence $s^1_{kk}U \in \mathcal{T}$ for all $k \in N_2$. Then $U = (s^1_{11} + s^1_{22})U = s^1_{11}U + s^1_{22}U \in \mathcal{T}$. ◊

Let $\mathcal{S}_p^*(\Gamma_{jj}) := \{\Delta_{jj} \subseteq \Gamma_{jj} \mid \Delta_{jj}$ is a 3-semiprime ideal of Γ_{jj} which satisfies $\Gamma_{jj_c} * \Gamma_{jj_c}^{-1}\Delta_{jj} \subseteq \Delta_{jj}\}$. From [4] we need two more results:

2.6 Proposition [4]. *Let Δ_{jj} be a 3-semiprime ideal of Γ_{jj}. Then*

(1) $$x\Gamma_{jj_c}\Gamma_{j_cj} \subseteq \Delta_{jj} \text{ implies } x \in \Delta_{jj};$$

(2) $$(\Gamma_{jj_c}^{-1}\Delta_{jj})\Gamma_{j_cj}^{-1} \text{ is a 3-semiprime ideal of } \Gamma_{j_cj_c}.$$

2.7 Proposition [4]. *There is a one-to-one correspondence, which preserves inclusions and intersections between $\mathcal{S}_p^*(\Gamma_{ii})$ and $\mathcal{S}_p^*(\Gamma_{i_ci_c})$* (given by the same map as in 2.3 above).

2.8 Proposition. *Let $i \in N_2$ be fixed. Let $\Delta = (\Delta_{ii}, \Delta_{ii_c}, \Delta_{i_ci}, \Delta_{i_ci_c})$ be an ideal of $\Gamma = (\Gamma_{ii}, \Gamma_{ii_c}, \Gamma_{i_ci}, \Gamma_{i_ci_c})$. Then*

(1) $$\Delta_{ii_c} \subseteq \Delta_{ii}\Gamma_{i_ci}^{-1} = \{x \in \Gamma_{ii_c} \mid x\Gamma_{i_ci} \subseteq \Delta_{ii}\};$$

(2) $$\Delta_{i_ci} = \Gamma_{ii_c}^{-1}\Delta_{ii} = \{x \in \Gamma_{i_ci} \mid \Gamma_{ii_c}x \subseteq \Delta_{ii}\} \text{ and } \Gamma_{ii_c} * \Gamma_{ii_c}^{-1}\Delta_{ii} \subseteq \Delta_{ii};$$

(3) $$\Delta_{i_ci_c} \subseteq \Gamma_{ii_c}^{-1}\Delta_{ii}\Gamma_{i_ci}^{-1} = \{x \in \Gamma_{i_ci_c} \mid \Gamma_{ii_c}x\Gamma_{i_ci} \subseteq \Delta_{ii}\};$$

(4) $$(\Delta_{ii}, \Delta_{ii_c}, \Delta_{i_ci}, \Delta_{i_ci_c}) = (\Delta_{ii}, \Delta_{ii}\Gamma_{i_ci}^{-1}, \Gamma_{ii_c}^{-1}\Delta_{ii}, \Gamma_{ii_c}^{-1}\Delta_{ii}\Gamma_{i_ci}^{-1})$$ *if and only if for $x \in \Gamma_{ii_c}$, $x\Gamma_{i_ci} \subseteq \Delta_{ii}$ implies $x \in \Delta_{ii_c}$.*

Proof. (1): Let $x \in \Delta_{ii_c}$. Then $x\Gamma_{i_ci} \subseteq \Delta_{ii_c}\Gamma_{i_ci} \subseteq \Delta_{ii}$; hence $x \in \Delta_{ii}\Gamma_{i_ci}^{-1} = \{x \in \Gamma_{ii_c} \mid x\Gamma_{i_ci} \subseteq \Delta_{ii}\}$ (by definition).

(2): Let $x \in \Delta_{i_ci}$. Then $\Gamma_{i_ci}\Gamma_{ii_c}x \subseteq \Gamma_{i_ci}(\Gamma_{ii_c}\Delta_{i_ci}) \subseteq \Gamma_{i_ci}\Delta_{ii} \subseteq \Gamma_{i_ci} * \Delta_{ii} \subseteq \Gamma_{i_ci}^{-1}\Delta_{ii}$ by the definition of the latter. By 1.11 we have $x \in \Gamma_{ii_c}^{-1}\Delta_{ii}$. Since $\Gamma_{i_ci} * \Delta_{ii} \subseteq \Delta_{i_ci}$ and Δ_{i_ci} is an ideal of the $\Gamma_{i_ci_c}$-Γ_{ii}-bimodule Γ_{i_ci}, we get $\Gamma_{i_ci}^{-1}\Delta_{ii} \subseteq \Delta_{i_ci}$. Hence $\Delta_{i_ci} = \Gamma_{ii_c}^{-1}\Delta_{ii}$ and so $\Gamma_{ii_c} * \Gamma_{ii_c}^{-1}\Delta_{ii} = \Gamma_{ii_c} * \Delta_{i_ci} \subseteq \Delta_{ii}$. For the second equality, let $x \in \Gamma_{i_ci}$ such that $\Gamma_{ii_c}x \subseteq \Delta_{ii}$. By 1.12, $x \in \Delta_{i_ci} = \Gamma_{ii_c}^{-1}\Delta_{ii}$ follows. Conversely, if $x \in \Delta_{i_ci}$, then $\Gamma_{ii_c}x \subseteq \Gamma_{ii_c}\Delta_{i_ci} \subseteq \Delta_{ii}$.

(3): Let $x \in \Delta_{i_ci_c}$. Then $x\Gamma_{i_ci} \subseteq \Delta_{i_ci_c}\Gamma_{i_ci} \subseteq \Delta_{i_ci}$; hence $x \in \Delta_{i_ci}\Gamma_{i_ci}^{-1} = \Gamma_{ii_c}^{-1}\Delta_{ii}\Gamma_{i_ci}^{-1}$ from (2) above. The equality $\Gamma_{ii_c}^{-1}\Delta_{ii}\Gamma_{i_ci}^{-1} = \{x \in \Gamma_{i_ci_c} \mid \Gamma_{ii_c}x\Gamma_{i_ci} \subseteq \Delta_{ii}\}$ is obvious from (1) and (2) above.

(4): If the equality holds, then clearly $x \in \Gamma_{ii_c}$ with $x\Gamma_{i_ci} \subseteq \Delta_{ii}$ implies $x \in \Delta_{ii}\Gamma_{i_ci}^{-1} = \Delta_{ii_c}$. Conversely, let $x \in \Delta_{ii}\Gamma_{i_ci}^{-1}$. Then $x\Gamma_{i_ci} \subseteq \Delta_{ii}$ and by the assumption, $x \in \Delta_{ii_c}$. Thus $\Delta_{ii_c} = \Delta_{ii}\Gamma_{i_ci}^{-1}$. Let

$y \in \Gamma_{ii_c}^{-1}\Delta_{ii}\Gamma_{i_ci}^{-1} = \{a \in \Gamma_{i_ci_c} \mid \Gamma_{ii_c}a\Gamma_{i_ci} \subseteq \Delta_{ii}\}$. Then $\Gamma_{ii_c}y\Gamma_{i_ci} \subseteq \Delta_{ii}$ and by the assumption, $\Gamma_{ii_c}y \subseteq \Delta_{ii_c}$. By 1.12 $y \in \Delta_{i_ci_c}$ follows. $\Diamond$

2.9 Proposition. *Let $\mathcal{T}$ be a 3-semiprime ideal of $M_2(\Gamma)$. For $i, j, k \in N_2$ and $x \in \Gamma_{ij}$:*

(1) $$x\Gamma_{ji}x \subseteq \mathcal{T}_{ij} \text{ implies } x \in \mathcal{T}_{ij};$$
in particular, $\mathcal{T}_{ii}$ is a 3-semiprime ideal of Γ_{ii}.

(2) $$x\Gamma_{jk}\Gamma_{kj} \subseteq \mathcal{T}_{ij} \text{ implies } x \in \mathcal{T}_{ij}.$$

Proof. (1): For any $U \in M_2(\Gamma)$, there is some $a_k \in \Gamma_{ki}$ $(k = i, i_c)$ by 1.7, such that $s_{ij}^x U s_{ij}^x = s_{ij}^x U s_{ii}^1 s_{ij}^x = s_{ij}^x U(s_{ii}^1 + s_{i_ci_c}^0)s_{ij}^x = s_{ij}^x(s_{ii}^{a_i} + s_{i_ci_c}^{a_{i_c}})s_{ij}^x = s_{ii}^{xa_jx} \in \mathcal{T}$ since $xa_jx \in x\Gamma_{ji}x \subseteq \mathcal{T}_{ij}$. Since $\mathcal{T}$ is 3-semiprime, $s_{ij}^x \in \mathcal{T}$ and hence $x \in \mathcal{T}_{ij}$ follows.

(2): Suppose $x\Gamma_{jk}\Gamma_{kj} \subseteq \mathcal{T}_{ij}$ but $x \notin \mathcal{T}_{ij}$. By (1) above, there is a $y \in \Gamma_{ji} = \overline{\Gamma_{jk}\Gamma_{ki}}$ (cf. 1.10) such that $xyx \notin \mathcal{T}_{ij}$. Assume $y = \sum_{r=1}^{n} \sigma_r a_r b_r$ where $\sigma_r \in \{+, -\}$, $a_r \in \Gamma_{jk}$ and $b_r \in \Gamma_{ki}$. Then $a_{r_0}b_{r_0}x \notin \mathcal{T}_{jj}$ for some $r_0 \in \{1, 2, \ldots, n\}$. Once again, by (1) above (with $i = j$), there is a $u \in \Gamma_{jj}$ such that $a_{r_0}b_{r_0}x\,u\,a_{r_0}b_{r_0}x \notin \mathcal{T}_{jj}$. But $a_{r_0}b_{r_0}x\,u\,a_{r_0}b_{r_0}x = (a_{r_0}b_{r_0})x(ua_{r_0})(b_{r_0}x) \in \Gamma_{ji}x\Gamma_{jk}\Gamma_{kj} \subseteq \Gamma_{ji}\mathcal{T}_{ij} \subseteq \mathcal{T}_{jj}$; a contradiction. $\Diamond$

2.10 Proposition. *Let $i \in N_2$ be fixed and let $\Delta = (\Delta_{ii}, \Delta_{ii_c}, \Delta_{i_ci}, \Delta_{i_ci_c})$ be an ideal of $\Gamma = (\Gamma_{ii}, \Gamma_{ii_c}, \Gamma_{i_ci}, \Gamma_{i_ci_c})$. Then Δ^* is a 3-semiprime ideal of $M_2(\Gamma)$ if and only if Δ satisfies:*

(1) *Δ_{ii} is a 3-semiprime ideal of Γ_{ii} and*

(2) $$x\Gamma_{i_ci} \subseteq \Delta_{ii} (x \in \Gamma_{ii_c}) \text{ implies } x \in \Delta_{ii_c}.$$

Proof. If $\mathcal{T} := \Delta^*$ is a 3-semiprime ideal of $M_2(\Gamma)$, then $\Delta_{ii} = \mathcal{T}_{ii}$ a 3-semiprime ideal of Γ_{ii} follows from Prop. 2.10(1). If $x \in \Gamma_{ii_c}$ such that $x\Gamma_{i_ci} \subseteq \Delta_{ii}$, then $x\Gamma_{i_ci}\Gamma_{ii_c} \subseteq \Delta_{ii}\Gamma_{ii_c} \subseteq \Delta_{ii_c} = \mathcal{T}_{ii_c}$ and by Prop. 2.10(2) we have $x \in \mathcal{T}_{ii_c} = \Delta_{ii_c}$.

Conversely, suppose (1) and (2) are satisfied. Let $U \in M_2(\Gamma)$ such that $UM_2(\Gamma)U \subseteq \Delta^*$. Suppose $U \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$. By 1.8 $U(s_{1k}^{a_{1k}} + s_{2k}^{a_{2k}}) = s_{tk}^{b_{1k}} + s_{2k}^{b_{2k}}$ for $k \in N_2$ and

$V := s_{kj}^y U(s_{1k}^{a_{1k}} + s_{2k}^{a_{2k}})s_{kj}^x U(s_{1k}^{a_{1k}} + s_{2k}^{a_{2k}}) \in \Delta^*$ for all $k, j \in N_2$, $x, y \in \Gamma_{kj}$.

But $V = s_{kj}^y(s_{1k}^{b_{1k}} + s_{2k}^{b_{2k}})s_{kj}^x(s_{1k}^{b_{1k}} + s_{2k}^{b_{2k}}) = s_{kk}^{yb_{jk}} s_{kk}^{xb_{jk}} = s_{kk}^{yb_{jk}xb_{jk}} \in \Delta^*$

and so $yb_{jk}xb_{jk} \in \Delta_{kk}$. Thus $\Gamma_{kj}b_{jk}\Gamma_{kj}b_{jk} \subseteq \Delta_{kk}$ for all $j,k \in N_2$. By (1), Δ_{ii} is a 3-semiprime ideal of Γ_{ii} and by Props. 2.6 and 2.10, also $\Delta_{i_ci_c} = \Gamma_{ii_c}^{-1}\Delta_{ii}\Gamma_{i_ci}^{-1}$ is a 3-semiprime ideal of $\Gamma_{i_ci_c}$. This means, for any $k \in N_2$, $(\Gamma_{kj}b_{jk})\Gamma_{kk}(\Gamma_{kj}b_{jk}) \subseteq \Delta_{kk}$ and since Δ_{kk} is a 3-semiprime ideal of $\Gamma_{kk}, \Gamma_{kj}b_{jk} \subseteq \Delta_{kk}$. By 1.12 we have $b_{jk} \in \Delta_{jk}$ for all $j,k \in N_2$. Hence $U\Gamma^+ \subseteq \Delta^+$; so $U \in \Delta^*$. $\lozenge$

2.11 Corollary. *Let $i \in N_2$ be fixed and let $\Gamma = (\Gamma_{ii}, \Gamma_{ii_c}, \Gamma_{i_ci}, \Gamma_{i_ci_c})$ be a morita context. Let $\mathcal{T}$ be a 3-semiprime ideal of $M_2(\Gamma)$. Then:*

$$\begin{aligned} \mathcal{T}_{ii_c} &= \mathcal{T}_{ii}\Gamma_{i_ci}^{-1} = \{x \in \Gamma_{ii_c} \mid x\Gamma_{i_ci} \subseteq \mathcal{T}_{ii}\}, \\ \mathcal{T}_{i_ci} &= \Gamma_{ii_c}^{-1}\mathcal{T}_{ii} = \{x \in \Gamma_{i_ci} \mid \Gamma_{ii_c}x \subseteq \mathcal{T}_{ii}\} \text{ and} \\ \mathcal{T}_{i_ci_c} &= \Gamma_{ii_c}^{-1}\mathcal{T}_{ii}\Gamma_{i_ci}^{-1} = \{x \in \Gamma_{i_ci_c} \mid \Gamma_{ii_c}x\Gamma_{i_ci} \subseteq \mathcal{T}_{ii}\}; \end{aligned} \tag{1}$$

$$\begin{aligned} \mathcal{T}_* &= (\mathcal{T}_{ii}, \mathcal{T}_{ii_c}, \mathcal{T}_{i_ci}, \mathcal{T}_{i_ci_c}) = (\mathcal{T}_{ii}, \mathcal{T}_{ii}\Gamma_{i_ci}^{-1}, \Gamma_{ii_c}^{-1}\mathcal{T}_{ii}, \Gamma_{ii_c}^{-1}\mathcal{T}_{ii}\Gamma_{i_ci}^{-1}) = \\ &= (\Gamma_{i_ci}^{-1}\mathcal{T}_{i_ci_c}\Gamma_{ii_c}^{-1}, \Gamma_{i_ci}^{-1}\mathcal{T}_{i_ci_c}, \mathcal{T}_{i_ci_c}\Gamma_{ii_c}^{-1}, \mathcal{T}_{i_ci_c}). \end{aligned} \tag{2}$$

Proof. (1): Let $\Delta := \mathcal{T}_* = (\mathcal{T}_{ii}, \mathcal{T}_{ii_c}, \mathcal{T}_{i_ci}, \mathcal{T}_{i_ci_c})$. By Prop. 2.5, $\Delta^* = (\mathcal{T}_*)^* = \mathcal{T}$. The result then follows from Props. 2.10 and 2.8. (2) follows by using (1) above twice; once for i and then for i_c. $\lozenge$

2.12 Corollary. *Let $i \in N_2$ be fixed. Let Δ_{ii} be an ideal of Γ_{ii} such that $\Gamma_{ii_c} * \Gamma_{ii_c}^{-1}\Delta_{ii} \subseteq \Delta_{ii}$. Let Δ be the ideal $\Delta = (\Delta_{ii}, \Delta_{ii}\Gamma_{i_ci}^{-1}, \Gamma_{ii_c}^{-1}\Delta_{ii}, \Gamma_{ii_c}^{-1}\Delta_{ii}\Gamma_{i_ci}^{-1})$ of the morita context Γ (*cf. Prop. 2.2*). Then Δ^* is a 3-semiprime ideal of $M_2(\Gamma)$ if and only if Δ_{ii} is a 3-semiprime ideal of Γ_{ii}. If any one of these two conditions holds, then $\Delta_{nj}\Gamma_{kj}^{-1} = \Delta_{nk} = \Gamma_{jn}^{-1}\Delta_{jk}$ for all $j,k,n \in N_2$.*

Proof. The sufficiency is clear from Prop. 2.10. Conversely, since $x\Gamma_{i_ci} \subseteq \Delta_{ii}$ implies $x \in \Delta_{ii}\Gamma_{i_ci}^{-1}$, once again Prop. 2.10 yields the result. For $x \in \Delta_{nj}\Gamma_{kj}^{-1}$, $x\Gamma_{kj} \subseteq \Delta_{nj}$ and so $x\Gamma_{kj}\Gamma_{jk} \subseteq \Delta_{nj}\Gamma_{jk} \subseteq \Delta_{nk}$. By the assumption and Prop. 2.9 we get $x \in \Delta_{nk}$. Conversely, $x \in \Delta_{nk}$ implies $x\Gamma_{kj} \subseteq \Delta_{nj}$ and so $x \in \Delta_{nj}\Gamma_{kj}^{-1}$. Since $\Gamma_{nj} * \Delta_{jk} \subseteq \Delta_{nk}$ and Δ_{nk} is an ideal of the Γ_{nn}-Γ_{kk}-bimodule Γ_{nk}, we have $\Gamma_{jn}^{-1}\Delta_{jk} \subseteq \Delta_{nk}$. Conversely, for $x \in \Delta_{nk}$, we have $\Gamma_{nj}\Gamma_{jn}x \subseteq \Gamma_{nj}\Delta_{jk} \subseteq \Gamma_{nj} * \Delta_{jk} \subseteq \Gamma_{jn}^{-1}\Delta_{jk}$. For $j = n$, $x \in \Gamma_{nn}x = \Gamma_{nj}\Gamma_{jn}x \subseteq \Gamma_{jn}^{-1}\Delta_{jk}$, and for $j = n_c$, by 1.11 we get $x \in \Gamma_{jn}^{-1}\Delta_{jk}$. Thus $\Gamma_{jn}^{-1}\Delta_{jk} = \Delta_{nk}$. $\lozenge$

2.13 Proposition. *Let $\Delta = (\Delta_{11}, \Delta_{12}, \Delta_{21}, \Delta_{22})$ be an ideal of the morita context $\Gamma = (\Gamma_{11}, \Gamma_{12}, \Gamma_{21}, \Gamma_{22})$. Then*

$$M_2\left(\frac{\Gamma}{\Delta}\right) \cong \frac{M_2(\Gamma)}{\Delta^*}.$$

Before proceeding with the proof, we need:

2.14 Lemma. *For each $U \in M_2(\Gamma)$, there is a $U_q \in M_2(\frac{\Gamma}{\Delta})$ with the property:*

$$\text{If } U \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix},$$

$$\text{then } U_q \begin{bmatrix} a_{11}+\Delta_{11} & a_{12}+\Delta_{12} \\ a_{21}+\Delta_{21} & a_{22}+\Delta_{22} \end{bmatrix} = \begin{bmatrix} b_{11}+\Delta_{11} & b_{12}+\Delta_{12} \\ b_{21}+\Delta_{21} & b_{22}+\Delta_{22} \end{bmatrix}.$$

Proof (by induction on $w(U)$). If $U \in M_2(\Gamma)$ with $w(U) = 1$, then $U = s_{kn}^x$ for some $k, n \in N_2$, $x \in \Gamma_{kn}$. Let $U_q := s_{kn}^y$ where $y = x + + \Delta_{kn}$. Then $U_q \in M_2(\frac{\Gamma}{\Delta})$ and it has the required property. Suppose for all $V \in M_2(\Gamma)$ with $w(V) < m$, $m \geq 2$, such a $V_q \in M_2(\frac{\Gamma}{\Delta})$ has been found. Let $U \in M_2(\Gamma)$ with $w(U) = m$. Then $U = U_1 + U_2$ or $U = U_1U_2$ where $U_1, U_2 \in M_2(\Gamma)$ with $w(U_1) < m$ and $w(U_2) < m$. If $U = U_1 + U_2$, let $U_q = (U_1)_q + (U_2)_q$ and if $U = U_1U_2$, let $U_q = = (U_1)_q(U_2)_q$. It follows readily that for both possibilities, U_q has the desired property. We also remark that, even if $U_1 + U_2 = U = U_1' + U_2'$ or $U_1+U_2 = U = U_1'U_2'$ respectively, where $w(U_1') < m$ and $w(U_2') < m$, then U_q is well-defined since for each case, $(U_1)_q+(U_2)_q = (U_1')_q+(U_2')_q$ or $(U_1)_q + (U_2)_q = (U_1')_q(U_2')_q$ respectively. ◊

Proof (of Prop. 2.13). For each $U \in M_2(\Gamma)$, the $U_q \in M_2(\frac{\Gamma}{\Delta})$ given by the Lemma is obviously uniquely determined by U; hence we have a well-defined function

$$\varphi\colon M_2(\Gamma) \to M_2\left(\frac{\Gamma}{\Delta}\right), \quad \text{given by} \quad \varphi(U) = U_q.$$

Since $(U_1+U_2)_q = (U_1)_q+(U_2)_q$ and $(U_1U_2)_q = (U_1)_q(U_2)_q$, it is a near-ring homomorphism. Let us abbreviate the elements of Γ^+ and $\left(\frac{\Gamma}{\Delta}\right)^+$ by $[a_{ij}]$ and $[a_{ij} + \Delta_{ij}]$ respectively $\Big($meaning of course, for example for $[a_{ij}], [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\Big)$. Note that $U_q[a_{ij} + \Delta_{ij}] = [b_{ij} + \Delta_{ij}]$ if and only if $U[a_{ij}] = [b_{ij} + d_{ij}]$ for some $d_{kn} \in \Delta_{kn}$, $k, n = 1, 2$. Hence we get $\ker\varphi = \{U \in M_2(\Gamma) | U_q = 0\} = \{U \in M_2(\Gamma) | U\Gamma^+ \subseteq \Delta^+\} = = \Delta^*$. Finally we show φ is surjective from which $\frac{M_2(\Gamma)}{\Delta^*} \cong M_2(\frac{\Gamma}{\Delta})$ will follow. Let $V \in M_2(\frac{\Gamma}{\Delta})$ and let U be an element of $M_2(\Gamma)$ which is obtained from V by replacing each $s_{ij}^{x+\Delta_{ij}}$ present in V by s_{ij}^{x}. Of course there may be many different such U's (x in s_{ij}^{x} can be replaced by other representative from $x+\Delta_{ij}$); for our purposes any one such U will do. A straightforward induction on $w(V)$ will show that $V = U_q$; hence φ is surjective. ◊

Let $i \in N_2$ be fixed. For $k = 1, 2$, let Δ_{ik} be a subgroup of Γ_{ik}. Then $\tau_i(\Delta_{i1}, \Delta_{i2}) := \{\tau_i(a, b) \mid a \in \Delta_{i1}, b \in \Delta_{i2}\}$ is a subgroup of Γ^+ (which is normal if each Δ_{ik} is normal in Γ_{ik}). Let $\mathcal{R}_i(\Delta_{i1}, \Delta_{i2}) := = \{U \in M_2(\Gamma) \mid U\Gamma^+ \subseteq \tau_i(\Delta_{i1}, \Delta_{i2})\}$.

2.15 Proposition. *$\mathcal{R}_i(\Delta_{i1}, \Delta_{i2})$ is a right invariant subgroup of $M_2(\Gamma)$. It is a right ideal of $M_2(\Gamma)$ if Δ_{ik} is normal in Γ_{ik} for $k = 1, 2$. If Δ_{ii} is a subnear-ring of Γ_{ii}, then*

$$\varphi_i\colon \mathcal{R}_i(\Delta_{i1}, \Delta_{i2}) \to \Delta_{ii}, \ \textit{defined by}\ \ \varphi_i(U) := \pi_{ii}(U(\pi_{ii}(1)))$$

is a near-ring homomorphism. It is surjective if $\Delta_{ii}\Gamma_{ij} \subseteq \Delta_{ij}$ for all $j = 1, 2$. Moreover, $\ker\varphi_i \subseteq \{U \in \mathcal{R}_i(\Delta_{i1}, \Delta_{i2}) \mid U\mathcal{R}_i(\Delta_{i1}, \Delta_{i2}) = 0\}$; in particular, $(\ker\varphi_i)^2 = 0$.

Proof. It is straightforward to see that $\mathcal{R}_i(\Delta_{i1}, \Delta_{i2})$ is a right invariant subgroup, which is a right ideal of $M_2(\Gamma)$ if the Δ_{ik}'s are normal. We show that φ_i is well defined, i.e. $\varphi_i(U) \in \Delta_{ii}$. Firstly note that for any $U \in \mathcal{R}_i(\Delta_{i1}, \Delta_{i2})$, $Us_{ii}^{1} = s_{ii}^{u}$ for some unique $u \in \Delta_{ii}$. Thus

$$\varphi_i(U) = \pi_{ii}(U(\tau_{ii}(1))) = \pi_{ii}(Us_{ii}^{1}(\tau_{ii}(1))) = \pi_{ii}(s_{ii}^{u}(\tau_{ii}(1))) = u \in \Delta_{ii}.$$

Let $U_1, U_2 \in \mathcal{R}_i(\Delta_{i1}, \Delta_{i2})$ with $U_1 s_{ii}^{1} = s_{ii}^{u_1}$ and $U_2 s_{ii}^{1} = s_{ii}^{u_2}$. Then $(U_1 + U_2)s_{ii}^{1} = s_{ii}^{u_1+u_2}$ and so $\varphi_i(U_1 + U_2) = u_1 + u_2 = \varphi(U_1) + \varphi(U_2)$. Furthermore, $\varphi_i(U_1U_2) = \pi_{ii}(U_1(U_2 s_{ii}^{1}\tau_{ii}(1))) = \pi_{ii}(U_1 s_{ii}^{1} s_{ii}^{u_2}\tau_{ii}(1)) = = \pi_{ii}(s_{ii}^{u_1} s_{ii}^{u_2}\tau_{ii}(1)) = \pi_{ii}(s_{ii}^{u_1u_2}\tau_{ii}(1)) = u_1u_2 = \varphi(U_1)\varphi(U_2)$. Let $K \in \in \ker\varphi_i$ and $U \in \mathcal{R}_i(\Delta_{i1}, \Delta_{i2})$. For any $a_{jk} \in \Gamma_{jk}$, $j, k \in N_2$, and for some $b_{i1} \in \Delta_{ii}$ and $b_{i2} \in \Delta_{i2}$, $KU \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = K(\tau_i(b_{i1}, b_{i2})) = = Ks_{ii}^{1}\tau_i(b_{i1}, b_{i2}) = s_{ii}^{k}\tau_i(b_{i1}, b_{i2}) = 0$ since $K \in \ker\varphi_i$ implies $k = = \varphi_i(K) = 0$ where $Ks_{ii}^{1} = s_{ii}^{k}$. Thus $(\ker\varphi_i)\cdot \mathcal{R}_i(\Delta_{i1}, \Delta_{i2}) = 0$. Finally, suppose $\Delta_{ii}\Gamma_{ij} \subseteq \Delta_{ij}$ for $j = 1, 2$. Let $d \in \Delta_{ii}$. Then $s_{ii}^{d} \in \in \mathcal{R}_i(\Delta_{i1}, \Delta_{i2})$ and $\varphi_i(s_{ii}^{d}) = d$. Thus φ_i is surjective. ◊

Once again, let $i \in N_2$ be fixed and for each $k = 1, 2$, let Δ_{ki} be a subgroup of Γ_{ki}. Let $\mathcal{C}_i(\Delta_{1i}, \Delta_{2i}) := \{s_{1i}^{x_1} + s_{2i}^{x_2} \mid x_k \in \Delta_{ki},\ k = 1, 2\}$. Note that if Δ_{ki} for $k = 1, 2$, then $\mathcal{C}_i(\Delta_{1i}, \Delta_{2i}) = \mathcal{C}_i(\Gamma_{1i}, \Gamma_{2i}) = C_i$ (cf. 1.7).

2.16 Proposition. *Suppose $\Gamma_{jk}\Delta_{ki} \subseteq \Delta_{ji}$ for all $k, j \in N_2$. Then*

$\mathcal{C}_i(\Delta_{1i},\Delta_{2i})$ is a left invariant subgroup of $M_2(\Gamma)$ and if $\gamma_i:\mathcal{C}_i(\Delta_{1i},\Delta_{2i})\to\Delta_{ii}$ is defined by $\gamma_i(s_{1i}^{x_1}+s_{2i}^{x_2}):=x_i$, then γ_i is a surjective near-ring homomorphism with $(\ker\gamma_i)^2=0$.

Proof. $\mathcal{C}_i(\Delta_{1i},\Delta_{2i})$ is clearly a subgroup of $M_2(\Gamma)$. Let $U\in M_2(\Gamma)$ and let $s_{1i}^{x_1}+s_{2i}^{x_2}\in\mathcal{C}_i(\Delta_{1i},\Delta_{2i})$. We show that $U(s_{1i}^{x_1}+s_{2i}^{x_2})\in\mathcal{C}_i(\Delta_{1i},\Delta_{2i})$ by induction on $w(U)$. If $w(U)=1$, then $U=s_{kj}^{y}$ for some $k,j\in N_2$, $y\in\Gamma_{kj}$. Then

$$s_{kj}^{y}(s_{1i}^{x_1}+s_{2i}^{x_2})=s_{ki}^{yx_j}=s_{ki}^{yx_j}+s_{k_ci}^{0}\in\mathcal{C}_i(\Delta_{1i},\Delta_{2i})$$

by 1.2 and since $yx_j\in\Gamma_{kj}\Delta_{ji}\subseteq\Delta_{ki}$.

Suppose $V(s_{1i}^{a}+s_{2i}^{b})\in\mathcal{C}_i(\Delta_{1i},\Delta_{2i})$ for all $V\in M_2(\Gamma)$ with $w(V)<m$ $(m\geq 2)$ and $s_{1i}^{a}+s_{2i}^{b}\in\mathcal{C}_i(\Delta_{1i},\Delta_{2i})$. Let $U\in M_2(\Gamma)$ with $w(U)=m$. Then $U=U_1+U_2$ or $U=U_1U_2$ for some $U_j\in M_2(\Gamma)$ with $w(U_j)<m$, $j=1,2$. Then $U(s_{1i}^{x_1}+s_{2i}^{x_2})=U_1(s_{1i}^{x_1}+s_{2i}^{x_2})+{}+U_2(s_{1i}^{x_1}+s_{2i}^{x_2})\in\mathcal{C}_i(\Delta_{1i},\Delta_{2i})$ by the induction assumption, or, by 1.7, $U(s_{1i}^{x_1}+s_{2i}^{x_2})=U_1(U_2(s_{1i}^{x_1}+s_{2i}^{x_2}))=U_1(s_{1i}^{y_1}+s_{2i}^{y_2})\in\mathcal{C}_i(\Delta_{1i},\Delta_{2i})$, once again by the induction assumption.

Note that by the assumptions on Δ_{1i} and Δ_{2i}, Δ_{ii} is a subnear-ring of Γ_{ii}. Clearly γ_i is a group homomorphism and $\gamma_i((s_{1i}^{x_1}+s_{2i}^{x_2})(s_{1i}^{y_1}+{}+s_{2i}^{y_2}))=\gamma_i(s_{1i}^{x_1y_i}+s_{2i}^{x_2y_i})=x_iy_i=\gamma_i(s_{1i}^{x_1}+s_{2i}^{x_2})\cdot\gamma_i(s_{1i}^{y_1}+s_{2i}^{y_2})$. It is clear that γ_i is surjective and $\ker\gamma_i=\{s_{1i}^{x_1}+s_{2i}^{x_2}\in\mathcal{C}_i(\Delta_{1i},\Delta_{2i})\mid x_i={}=0\}=\{s_{i_ci}^{x_{i_c}}\mid x_{i_c}\in\Delta_{i_ci}\}$. Thus, $(\ker(\gamma_i)^2=0$. ◊

3. Radical theory

Here we investigate the relationship between the radical of the near-ring $M_2(\Gamma)$ and the radicals of the near-rings Γ_{11} and Γ_{22}. We once again stress our assumption that $\Gamma=(\Gamma_{ii},\Gamma_{ii_c},\Gamma_{i_ci},\Gamma_{i_ci_c})$ is always a standard morita context for near-rings. We shall give two approaches to establish this relationship. The first is by placing additional conditions on a Kurosh–Amitsur radical, and the second will be by considering conditions on a class of near-rings such that the corresponding Hoehnke radical has the desired properties. Throughout this section $\mathcal{M}$ is a class of 2-semiprime near-rings. Let ϱ be the corresponding Hoehnke radical, i.e. $\varrho N=\cap(I$ an ideal of $N\mid N/I\in\mathcal{M})$ for all near-rings N.

Conditions on ϱ. Here we suppose that ϱ is a Kurosh–Amitsur radical. For more information on the relevant requirements for this to hold, [5] can be consulted. Of importance here, are the following:

Let $\mathcal{R}_\varrho := \{\text{near-rings } N \mid \varrho N = N\}$ be the radical class determined by ϱ. Then

3.1.1 $\mathcal{R}_\varrho$ is homomorphically closed;

3.1.2 $\varrho N \in \mathcal{R}_\varrho$ for all N;

3.1.3 $\mathcal{R}_\varrho$ is closed under extensions, i.e. if I is an ideal of N such that both I and N/I are in $\mathcal{R}_\varrho$, then $N \in \mathcal{R}_\varrho$;

3.1.4 if I is an ideal of N with $I \in \mathcal{R}_\varrho$, then $I \subseteq \varrho N$;

3.1.5 since $\mathcal{M}$ consists of 2-semiprime near-rings, $\mathcal{R}_\varrho$ contains all the near-rings with zero multiplication.

Examples of such radicals are the Jacobson radicals J_2 and J_3, the Brown-McCoy radical $\mathcal{G}$ and the equiprime radical e determined by the classes of 2-primitive, 3-primitive, simple near-rings with identity and the equiprime near-rings respectively. If a 2-primitive near-ring N has an identity, then it is 3-primitive and consequently the J_2 and J_3 radicals of any near-ring with identity coincide. Additional properties that the radical ϱ may satisfy are:

ϱ is *right strong* (resp. *left invariantly strong*) if whenever I is a right ideal (resp. left invariant subgroup) of N with $\varrho I = I$, then $I \subseteq \varrho N$.

ϱ is *hereditary on right ideals* (resp. *hereditary on left invariant subgroups*) if whenever I is a right ideal (resp. left invariant subgroup) of $N \in \mathcal{R}_\varrho$, then $I \in \mathcal{R}_\varrho$.

In the sequel, we let $\mathcal{T} := \varrho(M_2(\Gamma))$. Since $\mathcal{T}$ is an intersection of 3-semiprime ideals (= intersection of 2-semiprime ideals since $M_2(\Gamma)$ has an identity), $\mathcal{T}$ itself is 3-semiprime and thus full (cf. 2.5).

3.1.6 Proposition. *Suppose ϱ is right strong. Then $\varrho(\Gamma_{ii}) \subseteq \mathcal{T}_{ii}$ for all $i = 1, 2$.*

Proof. Let $i \in N_2$ be fixed. Let $\Delta_{ii} = \varrho(\Gamma_{ii})$ and let $\Delta_{ii_c} = \Gamma_{ii_c}$. Then Δ_{ik} is a normal subgroup of Γ_{ik} and $\Delta_{ii}\Gamma_{ik} \subseteq \Delta_{ik}$ for each $k = 1, 2$. By Prop. 2.15, $\varphi_i \colon \mathcal{R}_i(\Delta_{ii}, \Delta_{ii_c}) \to \Delta_{ii}$ is a surjective near-ring homomorphism with $K^2 = 0$ where $K = \ker \varphi_i$. (Note the inconsistency here, as well as in a few other places in the sequel of our notation; strictly speaking for $i = 2$, we should write $\mathcal{R}_i(\Delta_{ii_c}, \Delta_{ii})$ instead of $\mathcal{R}_i(\Delta_{ii}, \Delta_{ii_c})$.) By 3.1.5, 3.1.2 and 3.1.3 we get $\mathcal{R}_i(\Delta_{ii}, \Delta_{ii_c}) \in \mathcal{R}_\varrho$. Since $\mathcal{R}_i(\Delta_{ii}, \Delta_{ii_c})$ is a right ideal of $M_2(\Gamma)$ and ϱ is right strong, we get $\mathcal{R}_i(\Delta_{ii}, \Delta_{ii_c}) \subseteq$ $\subseteq \varrho(M_2(\Gamma)) = \mathcal{T}$. Let $x \in \varrho(\Gamma_{ii}) = \Delta_{ii}$. Then $s^x_{ii} \in \mathcal{R}_i(\Delta_{ii}, \Delta_{ii_c}) \subseteq \mathcal{T}$; hence $x \in \mathcal{T}_{ii}$. $\Diamond$

3.1.7 Proposition. *Suppose ϱ is hereditary on right ideals. Then $\mathcal{T}_{ii} \subseteq \varrho(\Gamma_{ii})$ for all $i = 1, 2$.*

Proof. Let $i \in N_2$ be fixed. For each $k \in N_2$, let $\Delta_{ik} = \mathcal{T}_{ik}$. Then Δ_{ik} is a normal subgroup of Γ_{ik} and $\Delta_{ii}\Gamma_{ik} \subseteq \Delta_{ik}$ for all $k = 1,2$. By Prop. 2.15, $\mathcal{R}_i(\Delta_{ii}, \Delta_{ii_c})$ is a right ideal of $M_2(\Gamma)$ and $\varphi_i \colon \mathcal{R}_i(\Delta_{ii}, \Delta_{ii_c}) \to$ $\to \Delta_{ii}$ is a surjective near-ring homomorphism. Let $U \in \mathcal{R}_i(\Delta_{ii}, \Delta_{ii_c})$. Then $U\Gamma^+ \subseteq \tau_i(\Delta_{ii}, \Delta_{ii_c}) \subseteq (\mathcal{T}_*)^+$. Thus $U \in (\mathcal{T}_*)^* = \mathcal{T}$; hence $\mathcal{R}_i(\Delta_{ii}, \Delta_{ii_c}) \subseteq \mathcal{T} = \varrho(M_2(\Gamma)) \in \mathcal{R}_\varrho$. By our assumption on ϱ, $\mathcal{R}_i(\Delta_{ii}, \Delta_{ii_c}) \in \mathcal{R}_\varrho$ and thus also $\mathcal{T}_{ii} = \Delta_{ii} \in \mathcal{R}_\varrho$ (by 3.1.1). Since $\mathcal{T}_{ii}$ is an ideal of Γ_{ii}, we have $\mathcal{T}_{ii} \subseteq \varrho(\Gamma_{ii})$ by 3.1.4. ◊

Since $\mathcal{T} = \varrho(M_2(\Gamma))$ is a 3-semiprime ideal of $M_2(\Gamma)$, we have by Cor. 2.11(2)

$$\begin{aligned}\mathcal{T}_* &= (\mathcal{T}_{ii}, \mathcal{T}_{ii_c}, \mathcal{T}_{i_ci}, \mathcal{T}_{i_ci_c}) = \\ &= (\mathcal{T}_{ii}, \mathcal{T}_{ii}\Gamma_{i_ci}^{-1}, \Gamma_{ii_c}^{-1}\mathcal{T}_{ii}, \mathcal{T}_{i_ci_c}) = \\ &= (\mathcal{T}_{ii}, \Gamma_{i_ci}^{-1}\mathcal{T}_{i_ci_c}, \mathcal{T}_{i_ci_c}\Gamma_{ii_c}^{-1}, \mathcal{T}_{i_ci_c}).\end{aligned}$$

Let $\varrho_1(\Gamma) := (\varrho(\Gamma_{11}), \varrho(\Gamma_{11})\Gamma_{21}^{-1}, \Gamma_{12}^{-1}\varrho(\Gamma_{11}), (\Gamma_{12}^{-1}\varrho(\Gamma_{11}))\Gamma_{21}^{-1})$ and let

$$\varrho_2(\Gamma) := ((\Gamma_{21}^{-1}\varrho(\Gamma_{22}))\Gamma_{12}^{-1}, \Gamma_{21}^{-1}\varrho(\Gamma_{22}), \varrho(\Gamma_{22})\Gamma_{12}^{-1}, \varrho(\Gamma_{22})).$$

In general, neither of these need to be an ideal of the morita context Γ and they need not be equal. However, if $\varrho_1(\Gamma) = \varrho_2(\Gamma)$, then $\varrho_1(\Gamma)$ is an ideal of Γ (cf. 2.2) and in this case we say the radical of Γ exists and call it the *radical of the morita context* Γ. We denote it by $\varrho(\Gamma)$.

3.1.8 Corollary. *Suppose ϱ is right strong and hereditary on right ideals. Then $\varrho(\Gamma_{jj}) = \mathcal{T}_{jj}$ for all $j \in N_2$ and $\varrho(\Gamma) = \varrho_1(\Gamma) = \varrho_2(\Gamma)$; hence $\varrho(M_2(\Gamma)) = \varrho(\Gamma)^*$.*

Proof. By 3.1.6 and 3.1.7 we have $\varrho(\Gamma_{jj}) = \mathcal{T}_{jj}$ and by the discussion preceding the corollary, we get $\varrho_1(\Gamma) = \mathcal{T}_* = \varrho_2(\Gamma)$. Hence $\varrho(\Gamma)$ exists and $\varrho(\Gamma) = \mathcal{T}_*$. Thus $\varrho(M_2(\Gamma)) = \mathcal{T} = (\mathcal{T}_*)^* = (\varrho(\Gamma))^*$. ◊

3.1.9 Proposition. *Suppose ϱ is hereditary on left invariant subgroups. Then $\mathcal{T}_{ii} \subseteq \varrho(\Gamma_{ii})$ for all $i \in N_2$.*

Proof. Let $i \in N_2$ be fixed and for each $k \in N_2$, let $\Delta_{ki} = \mathcal{T}_{ki}$. Then Δ_{ki} is a subgroup of Γ_{ki} and $\Gamma_{jk}\Delta_{ki} \subseteq \Delta_{ji}$ for all $k, j \in N_2$. By Prop. 2.16, $\mathcal{C}_i(\Delta_{ii}, \Delta_{i_ci})$ is a left invariant subgroup of $M_2(\Gamma)$ and $\gamma_i \colon \mathcal{C}_i(\Delta_{ii}, \Delta_{i_ci}) \to \Delta_{ii}$ is a surjective near-ring homomorphism. Now $\mathcal{C}_i(\Delta_{ii}, \Delta_{i_ci}) \subseteq \mathcal{T} = \varrho(M_2(\Gamma))$: Indeed, for all $n, m \in N_2$, let $a_{nm} \in$ $\in \Gamma_{nm}$. For $x_i \in \Delta_{ii}$ and $x_{i_c} \in \Delta_{i_ci}$, we have

$$\left(s_{ii}^{x_i} + s_{i_ci}^{x_{i_c}}\right)\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{bmatrix} = \tau_i(x_ia_{i1}, x_ia_{i2}) + \tau_{i_c}(x_{i_c}a_{i1}, x_{i_c}a_{i2}) \in (\mathcal{T}_*)^+$$

since $x_ka_{ij} \in \Delta_{ki}\Gamma_{ij} = \mathcal{T}_{ki}\Gamma_{ij} \subseteq \mathcal{T}_{kj}$ for all $k, j \in N_2$. Thus $s_{ii}^{x_i} + s_{i_ci}^{x_{i_c}} \in$

$\in (\mathcal{T}_*)^* = \mathcal{T}$. Since ϱ is hereditary on left invariant subgroups, we get $\mathcal{C}_i(\Delta_{ii}, \Delta_{i_c i})$ and thus also $\Delta_{ii} = \mathcal{T}_{ii}$ in $\mathcal{R}_\varrho$ (by 3.1.1). Thus $\mathcal{T}_{ii} \subseteq \varrho(\Gamma_{ii})$ by 3.1.4. ◊

3.1.10 Proposition. *Suppose ϱ is left invariantly strong. For $i \in N_2$ fixed, let $\Delta_{ii} = \varrho(\Gamma_{ii})$ and let $\Delta_{i_c i}$ be the subgroup of $\Gamma_{i_c i}$ generated by $\Gamma_{i_c i}\Delta_{ii}$. If $\Gamma_{ii_c}\Delta_{i_c i} \subseteq \Delta_{ii}$, then $\varrho(\Gamma_{ii}) \subseteq \mathcal{T}_{ii}$.*

Proof. Firstly note that $\Gamma_{jk}\Delta_{ki} \subseteq \Delta_{ji}$ for all $j, k \in N_2$ by Proposition 2.16, $\mathcal{C}_i(\Delta_{ii}, \Delta_{i_c i})$ is a left invariant subgroup of $M_2(\Gamma)$ and $\gamma_i \colon \mathcal{C}_i(\Delta_{ii}, \Delta_{i_c i}) \to \Delta_{ii}$ is a surjective near-ring homomorphism with $K^2 = 0$ where $K = \ker\theta$. By 3.1.5, 3.1.2 and 3.1.3 we get $\mathcal{C}_i(\Delta_{ii}, \Delta_{i_c i}) \in \in \mathcal{R}_\varrho$. By the assumption on ϱ, we get $\mathcal{C}_i(\Delta_{ii}, \Delta_{i_c i}) \subseteq \varrho(M_2(\Gamma)) = \mathcal{T}$. For $x \in \varrho(\Gamma_{ii}) = \Delta_{ii}$, $s^x_{ii} \in \mathcal{C}_i(\Delta_{ii}, \Delta_{i_c i}) \subseteq \mathcal{T}$; hence $x \in \mathcal{T}_{ii}$. ◊

As in 3.1.8, we get:

3.1.11 Corollaries. *If*

(i) *ϱ is right strong and hereditary on left invariant subgroups*

or if

(ii) *ϱ is hereditary on left invariant subgroups and left invariantly strong such that $\Gamma_{jj_c}\Delta_{j_c j} \subseteq \varrho(\Gamma_{jj})$ where $\Delta_{j_c j}$ is the subgroup of $\Gamma_{j_c j}$ generated by $\Gamma_{j_c j}\varrho(\Gamma_{jj})$ for all $j = 1, 2$*

then

$\varrho(\Gamma_{ii}) = \mathcal{T}_{ii}$ for all $i \in N_2$, $\varrho(\Gamma)$ exists and $\varrho(M_2(\Gamma)) = (\varrho(\Gamma))^*$. ◊

3.2 Conditions on $\mathcal{M}$. Throughout this section, $\Gamma = (\Gamma_{11}, \Gamma_{12}, \Gamma_{21}, \Gamma_{22})$ is a fixed standard morita context and $\mathcal{M}$ is a class of 2-semiprime near-rings with ϱ the corresponding Hoehnke radical. We write $\mathcal{T}$ for $\varrho(M_2(\Gamma))$.

3.2.1 Proposition. *Suppose $\mathcal{M}$ satisfies:*

(I) *If A is an ideal of $M_2(\Gamma)$ with $M_2(\Gamma)/\mathcal{A} \in \mathcal{M}$, then $\Gamma_{ii}/\mathcal{A}_{ii} \in \mathcal{M}$ for $i \in N_2$.*

Then $\varrho(\Gamma_{ii}) \subseteq \mathcal{T}_{ii}$.

Proof. Let $x \in \varrho(\Gamma_{ii})$ and let $\mathcal{A}$ be an ideal of $M_2(\Gamma)$ with $M_2(\Gamma)/\mathcal{A} \in \in \mathcal{M}$. By condition (I) $x \in \varrho(\Gamma_{ii}) \subseteq \mathcal{A}_{ii}$ and so $s^x_{ii} \in \mathcal{A}$. Since this holds for all such ideals $\mathcal{A}$ of $M_2(\Gamma)$, $s^x_{ii} \in \varrho(M_2(\Gamma)) = \mathcal{T}$. Thus $x \in \mathcal{T}_{ii}$. ◊

3.2.2 Proposition. *Suppose $\mathcal{M}$ and Γ satisfy:*

(II) *For $i \in N_2$, if Δ_{ii} is an ideal of Γ_{ii} with $\Gamma_{ii}/\Delta_{ii} \in \mathcal{M}$, then*

$$\Gamma_{ii_c} * \Gamma_{ii_c}^{-1}\Delta_{ii} \subseteq \Delta_{ii} \text{ and } M_2(\Gamma)/\Delta^* \in \mathcal{M}$$

where $\Delta^* = (\Delta_{ii}, \Delta_{ii}\Gamma_{i_ci}^{-1}, \Gamma_{ii_c}^{-1}\Delta_{ii}, \Gamma_{ii_c}^{-1}\Delta_{ii}\Gamma_{i_ci}^{-1})$.
Then $\mathcal{T}_{ii} \subseteq \varrho(\Gamma_{ii})$.
Proof. Let $x \in \mathcal{T}_{ii}$. Then $s_{ii}^x \in \mathcal{T}$. Let Δ_{ii} be an ideal of Γ_{ii} with $\Gamma_{ii}/\Delta_{ii} \in \mathcal{M}$. By condition (II) we get $s_{ii}^x \in \mathcal{T} = \varrho(M_2(\Gamma)) \subseteq \Delta^*$; hence $x \in (\Delta^*)_{ii} = \Delta_{ii}$. Thus $x \in \varrho(\Gamma_{ii})$. ◊
3.2.3 Theorem. *Suppose* $\mathcal{M}$ *and* Γ *satisfy conditions* (I) *and* (II). *Then* $\varrho(\Gamma_{ii}) = \mathcal{T}_{ii}$ *for* $i \in N_2$, $\varrho(\Gamma)$ *exists and* $\varrho(M_2(\Gamma)) = (\varrho(\Gamma))^*$.
Proof. By the previous two results and Cor. 2.11, we have that $\varrho(\Gamma)$ exists and $\varrho(\Gamma) = \mathcal{T}_*$. Thus $\varrho(M_2(\Gamma)) = \mathcal{T} = (\mathcal{T}_*)^* = (\varrho(\Gamma))^*$. ◊
3.2.4 Proposition. *The conditions* (I) *and* (II) *are equivalent to* (A) *and* (B) *where:*

(A) *For* $i \in N_2$, *if* Δ_{ii} *is an ideal of* Γ_{ii} *with* $\Gamma_{ii}/\Delta_{ii} \in \mathcal{M}$, *then* $\Gamma_{ii_c} * \Gamma_{ii_c}^{-1}\Delta_{ii} \subseteq \Delta_{ii}$ *and* $\Gamma_{i_ci_c}/\Gamma_{ii_c}^{-1}\Delta_{ii}\Gamma_{i_ci}^{-1} \in \mathcal{M}$.

(B) *Let* $\Delta = (\Delta_{11}, \Delta_{12}, \Delta_{21}, \Delta_{22})$ *be an ideal of* Γ. *Then* $M_2(\Gamma/\Delta) \in \mathcal{M}$ *if and only if* $\Gamma_{ii}/\Delta_{ii} \in \mathcal{M}$ *for* $i \in N_2$.

Proof. Suppose (I) and (II) hold. Let $i \in N_2$ and suppose Δ_{ii} is an ideal of Γ_{ii} with $\Gamma_{ii}/\Delta_{ii} \in \mathcal{M}$. By (II), $\Gamma_{ii_c} * \Gamma_{ii_c}^{-1}\Delta_{ii} \subseteq \Delta_{ii}$ and so for $\Delta = (\Delta_{ii}, \Delta_{ii}\Gamma_{i_ci}^{-1}, \Gamma_{ii_c}^{-1}\Delta_{ii}, \Gamma_{ii_c}^{-1}\Delta_{ii}\Gamma_{i_ci}^{-1})$, Δ^* is an ideal of $M_2(\Gamma)$ for which $M_2(\Gamma)/\Delta^* \in \mathcal{M}$. By (I) we then get $\Gamma_{i_ci_c}/\Gamma_{ii_c}^{-1}\Delta_{ii}\Gamma_{i_ci}^{-1} \in \mathcal{M}$ and so (A) holds.

Let $\Delta = (\Delta_{11}, \Delta_{12}, \Delta_{21}, \Delta_{22})$ be an ideal of Γ. If $M_2(\Gamma/\Delta) \in \mathcal{M}$, then $M_2(\Gamma)/\Delta^* \in \mathcal{M}$ (by Prop. 2.13) and from (I) we get $\Gamma_{ii}/\Delta_{ii} \in \mathcal{M}$. If $\Gamma_{ii}/\Delta_{ii} \in \mathcal{M}$, then (II) gives $M_2(\Gamma/\Delta) \cong M_2(\Gamma)/\Delta^* \in \mathcal{M}$ which shows the validity of (B).

Conversely, suppose (A) and (B) hold. Let $\mathcal{A}$ be an ideal of $M_2(\Gamma)$ such that $M_2(\Gamma)/\mathcal{A} \in \mathcal{M}$. Then $M_2(\Gamma/\mathcal{A}_*) \in \mathcal{M}$ and by (B), $\Gamma_{ii}/\mathcal{A}_{ii} = (\Gamma/\mathcal{A}_*)_{ii} \in \mathcal{M}$. Thus (I) holds. Let Δ_{ii} be an ideal of Γ_{ii} with $\Gamma_{ii}/\Delta_{ii} \in \mathcal{M}$. By (A), $\Gamma_{ii_c} * \Gamma_{ii_c}^{-1}\Delta_{ii} \subseteq \Delta_{ii}$ and $\Gamma_{i_ci_c}/\Gamma_{ii_c}^{-1}\Delta_{ii}\Gamma_{i_ci}^{-1} \in \mathcal{M}$. For the ideal

$$\Delta = (\Delta_{11}, \Delta_{12}, \Delta_{21}, \Delta_{22}) := (\Delta_{11}, \Delta_{11}\Gamma_{21}^{-1}, \Gamma_{12}^{-1}\Delta_{11}, \Gamma_{12}^{-1}\Delta_{11}\Gamma_{12}^{-1}) =$$
$$= (\Gamma_{21}^{-1}\Delta_{22}\Gamma_{12}^{-1}, \Gamma_{21}^{-1}\Delta_{22}, \Delta_{22}\Gamma_{12}^{-1}, \Delta_{22})$$

we then have $\Gamma_{ii}/\Delta_{ii} \in \mathcal{M}$ for $i \in N_2$. By (B), $M_2(\Gamma)/\Delta^* = M_2(\Gamma/\Delta) \in \mathcal{M}$ which yields (II). ◊

Contrary to the ring case the conditions (I) and (II) for the near-ring case can apparently not be expressed in terms of standard morita context without reference to ideals (e.g. in the case of (II), $\Gamma_{ii} \in \mathcal{M}$ implies $M_2(\Gamma) \in \mathcal{M}$). The reason being that for an ideal Δ_{ii} of Γ_{ii}, $\Delta = (\Delta_{11}, \Delta_{11}\Gamma_{21}^{-1}, \Gamma_{12}^{-1}\Delta_{11}, \Gamma_{12}^{-1}\Delta_{11}\Gamma_{12}^{-1})$ is not necessarily an ideal of Γ (cf. Prop. 2.2).

4. Examples

4.1. Let $\mathcal{M}$ be a class of 2-semiprime near-rings which has the matrix extension property, i.e. if A is a near-ring with identity, then $A \in \mathcal{M}$ if and only if $M_n(A) \in \mathcal{M}$ where $M_n(A)$ is the $n \times n$ matrix near-ring over A. Let N be a 0-symmetric near-ring with identity. Then $\Gamma := = (N, N^+, N^+, N)$ is a standard morita context (all multiplications are just the near-ring multiplication). In this case $M_2(\Gamma) \cong M_2(N)$ (cf. [3]) and Γ and $\mathcal{M}$ clearly satisfy the conditions (A) and (B). Hence $\varrho(M_2(\Gamma)) = (\varrho(\Gamma))^*$. But $\varrho(M_2(\Gamma)) \cong \varrho(M_2(N))$ and

$$(\varrho(\Gamma))^* \cong \left\{ U \in M_2(N) \mid U \begin{bmatrix} a \\ b \end{bmatrix} \subseteq \begin{bmatrix} \varrho(N) \\ \varrho(N) \end{bmatrix} \right\} = (\varrho(N))^*.$$

Hence $\varrho(M_2(N)) = (\varrho(N))^*$, confirming a well-known result (cf. [7]).

4.2. Let $\mathcal{M}$ be the class of all 2-semiprime near-rings. Let $\Gamma = = (\Gamma_{11}, \Gamma_{12}, \Gamma_{21}, \Gamma_{22})$ be a standard morita context. By Prop. 2.9, if $\mathcal{A}$ is a 2-semiprime ideal (= 3-semiprime ideal) of $M_2(\Gamma)$, then $\mathcal{A}_{ii}$ is a 2-semiprime ideal of Γ_{ii} and so condition (I) is satisfied. If the context Γ has the property that for each $i \in N_2$, whenever Δ_{ii} is a 3-semiprime ideal of Γ_{ii}, then $\Gamma_{ii_c} * \Gamma_{ii_c}^{-1}\Delta_{ii} \subseteq \Delta_{ii}$, then also condition (II) is satisfied by Cor. 2.12.

Let us mention that the context $\Gamma = (\Gamma_{11}, \Gamma_{12}, \Gamma_{21}, \Gamma_{22}) := := (N, G, H, M_n(N))$ (cf. [4]) has the property that for any ideal Δ_{ii} of Γ_{ii}, $\Gamma_{ii_c} * \Gamma_{ii_c}^{-1}\Delta_{ii} \subseteq \Delta_{ii}$.

4.3. Let $\mathcal{M}$ be the class of 3-primitive near-rings. Then $\varrho = J_3$. Anderson, Kaarli and Wiegandt [2] have shown that J_3 is a right strong radical (a note of caution, they deal with *left near-rings* and consequently show that J_3 is *left strong*). Thus $J_3(\Gamma_{ii}) \subseteq \mathcal{T}_{ii}$ where $\mathcal{T} = \varrho(M_2(\Gamma))$ for any standard morita context Γ. This result also follows from condition (I) which we now verify:

4.3.1 Proposition. *The class of* 3*-primitive near-rings satisfies condition* (I).

Proof. Condition (I) will follow from: $M_2(\Gamma)$ 2-primitive implies Γ_{ii} 2-primitive for any standard morita context $\Gamma = (\Gamma_{11}, \Gamma_{12}, \Gamma_{21}, \Gamma_{22})$

(by using Prop. 2.13 and the fact that the concepts 3-primitivity and 2-primitivity coincide on near-rings with identity). If $M_2(\Gamma)$ is 2-primitive, there is a faithful $M_2(\Gamma)$-group G which has no non-trivial $M_2(\Gamma)$-subgroups. Let $H = s^1_{ii}G$. Then H is a subgroup of G, for if $h_1 = s^1_{ii}g_1$ and $h_2 = s^1_{ii}g_2$ are elements of H, then $h_1 - h_2 = s^1_{ii}(g_1 - g_2) \in H$. Here we have used the fact that for any distributive element $U \in M_2(\Gamma)$, $U(g_1 + g_2) = Ug_1 + Ug_2$. Indeed, if $g_1 = 0$ or $g_2 = 0$ it clearly holds. Suppose thus $g_1 \neq 0$ and $g_2 \neq 0$. Then $M_2(\Gamma)g_1 = G = M_2(\Gamma)g_2$ and so $g_1 = Vg_1$ and $g_2 = Wg_1$ for some $V, W \in M_2(\Gamma)$. Then $U(g_1+g_2) = U((V + W)g_1) = (U(V + W))g_1 = (UV)g_1 + (UW)g_1 = Ug_1 + Ug_2$.

We now show that H is a faithful Γ_{ii}-group of type 2. Define $\Gamma_{ii} \times H \to H$ by

$$(a, s^1_{ii}g) \longmapsto s^a_{ii}g \quad \text{for all} \quad a \in \Gamma_{ii},\ g \in G.$$

It is well-defined since $s^a_{ii}g = s^1_{ii}(s^a_{ii}g) \in s^1_{ii}G = H$. Also, $(a+b)(s^1_{ii}g) = as^1_{ii}g + bs^1_{ii}g$ and $(ab)s^1_{ii}g = s^{ab}_{ii}g = s^a_{ii}(s^1_{ii}(s^b_{ii}g)) = a(b(s^1_{ii}g)) = a(b(s^1_{ii}g))$. Thus H is a Γ_{ii}-group. Moreover, if $aH = 0$, then $s^a_{ii} \in (0 : G)_{M_2(\Gamma)} = 0$. Thus $a = 0$ and so H is faithful. Finally, for $0 \neq s^1_{ii}g \in H$, we show $\Gamma_{ii}(s^1_{ii}g) = H$: Let $0 \neq s^1_{ii}g' \in H$. Then $s^1_{ii}g' \in G = M_2(\Gamma)(s^1_{ii}g)$; say $s^1_{ii}g' = Us^1_{ii}g$ for some $U \in M_2(\Gamma)$. Then, for some $a_j \in \Gamma_{ji}$ $(j = 1, 2)$ we have $s^1_{ii}g' = s^1_{ii}(s^1_{ii}g') = s^1_{ii}(Us^1_{ii}g) = s^1_{ii}U(s^1_{ii} + s^0_{i_ci})g = s^1_{ii}(s^{a_i}_{ii} + s^{a_{i_c}}_{i_ci})g = s^{a_i}_{ii}g = a_i(s^1_{ii}g) \in \mathcal{G}_{ii}(s^1_{ii}g)$. As the other inclusion is obvious, we have $H = \Gamma_{ii}(s^1_{ii}g)$. Thus Γ_{ii} is a 2-primitive near-ring. $\Diamond$

4.4. Let $\mathcal{M}$ be the class of equiprime near-rings. Recall, a near-ring N is *equiprime* if $anx = any$ for all $n \in N$ implies $a = 0$ or $x = y$. Then $\varrho = e$ is the equiprime radical.

4.4.1 Proposition. *The equiprime radical is right strong.*

Proof. Let I be a right ideal of the near-ring N with $e(I) = I$. Let P be any equiprime ideal of N. We show $K := \{x \in I \mid xI \subseteq P)$ is an equiprime ideal of I. It is clearly an ideal. Let $a, x, y \in I$ such that $aix - aiy \in K$ for all $i \in I$. Then $aixj - aiyj \in P$ for all $i, j \in I$. If $a \notin K$, then $ai_0 \notin P$ for some $i_0 \in I$. Suppose also $x - y \notin K$. Then $xj_0 - yj_0 \notin P$ for some $j_0 \in I$. Since P is an equiprime ideal of N, there is an $n_0 \in N$ such that $(ai_0)n_0(xj_0) - (ai_0)n_0(yj_0) \notin P$. But $(ai_0)n_0(xj_0) - (ai_0)n_0(yj_0) = a(i_0n_0)xj_0 - a(i_0n_0)yj_0 \in P$ since $IN \subseteq I$, which is a contradiction. Thus K is an equiprime ideal of I and so $I = e(I) \subseteq K$; i.e. $I^2 \subseteq P$. Let $a \in I$. Then $aNa = (aN)a \subseteq I^2 \subseteq P$ and since P is an equiprime ideal, it is 3-prime and so $a \in P$. Thus

$I \subseteq P$ and we conclude that $I \subseteq e(N)$. $\Diamond$

This result yields $e(\Gamma_{ii}) \subseteq \mathcal{T}_{ii}$ where $\mathcal{T} = e(M_2(\Gamma))$. It also follows from condition (I) which we now verify.

4.4.2 Proposition. *The class of equiprime near-rings satisfy condition* (I).

Proof. Let $\mathcal{A}$ be an equiprime ideal of $M_2(\Gamma)$. We show $\mathcal{A}_{ii}$ is an equiprime ideal of Γ_{ii}. Let $a, b, c \in \Gamma_{ii}$ such that $anb - anc \in \mathcal{A}_{ii}$ for all $n \in \Gamma_{ii}$, i.e. $s_{ii}^{anb-anc} \in \mathcal{A}$ for all $n \in \Gamma_{ii}$. Suppose both a and $b - c$ are not in $\mathcal{A}_{ii}$. Then both s_{ii}^a and $s_{ii}^b - s_{ii}^c$ are not in $\mathcal{A}$ and consequently there is a $U \in M_2(\Gamma)$ such that $s_{ii}^a U s_{ii}^b - s_{ii}^a U s_{ii}^c \notin \mathcal{A}$. Now $Us_{ii}^1 = s_{1i}^{x_1} + s_{2i}^{x_2}$ for some $x_j \in \Gamma_{ji}$, $j = 1, 2$ (cf. 1.7), and so $s_{ii}^{ax_ib-ax_ic} = s_{ii}^a(s_{1i}^{x_1}+s_{2i}^{x_2})s_{ii}^b - s_{ii}^a(s_{1i}^{x_1}+s_{2i}^{x_2})s_{ii}^c = s_{ii}^a U s_{ii}^1 s_{ii}^b - s_{ii}^a U s_{ii}^1 s_{ii}^c =$ $= s_{ii}^a U s_{ii}^b - s_{ii}^a U s_{ii}^c \notin \mathcal{A}$; a contradiction. $\Diamond$

4.4.3 Proposition. *Let* $\Gamma = (\Gamma_{11}, \Gamma_{12}, \Gamma_{21}, \Gamma_{22})$ *be a standard morita context such that* $\Gamma_{ii_c} * \Gamma_{ii_c}^{-1}\Delta_{ii} \subseteq \Delta_{ii}$ *for any equiprime ideal* Δ_{ii} *of* Γ_{ii}. *Then* Γ *and* $\mathcal{M}$ *satisfy condition* (II).

Proof. Let Δ_{ii} be an equiprime ideal of Γ_{ii}. We show that Δ^* is an equiprime ideal of $M_2(\Gamma)$ where Δ is the ideal of $\Gamma = (\Gamma_{ii}, \Gamma_{ii_c}, \Gamma_{i_ci}, \Gamma_{i_ci_c})$, cf. Prop. 2.2 and our assumption, defined by $\Delta = (\Delta_{ii}, \Delta_{ii_c}, \Delta_{i_ci}, \Delta_{i_ci_c}) :=$ $(\Delta_{ii}, \Delta_{ii}\Gamma_{i_ci}^{-1}, \Gamma_{ii_c}^{-1}\Delta_{ii}, \Gamma_{ii_c}^{-1}\Delta_{ii}\Gamma_{i_ci}^{-1})$. For this we need a preliminary result:

4.4.4 Lemma. *For* $n, k, j \in N_2$, *if* $a \in \Gamma_{nk}$, $x, y \in \Gamma_{kj}$ *and* $abx - aby \in$ $\in \Delta_{nj}$ *for all* $b \in \Gamma_{kk}$, *then* $a \in \Delta_{nk}$ *or* $x - y \in \Delta_{kj}$. *In particular, for* $n = k = j = i_c$, $\Delta_{i_ci_c} = \Gamma_{ii_c}^{-1}\Delta_{ii}\Gamma_{i_ci}^{-1}$ *is an equiprime ideal of* $\Gamma_{i_ci_c}$.

Proof. As every equiprime ideal is 3-semiprime, we may use the second part of Cor. 2.12 (which we often do without any further mentioning). If $a \notin \Delta_{nk} = \Gamma_{in}^{-1}\Delta_{ik}$, then $ua \notin \Delta_{ik} = \Delta_{ii}\Gamma_{ki}^{-1}$ for some $u \in \Gamma_{in}$. Thus $uav \notin \Delta_{ii}$ for some $v \in \Gamma_{ki}$. For any $q \in \Gamma_{ik}$, $c \in \Gamma_{ji}$ and $d \in \Gamma_{ii}$, $(uav)d(qxc) - (uav)d(qyc) = [u(avdqx - avdqy) + avdqy) - uavdqy]c \in$ $\in (\Gamma_{in} * \Delta_{nj})\Gamma_{ji} \subseteq \Delta_{ij}\Gamma_{ji} \subseteq \Delta_{ii}$. Since Δ_{ii} is an equiprime ideal of Γ_{ii}, we have $qxc - qyc \in \Delta_{ii}$ for all $q \in \Gamma_{ik}$, $c \in \Gamma_{ji}$. Thus $qx - qy \in$ $\in \Delta_{ii}\Gamma_{ji}^{-1} = \Delta_{ij}$ for all $q \in \Gamma_{ik}$. Since $1 = 1_{\Gamma_{kk}} \in \Gamma_{kk} = \overline{\Gamma_{ki}\Gamma_{ik}}$, we have $1 = \sum_{t=1}^{m} \sigma_t g_t h_t$ where $\sigma_t \in \{+, -\}$, $g_t \in \Gamma_{ki}$ and $h_t \in \Gamma_{ik}$. Now

$$x - y = 1x - 1y = \sigma_1 g_1 h_1 x + \ldots + \sigma_m g_m h_m x - \sigma_m g_m h_m y - \ldots - \sigma_1 g_1 h_1 y.$$

For each t, $\sigma_t g_t h_t x - \sigma_t g_t h_t y = (\sigma_t g_t)[(h_t x - h_t y) + h_t y] - (\sigma_t g_t) h_t y \in$ $\in \Gamma_{ki} * \Delta_{ij} \subseteq \Delta_{kj}$, which is normal in Γ_{kj}, and we may conclude that $x - y \in \Delta_{kj}$. $\Diamond$

Proof (of 4.4.3). We abbreviate an element $\begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix}$ of Γ^+ by $[x_{ij}]$. Let $A, B, C \in M_2(\Gamma)$ such that $AUB - AUC \in \Delta^*$ for all $U \in M_2(\Gamma)$. Suppose $A \notin \Delta^*$ and $B - C \notin \Delta^*$. Then $[a_{ij}] = A[z_{ij}] \notin \Delta^+$ and $[b_{ij} - c_{ij}] = B[x_{ij}] - C[x_{ij}] = (B - C)[x_{ij}] \notin \Delta^+$ for some $[z_{ij}], [x_{ij}] \in \Gamma^+$. Suppose $a := a_{kj} \notin \Delta_{kj}$ and $b - c = b_{pq} \notin \Delta_{pq}$. Now $a \notin \Delta_{kj} = \Delta_{kp}\Gamma_{jp}^{-1}$ implies $au \notin \Delta_{kp}$ for some $u \in \Gamma_{jp}$. From the above Lemma, we know there is a $d \in \Gamma_{pp}$ such that $audb - audc \notin \Delta_{kq}$. Let $V := (s_{1j}^{z_{ij}} + s_{2j}^{z_{2j}})s_{jp}^{ud} \in M_2(\Gamma)$. Since $AVB - AVC \in \Delta^*$, also $s_{kk}^1 AVB - s_{kk}^1 AVC \in \Delta^*$. Thus $(s_{kk}^1 AVB - s_{kk}^1 AVC)[x_{ij}] = s_{kk}^1 (s_{1j}^{a_{1j}} + s_{2j}^{a_{2j}})s_{jp}^{ud}[b_{ij}] - s_{kk}^1 (s_{1j}^{a_{1j}} + s_{2j}^{a_{2j}})s_{jp}^{ud}[c_{ij}] = s_{kp}^{aud}[b_{ij}] - s_{kp}^{aud}[c_{ij}] = \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix} \in \Delta^+$ where $y_{kt} = audb_{pt} - audc_{pt}$ and $y_{k_c t} = 0$ for $t = 1, 2$. In particular, for $t = q$, we get $y_{kq} = audb - audc \in \Delta_{kq}$ — a contradiction. ◊

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CAUCHY'S EQUATION ON THE GRAPH OF AN INVOLUTION

Bogdan **CHOCZEWSKI***

Institute of Mathematics, University of Mining and Metallurgy, al. Mickiewicza 30, 30-059 Kraków, Poland

Ilie **COROVEI**

Institute of Mathematics, Technical University, Cluj-Napoca, Romania

Constantin **RUSU**

Institute of Mathematics, Technical University, Cluj-Napoca, Romania

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Abstract: Description of the form of solutions $f\colon \mathbb{R}\to\mathbb{R}$ of the functional equation $f(x+\alpha(x)) = f(x) + f(\alpha(x))$ is given in the case where $\alpha\colon \mathbb{R}^+\to\mathbb{R}^+$ is an involution ($\alpha\circ\alpha = \mathrm{id}$). When $\alpha(x) = \frac{1}{x}$, formal power series to the above equation are also determined.

The present paper was motivated by the following problem proposed by K. Lajkó on the XX. International Symposium on Functional Equations (Oberwolfach, 1982, cf. [3]): under what conditions the functions

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$$(*) \qquad f(x) = ax^2 + bx + 2a, \qquad a, b \in \mathbb{R},$$

are the only solutions of the equation

$$(1) \qquad f(x + 1/x) = f(x) + f(1/x), \qquad x \in \mathbb{R}^+ \quad (:= (0, +\infty))?$$

We give another proof of Lajkó's conjecture that $(*)$ is the only solution of (1) in the class of formal power series, some remarks on the general solution of the equation

$$(2) \qquad f(x + \alpha(x)) = f(x) + f(\alpha(x)), \quad \alpha(\alpha(x)) = x, \quad x \in \mathbb{R}^+,$$

and a theorem on the form of solutions of the equation

$$(3) \qquad f(\phi(x) + \psi(x)) = f(\phi(x)) + f(\psi(x)), \quad x \in \mathbb{R}^+,$$

with some specified ϕ and ψ.

The equations (1)–(3) are Cauchy's equations restricted to a graph of a given function which were recently studied by many authors, cf., e.g., W. Jarczyk [2] and the references quoted therein. In particular, equation (2) has been dealt with by J. Matkowski and M. Sablik [4], cf. the last section of our paper.

1. Let $\mathcal{F}$ be the linear space (over $\mathbb{R}$) of all formal power series

$$\mathcal{F} := \Big\{ f(x) = \sum_{k=-\infty}^{\infty} a_k x^k,\ a_k \in \mathbb{R} \Big\}$$

and consider the mapping $F\colon \mathcal{F} \to \mathcal{F}$, given by

$$[F(f)](x) = f(x + 1/x) - f(x) - f(1/x),\ f \in \mathcal{F},\ x \in \mathbb{R}^+.$$

The mapping F is linear and solving (1) means determining $\ker F$. We have $F(1) = -1$, $F(x) = 0$, $F(x^2) = 2$, so that the series $(*)$ belongs to $\ker F$. Let us examine $F(x^k)$ where $k \in \mathbb{Z} \setminus \{0, 1, 2\}$. Since, for $k \in \mathbb{N} \setminus \{0, 1, 2\}$ we have

$$F(x^k) = 2x^k + 2x^{-k} + \sum_{i=1}^{k-1} \binom{k}{i} x^{2i-k}$$

and for $k = -m$, $m \in \mathbb{N}$,

$$F(x^{-m}) = \Big(\sum_{i=0}^{\infty} x^{2i+1} \Big)^m + x^{-m} + x^m,$$

we see that any system $\{F(x^{k_1}), \dots, F(x^{k_p})\}$, $k_i \in \mathbb{Z} \setminus \{0, 1, 2\}$ is linearly independent over $\mathbb{R}$. Thus the series $(*)$ are the only solutions of

(1) in $\mathcal{F}$.

2. We shall deal with equation (2) under the following hypotheses:

(H) The function $\alpha\colon \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous, strictly decreasing involution ($\alpha \circ \alpha = \mathrm{id}$), mapping bijectively $\mathbb{R}^+$ onto itself, and the function $g\colon \mathbb{R}^+ \to \mathbb{R}^+$, given by

$$g(x) := x + \alpha(x),\ x \in R^+,$$

is strictly increasing on $[c, +)$, where $c = \alpha(c)$.

Let us observe that c is the only fixed point of α, we have $\lim\limits_{x-0} \alpha(x) = +\infty$, $\lim\limits_{x-\infty} \alpha(x) = 0$, g maps $[c, +\infty)$ bijectively onto $[2c, +\infty)$, and the sequence

$$x_n = g^n(c), \quad n \in \mathbb{N} \cup \{0\}$$

(where g^n denotes the n-th iterate of the function g) is strictly increasing and unbounded, so that

$$[2c, +\infty) = \bigcup_{k=1} [x_k, x_{k+1}). \tag{4}$$

Let $g_i := g \mid_{[x_{i-1}, x_i)}$. Thus g_i is an increasing continuous bijective function from $[x_{i-1}, x_i)$ onto $[x_i, x_{i+1})$. Let us put

$$h_i := g_i^{-1}.$$

Therefore $h_i\colon [x_{i-1}, x_i) \to [x_i, x_{i+1})$.

We have the following

Theorem 1. *Under hypotheses* (H), *if* $f_0\colon (0, 2c) \to \mathbb{R}$ *is an arbitrary function, then the function*

$$f(x) = \begin{cases} f_0(x) & \text{if } x \in (0, 2c), \\ f_0 \circ h_1 \circ \ldots \circ h_i(x) + f_0 \circ \alpha \circ h_1 \circ \ldots \circ h_i(x) + f_0 \circ \alpha \circ h_2 \circ \\ \circ \ldots \circ h_i(x) + \ldots + f_0 \circ \alpha \circ h_i(x) & \text{if } x \in [x_i, x_{i+1}),\ i \in \mathbb{N} \end{cases} \tag{5}$$

is a solution of equation (2). *If* f_0 *is continuous then so is* f.

Proof. a) $x \in [c, +\infty)$. Let first $x \in [c, 2c)$. Then $g(x) = g_1(x) \in [x_1, x_2)$, $\alpha(x) \in (0, c)$ and from (4) we get

$$f(x + \alpha(x)) = f \circ g(x) = f_0 \circ h_1 \circ g(x) + f_0 \circ \alpha \circ h_1 \circ g(x) = \\ = f_0(x) + f_0(\alpha(x)) = f(x) + f(\alpha(x))$$

and (1) is satisfied. Suppose now that f given by (4) satisfies (2) whenever $x \in [x_0, x_n)$ and let $x \in [x_n, x_{n+1})$. We have $g_{n+1}(x) = x + \alpha(x) \in [x_{n+1}, x_{n+2})$, and from (5) we obtain (since $h_{n+1} \circ g_{n+1} = \mathrm{id}$)

$$f(x+\alpha(x)) = f\circ g_{n+1}(x) = f_0\circ h_1\circ\ldots\circ h_n(x) + f_0\circ\alpha\circ h_1\circ$$
$$\circ\ldots\circ h_n(x)+\ldots+f_0\circ\alpha\circ h_n(x)+f_0\circ\alpha(x) = f(x)+f\circ\alpha(x).$$

Induction completes the proof in the case where $x\in[c,+\infty)$.

b) $x\in(0,c)$. Then $y=\alpha(x)\in(c,+\infty)$. Moreover $\phi(y)=\phi\circ\circ\phi(x)=x$, since ϕ is an involution. Thus we may apply case a) and write

$$f(x+\alpha(x)) = f(\alpha(y)+y) = f(\alpha(y))+f(y) = f(x)+f(\alpha(x)).$$

The continuity of f follows from (H) and the continuity of f_0. ◊

Now we are going to prove that the extension of f_0 to f, given by (5), is unique.

Theorem 2. *Let* (H) *be satisfied and let* $f_0\colon(0,2c)\to\mathbb{R}$ *be any function. There exists the unique solution* f *of equation* (2) *which coincides with* f_0 *on* $(0,2c)$.

Proof. Suppose we are given two solutions: f and f^* of (2) such that

$$f(x) = f^*(x) \quad \text{for} \quad x\in(0,2c) \tag{6}$$

and that there is a $t\in(0,2c)$ such that $f(t)\neq f^*(t)$. Because of (4), $x\in[x_i,x_{i+1})$ for some $i\in\mathbb{N}$, thus $h(t)\in[x_{i-1},x_i)$ and

$$t = g_i\circ h_i(t) = h_i(t)+\alpha\circ h_i(t) \tag{7}$$

and $\alpha\circ h_i(t)\in(0,c)$. We now use (2) and (7) to get

$$f(t) = f\circ h_i(t)+f\circ\alpha\circ h_i(t)\neq f^*(t) = f^*\circ h_i(t)+f^*\circ\alpha\circ h_i(t).$$

But the second terms here are equal, because of (5), so we end up with $f\circ h_i(t)\neq f^*\circ h_i(t)$. Now, by the same argument with $h_i(t)$ in place of t we arrive at $f\circ h_{i-1}\circ h_i(t)\neq f^*\circ h_{i-1}\circ h_i(t)$ and eventually at $f\circ h_1\circ\ldots\circ h_i(t)\neq f^*\circ h_1\circ\ldots\circ h_i(t)$. Since $h_1\circ\ldots\circ h_i(t)$ here belongs to $[x_0,x_1]=[c,2c)\subset(0,2c)$, we get a contradiction with (5). ◊

As a consequence of this theorem we get the following

Corollary. *If* (H) *holds then every solution of equation* (2) *is given by the construction* (4).

Proof. Indeed, let f^* be a solution of (2). Take the solution f of (2) given by (4) with $f_0=f^*\mid_{(0,2c)}$. According to Th. 2, since f and f^* coincide on $(0,2c)$, there is $f=f^*$. ◊

Remarks. (1) The interval $(0,2c)$ is the maximal set on which one may arbitrarily prescribe a solution to (2). Indeed, given any set $M\subset(0,2c)$ let us take an $x_0\in(0,2c)\setminus M$ such that $f(x_0)$ can be determined with the use of the values of f given on M. Since f satisfies (2), the following may happen: either there is an x such that $x_0 = x + \alpha(x) = g(x) < < 2c$, contrary to $g(x) \geq 2c$; or (2) is satisfied either with x_0, then $x_0 + \alpha(x_0) > 2c$ or for an y such that $x_0 = \alpha(y)$, then $y + \alpha(y) > 2c$. In the latter case both arguments do not belong to M, so that the value of f is not defined, contrary to the hypothesis.

(2) Since the function $\alpha(x) = 1/x$, $x \in \mathbb{R}^+$, satisfies hypothesis (H), Th. 1 determines also the general solution of equation (1).

(3) The solutions of (2) defined on $(-\infty, 0)$, respectively on $\mathbb{R}\setminus\{0\}$, can be described in a similar way as those defined on $\mathbb{R}^+$.

3. The following result by J. Matkowski and M. Sablik ([4], Th. 4) yields another construction of the general solution to (2) under more general assumptions than (H).

Proposition. *Let* $\alpha\colon \mathbb{R}^+ \to \mathbb{R}^+$ *be an involution satisfying* $\alpha(c) = c$ *and* $\alpha((0,c)) \subset (c, +\infty)$ *and* $\alpha((c,+\infty)) \subseteq (0,c)$. *Then every function* $f_0\colon (c,+\infty) \to \mathbb{R}$ *such that*

$$f_0(2c) = 2f_0(c) \tag{7}$$

can uniquely be extended to a solution $f\colon \mathbb{R}^+ \to \mathbb{R}$ *of equation* (2). *Moreover, if* α *and* f_0 *are continuous then so is* f.

An analogous result can be obtained for equation (3), i.e., for the Cauchy equation on the graph of a parametrically given curve $(\phi(x), \psi(x))$.

Theorem 3. *Let* $\phi, \psi\colon \mathbb{R}^+ \to \mathbb{R}^+$ *satisfy* $\phi(c) = c$, $\phi((0,c)) \subset (c,+\infty)$, $\phi((c,+\infty)) \subset (0,c)$ *and*

(8) $\phi \circ \psi(x) = \phi(x)$ *for* $x \in (0,c]$ *and* $\psi \circ \psi(x) = \phi(x)$ *for* $x \in (c,+\infty)$.

Then every function $f_0\colon [c,+\infty) \to \mathbb{R}$ *satisfying* (7) *can uniquely be extended to a solution* $f\colon \mathbb{R}^+ \to \mathbb{R}$ *of equation* (3). *Moreover, if* ϕ, ψ *and* f_0 *are continuous then so is* f.

Proof. Given an f_0 as claimed we define f as follows

$$f(x) = \begin{cases} f_0(x + \phi(x)) - f_0(\phi(x)) & \text{if } x \in (0,c), \\ f_0(x) & \text{if } x \in [c,+\infty) \end{cases} \tag{9}$$

The function f is well defined since $x \in (0,c)$ implies $\phi(x) > c$ and $x + \phi(x) > c$. If $x \in (0,c)$, then from (8) we get $\psi(x) \in (0,c)$ so that $\phi(x) + \psi(x) \in (c,+\infty)$ and

$$f \circ \phi(x) + f \circ \psi(x) = f_0 \circ \phi(x) + f_0(\psi(x) + \phi \circ \psi(x)) - f_0 \circ \phi \circ \psi(x) = \\ = f_0(\phi(x) + \psi(x)) = f(\phi(x) + \psi(x)).$$

If $x \in (c, +\infty)$, then $\phi(x) \in (0, c)$, $\phi \circ \phi(x) \in (c, +\infty)$, whence $\psi(x) \in \in (c, +\infty)$, $\phi(x) + \psi(x) \in (c, +\infty)$. Therefore

$$f \circ \phi(x) + f \circ \psi(x) = f_0\phi(x) + \phi \circ \phi(x)) - f_0 \circ \phi \circ \phi(x) + f_0 \circ \psi(x) = \\ = f_0(\phi(x) + \psi(x)) = f(\phi(x) + \psi(x)).$$

For $x = c$, (3) results from (7).

If the functions involved in its definition are continuous, then the continuity of f given by (9) is obvious for $x \neq c$, whereas for $x = c$ it results from (7) and (9). ◊

Concluding remark. The question (cf. [1]) under what conditions equation (2) has linear solutions only (or a finite-parameter family of solutions) remains unanswered. In this connection, during the XXXI International Symposium on Functional Equations (August 1993, Debrecen), J. Matkowski proposed the following, more adequate, problem:

Consider the system of functional equations (2) with two given involutions and establish conditions under which the only solution to the system is the identity function.

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COMMUTATIVITY RESULTS FOR RINGS THROUGH STREB'S CLASSIFICATION

Hamza A. S. **ABUJABAL**

Department of Mathematics, Faculty of Science, King Abdul Aziz University, P.O. Box 31464, Jeddah 21497, Saudi Arabia

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Abstract: An associative ring R is commutative if (and only if) for each $x, y \in R$, there exist integers $m > 0$, $n \geq 0$ and $f(X), g(X), h(X) \in X^2\mathbb{Z}\,[X]$ with $f(1) = \pm 1$ such that $[x, yx^m - f(y)x^n] = 0$ and $[x - g(x), y - h(y)] = 0$. Further, we extend this result for one sided s-unital rings.

Throughout this paper, R will denote an associative ring with center $Z(R)$, and $C(R)$ the commutator ideal of R. Let $N(R)$ be the set of nilpotent elements in R, and let $N^*(R)$ be the subset of $N(R)$ consisting of all elements in R which square to zero. A ring R is called *left* (resp. *right*) *s-unital* if $x \in Rx$ (resp. $x \in xR$) for every $x \in R$. Further, R is called *s-unital* if $x \in Rx \cap xR$ for all $x \in R$. If R is s-unital (resp. left or right s-unital), then for any finite subset F of R, there exists an element $e \in R$ such that $ex = xe = x$ (resp. $ex = x$ or $xe = x$) for all $x \in F$. Such an element e will be called a *pseudo* (resp. *pseudo left* or *pseudo right*) *identity* of F in R. We denote by $\mathbb{Z}{<}X, Y{>}$ the polynomial ring over $\mathbb{Z}$ the ring of integers, in the non-cummuting indeterminates X, and Y. As usual $\mathbb{Z}[X]$ is the totality of polynomials in X with coefficients in $\mathbb{Z}$ and for any $x, y \in R$, $[x, y] = xy - yx$. For any positive integer d, we consider the following ring property:

$\mathbf{Q}(d)$: if $x, y \in R$, and $d[x,y] = 0$, then $[x,y] = 0$.

By $GF(q)$, we mean the Galois field (finite field) with q elements, and $(GF(q))_2$ the ring of all 2×2 matrices over $GF(q)$. Set $e_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $e_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, and $e_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ in $(GF(p))_2$ for a prime p.

In [7, Prop. 2], Komatsuo et al. proved the following important result:

Proposition 1. *Let R be a ring generated by two elements such that the commutator ideal $C(R)$, is the heart of R and $C(R)R = RC(R) = 0$. Then R is nilpotent.*

In view of Prop. 1, we see that Streb's Theorem of [8] can be stated as follows:

Theorem S. *Let R be a non-commutative ring ($R \neq Z(R)$). Then there exists a factor subring of R which is of type* (a)$_{\text{i}}$, (a)$_{\text{ii}}$, (b), (c), (d), (e), (f) *or* (g):

(a)$_{\text{i}}$ $\begin{bmatrix} GF(p) & GF(p) \\ 0 & GF(p) \end{bmatrix}$, *$p$ a prime.*

(a)$_{\text{ii}}$ $\begin{bmatrix} 0 & GF(p) \\ 0 & GF(p) \end{bmatrix}$, *$p$ a prime.*

(b) $M_\eta(\mathbf{K}) = \left\{ \begin{bmatrix} a & b \\ 0 & \eta(a) \end{bmatrix} \mid a, b \in \mathbf{K} \right\}$, *where $\mathbf{K}$ is a finite field with a non-trivial automorphism η.*

(c) *A non-commutative division ring.*

(d) *A non-commutative ring with no non-zero divisors of zero.*

(e) *A finite nilpotent ring S such that $C(S)$ is the heart of S and $SC(S) = C(S)S = 0$.*

(f) *A ring S generated by two elements of finite additive order such that $C(S)$ is the heart of S, $SC(S) = C(S)S = 0$, and $N(S)$ is a commutative nilpotent ideal of S.*

(g) *A simple radical ring with no non-zero divisors of zero.*

Further, from the proof of [8, Korollar 1], we have the following:

Theorem ST. *Let R be a non-commutative ring with 1. Then there exists a factor subring of R which is of type* (a)$_{\text{i}}$, (b), (c), (d), (d)$'$, (e)$'$ *or* (e)$''$:

(a)$_{\text{i}}$ $\begin{bmatrix} GF(p) & GF(p) \\ 0 & GF(p) \end{bmatrix}$, *$p$ a prime.*

(b) $M_\eta(\mathbf{K}) = \left\{ \begin{bmatrix} a & b \\ 0 & \eta(a) \end{bmatrix} \mid a, b \in \mathbf{K} \right\}$, *where $\mathbf{K}$ is a finite field with a non-trivial automorphism η.*

(c) *A non-commutative division ring.*
(d) *A non-commutative ring with no non-zero divisors of zero.*
(d)′ $T = <1> + S$ *is a finite integral domain, where* S *is a simple radical ring.*
(d) *A non-commutative ring with no non-zero divisors of zero.*
(e)′ $T = <1> + S$, *where* S *is a finite nilpotent ring such that* $C(S)$ *is the heart of* S *and* $SC(S) = C(S)S = 0$.
(e)″ $T = < 1 > + S$, *where* S *is a non-commutative subring of* T *such that* $S[S,S] = [S,S]S = 0$.

Now Th. S and Th. ST give the following Meta Theorem which plays an important role in our subsequent study.

Lemma 1 (Meta Theorem). *Let* **P** *be a ring property which is inherited by factor subrings. If no rings of type* (a)$_i$, (a)$_{ii}$, (b), (c), (e), *or* (g), (f) *(resp.* (a)$_i$, (b), (c), (d), (d)′, (e)′ *or* (e)″*) satisfy* **P**, *then every ring (resp. every ring with unity* 1*) satisfying* **P** *is commutative.*

Our objective is to prove the following results.

Theorem 1. *Let* R *be a ring. Then* R *is commutative if (and only if) for each* $x, y \in R$, *there exist integers* $m > 0$, $n \geq 0$, *and* $f(X), g(X), h(X) \in X^2\mathbb{Z}[X]$ *with* $f(1) = \pm 1$ *such that* $[x, yx^m - f(y)x^n] = 0$ *and* $[x - g(x), y - h(y)] = 0$.

Theorem 2. *Let* R *be a right s-unital ring, and let* m *and* n *be non-negative integers. Assume that for each* $y \in R$, *there exists* $f(X) \in X^2\mathbb{Z}[X]$ *such that* $[x, yx^m - f(y)x^n] = 0$ *for all* $x \in R$. *Then* R *is commutative.*

Theorem 3. *Let* R *be a right (or left) s-unital ring. Then the following are equivalent:*

(i) R *is commutative.*
(ii) *For each* x, y *in* R, *there exist non-negative integers* $m > 0$, $n \geq 0$ *and* $f(X) \in X^2\mathbb{Z}[X]$ *with* $f(1) = \pm 1$ *such that* $[x, yx^m - f(y)x^n] = 0$, *and for each* $x \in R$, *either* $x \in Z(R)$, *or there exists* $g(X) \in X^2\mathbb{Z}[X]$ *such that* $x - g(x) \in N(R)$.
(iii) *For each* $y \in R$, *there exists* $f(X) \in X^2\mathbb{Z}[X]$ *with* $f(1) = \pm 1$ *such that* $[x, yx^m - f(y)x^n] = 0$ *for all* $x \in R$, *provided* m, n *are fixed non-negative integers.*

Theorem 4. *Let* R *be a right s-unital ring. Suppose that* R *satisfies a polynomial identity*

$$[f(X), Y]X^m + \lambda(X,Y)[X, g(Y)]\lambda^*(X,Y) = 0,$$

where m *is a non-negative integer,* $\lambda(X,Y)$ *and* $\lambda^*(X,Y)$ *are monic monomials in* $\mathbb{Z}<X,Y>$, $f(X)$ *and* $g(X)$ *are polynomials in* $X\mathbb{Z}[X]$ *with*

$f(1)=\pm 1$ and $g(1)=\pm 1$, and every monomial of $\lambda(X,Y)g(Y)\lambda^*(X,Y)$ has degree ≥ 2 in Y. Suppose that $n=(f'(1),g'(1))$ is non-zero, where $f'(X)$ and $g'(X)$ are the usual derivatives of $f(X)$ and $g(X)$ respectively. If R satisfies the property $\mathbf{Q}(n)$, then R is commutative.

Following [4], let **P** be a ring property. If **P** is inherited by every subring and every homomorphic image, then **P** is called an **h**-property. More weakly, if **P** is inherited by every finitely generated subring and every natural homomorphic image modulo the annihilator of a central element, then **P** is called an **H**-property.

A ring property **P** such that a ring R has the property **P** if and only if all its finitely generated subrings have **P**, is called an **F**-property.

Lemma 2 ([4, Prop. 1]). *Let **P** be an **H**-property, and let **P**$'$ be an **F**-property. If every ring R with unity 1 having the property **P** has the property **P**$'$, then every s-unital ring having **P** has **P**$'$.*

Lemma 3 ([3, Th.]). *If for every x, y in a ring R, we can find a polynomial $p_{x,y}(t)$ with integer coefficients which depend on x and y such that $[x^2p_{x,y}(x)-x,y]=0$, then R is commutative.*

Lemma 4 ([1, Lemma]). *Let R be a ring with unity 1. If for each $x,y\in$ $\in R$, there exists an integer $m=m(x,y)\geq 1$ such that $x^m[x,y]=0$, or $[x,y]x^m=0$, then necessarily $[x,y]=0$.*

Lemma 5 ([5, Th.]). *Let f be a polynomial in non-commuting indeterminates $x_1,x_2,\ldots,x_n$ with coprime integer coefficients. Then the following statements are equivalent:*

(1) *For any ring R satisfying $f=0$, $C(R)$ is a nil ideal.*

(2) *For every prime p, $(GF(p))_2$ fail to satisfy $f=0$.*

In [2], Chacron defined the cohypercenter $C'(R)$ of a ring R as the set of all elements $a\in R$ such that for each $x\in R$ there holds $[a,x-f(x)]=0$ with some $f(X)\in X^2\mathbb{Z}[X]$, which is a commutative subring of R ([2, Remark 12]). Indeed Chacron proved the following result:

Theorem C (Chacron, [2]). *Suppose that R satisfies the following condition:*

(C) *For each $x,y\in R$, there exist $f(X),g(X)\in X^2\mathbb{Z}[X]$ such that $[x-f(x),y-g(y)]=0$.*

Then we have the following:

(1) *$C'(R)$ is a commutative subring of R containing $N(R)$;*

(2) *$N(R)$ is a commutative ideal of R containing $C(R)$;*

(3) *$N(R)[C'(R),R]=[C'(R),R]N(R)=0$ and $[C'(R),R]\subseteq N^*(R)$.*

In this paper, we hall study rings satisfying condition (C) of Th. C by making use of the recent result of W. Streb [8], which we called *Streb's classification.*

Theorem SC (Streb [8]). *Suppose that a ring R satisfies the following condition:*

(SC) *For each $x, y \in R$, there exists a polynomial $f(X,Y) \in \in \mathbb{Z}<X,Y>[X,Y]\mathbb{Z}<X,Y>$ each of whose monomial terms is of length ≥ 3 such that $[x,y] = f(x,y)$.*

Then there exists no factor subring of R which is of type (e) *or* (f). *Therefore, if R is non-commutative, then there exists a factor subring of R which is of type* (a), (b), (c) *or* (d).

The next result is crucial in our subsequent study is immediate by Th. C, and Th. SC.

Theorem KT. *Suppose that a ring R satisfies* (C). *Then there exists no factor subring of R which is of type* (c), (d), (e) *or* (f). *Therefore, if R is non-commutative, then there exists a factor subring of R which is of type* (a) *or* (b).

Proof of Th. 1. Let p be prime. Consider the ring $\begin{bmatrix} GF(p) & GF(p) \\ 0 & GF(p) \end{bmatrix}$. Set $x = e_{22}$ and $y = e_{12}$ in our hypothesis to obtain

$$[e_{22}, e_{12}e_{22}^m - f(e_{12})e_{22}^n] \neq 0$$

for all integers $m > 0$, $n \geq 0$ and $f(X) \in X^2\mathbb{Z}[X]$ with $f(1) = \pm 1$. Further, consider the ring $M_\eta(\mathbf{K})$, a ring of type (b). Let $x = \begin{bmatrix} \gamma & 0 \\ 0 & \eta(\gamma) \end{bmatrix}$, $(\eta(\gamma) \neq \gamma)$ and $y = e_{12}$. Then

$$[x, yx^m - f(y)x^n] = [x,y]x^m = y(\gamma - \eta(\gamma))\gamma^m \neq 0$$

for all integers $m > 0$, $n \geq 0$ and $f(X) \in X^2\mathbb{Z}[X]$. Hence, R is commutative by Th. KT. ◊

Corollary 1. *Suppose that for each $x, y \in R$, there exist integers $l > 1$, $m > 0$, $n \geq 0$, and $f(X), g(X) \in X^2\mathbb{Z}[X]$ such that $[x, yx^m - y^l x^n] = 0$ and $[x - f(x), y - g(y)] = 0$. Then R is commutative.*

Lemma 6. *If R is a right s-unital and not left s-unital, then R has a factor subring of type* $(a)_1$.

Proof. There exists $x \in R$ such that $x \notin xR$, (R is not left s-unital). Let $e, f \in R$ such that $xe = x$ and $ef = e$. Then $xf = x$. Put $y = x - - fx$. Then $y \neq 0$, $y^2 = 0$, $ye = y$ and $ey = 0$. Let M be an ideal of $< e, y >$ which is maximal with respect to $y \notin M$. Put $I =< e, y > /M$,

$\bar{e} = e + M$, $\bar{y} = y + M$. Thus $\bar{y}\bar{e} = \bar{y}$ and $\bar{e}\bar{y} = 0 = \bar{y}^2$. So we have $I = \langle\bar{e}\rangle + \bar{y}\mathbb{Z}$ and $\bar{y}\mathbb{Z}$ is the smallest non-zero ideal of I. Hence $\bar{y}\mathbb{Z}$ is an irreducible right $\langle\bar{e}\rangle$-module. Next, we can see that $A = \{s \in \langle\bar{e}\rangle \mid \bar{y}s = 0\}$ is an ideal of I which does not contain $\bar{y}$, so $A = 0$. Therefore $\langle\bar{e}\rangle$ is a commutative primitive ring and so a field. Since $\bar{e}^2 - \bar{e} \in A = 0$, $I = \bar{e}\mathbb{Z} \oplus \bar{y}\mathbb{Z}$ is of type $(a)_i$. ◊

Proof of Th. 2. Trivially, we can check that no rings of type $(a)_i$ or (b) satisfy our hypothesis. In view of Lemma 6, R is s-unital. Hence, by Lemma 2, we may assume that R with 1. If $m = n = 0$, then $[x, y - f(y)] = 0$. Therefore, R is commutative by Lemma 3. Henceforth, we may assume that $m > 0$, or $n > 0$. Then $x = e_{22}$ and $y = e_{12}$ in $(GF(p))_2$ p prime, fails to satisfy $[x, y]x^m = [x, f(y)]x^n$. Hence, by Lemma 5, R has no factor subrings of type (d). Further, suppose that R has a factor subring T of type (e)'. Take $s, t \in S$ such that $[s, t] \neq 0$. Then there exists $f(X) \in X^2\mathbb{Z}[X]$ such that $[s, t] = [s, t](s + 1)^m - [s, f(t)](s + 1)^n = 0$, which is a contradiction. Therefore, R is commutative by Lemma 1. ◊

Lemma 7. *Let R be a ring with 1. Suppose that for each $x, y \in R$, there exists non-negative integers m, n and $f(X) \in X^2\mathbb{Z}[X]$ such that $[x, yx^m - f(y)x^n] = 0$. Then $N(R) \subseteq Z(R)$.*

Proof. Suppose that $a \in N(R)$, and $a \in R$. Then $[x, a]x^{m_1} = [x, f_1(a)]x^{n_1}$, for $m_1 \geq 0$, $n_1 \geq 0$, and some $f_1(X) \in X^2\mathbb{Z}[X]$. Also, $[x, f_1(a)]x^{m_2} = [x, f_2(f_1(a))]x^{n_2}$, for some $m_2 \geq 0$, $n_2 \geq 0$, and some $f_2(X) \in X^2\mathbb{Z}[X]$. Thus

$$[x, a]x^{m_1+m_2} = [x, f_2(f_1(a))]x^{n_1+n_2}.$$

Continuing this process, we can see that

$$[x, a]x^{m_1+\ldots+m_t} = [x, f_t(\cdots f_1(a)\cdots)]x^{n_1+\cdots+n_t},$$

for some $m_k \geq 0$, $n_k \geq 0$ and some $f_k(X) \in X^2\mathbb{Z}[X]$, $k = 1, \cdots, t$. Since $a \in N(R)$, for sufficiently large t, we get

$$[x, a]x^{m_1+\cdots+m_t} = 0,$$

and so

$$[x, a](x + 1)^{m_1+\cdots+m_t} = 0,$$

for $m_1 + \cdots + m_t \geq 0$. By Lemma 4, $[x, a] = 0$. Thus, $N(R) \subseteq Z(R)$. ◊

Proof of Theorem 3. It suffices to show that each of (ii) and (iii) implies (i).

(ii)⇒(i): Consider the ring $(GF(p))_2$, p a prime. Then we see that $[e_{22}, e_{12}e_{22}^m - f(e_{12})e_{22}^n] = e_{12} \neq 0$, for any integers $m > 0$, $n \geq 0$ and $f(X) \in X^2\mathbb{Z}[X]$ with $f(1) = \pm 1$. Accordingly, R has no factor subrings of type $(a)_i$. Thus in view of Lemma 6 and its dual, R is s-unital. By Lemma 2, we may assume that R has unity 1. Since $N(R) \subseteq Z(R)$, by Lemma 7, R satisfies all the hypotheses of Th. 1. Therefore, R is commutative.

(iii)⇒(i): In case $m > 0$, we have shown above that R has no factor subrings of type $(a)_{iii}$. If $m = 0$, then we consider in $(GF(p))_2$, p a prime, $x = e_{22}$ and $y = e_{12}$ in our hypotheses to obtain $[e_{22}, e_{12}e_{22}^m - f(e_{12})e_{22}^n] \neq 0$ for any integer $n \geq 0$ and $f(X) \in X^2\mathbb{Z}[X]$. Hence, R has no factor subrings of type $(a)_i$. In view of the dual of Lemma 6, if R is left s-unital, then R is also right s-unital. By Th. 2, R is commutative. ◊

Corollary 2. *If R is a right (or left) s-unital ring, then the following conditions are equivalent:*

(1) *R is commutative.*

(2) *For each $x, y \in R$, there exist integers $l > 1$, $m > 0$, $n \geq 0$ such that $[x, yx^m - y^l x^n] = 0$, and for each $x \in R$, either $x \in Z(R)$ or there exists $f(X) \in X^2\mathbb{Z}[X]$ such that $x - f(x) \in N(R)$.*

(3) *For each $y \in R$, there exists an integer $l > 1$ such that $[x, yx^m - y^l x^n] = 0$, for all $x \in R$, where m, n are fixed non-negative integers.*

Following Kobayashi [6], let Θ be the additive mapping of $\mathbb{Z}\langle X, Y\rangle$ to $\mathbb{Z}$ defined as follows: For each monic monomial $X_1, \cdots, X_t$, (X_i is either X or Y), $\Theta(X_1, \cdots, X_t)$ is the number of pairs (i, j) such that $1 \leq i < j \leq t$ and $X_i = X$, $X_j = Y$. Trivially, one can see that, for any $f(X, Y) \in \mathbb{Z}\langle X, Y\rangle$, $\Theta(f(X, Y))$ equals the coefficient of XY occurring in $f(X+1, Y+1)$.

Let $\mathbf{N}$ be the set of all non-negative integers, $F(X, Y) \in \mathbb{Z}\langle X, Y\rangle$, and $(m, n) \in \mathbf{N} \times \mathbf{N}$. Then (m, n)-component of F, the sum of all monomials of degree (m, n), that is, of degree m with respect to X, and of degree n with respect to Y, is denoted by $F_{m,n}$.

Using the above definition, we state the following:

Lemma 8 ([6, Th.]). *Let R be a ring with unity 1, and let $F(X, Y)$ be a polynomial in $\mathbb{Z}\langle X, Y\rangle$ of total degree d. Suppose that the greatest common divisor of $\{(m-1)!(n-1)!\Theta(F_{m,n}) \mid m + n = d,\ m, n > 0\}$ is positive. If R satisfies the identity $F(X, Y) = 0$, then R satisfies the* *identity* $l(XY - YX) = 0$. *Therefore, if moreover* R *has* $\mathbf{Q}(l)$, *then* R *is commutative.*

Proof of Th. 4. By Lemma 1, it is enough to show that R has no factor subrings of type $(a)_{ii}$, (b), (d) or (f). It is easy to see that no rings of type $(a)_{ii}$ satisfy

$$[f(X), Y]X^m + \lambda(X,Y)[X, g(Y)]\lambda^*(X,Y) = 0,$$

where m is a non-negative integer. In view of Lemma 5, we also see that R has no factor subrings of type (d). Further, by Lemma 6, R is s-unital. Hence, in view of Lemma 2, we may assume that R has unity 1.

The sum of all monomials which have the maximal degree in

$$[f(X), Y]X^m + \lambda(X,Y)[X, g(Y)]\lambda^*(X,Y)$$

is one of the following:

$$a[X^k, Y]X^m, \qquad b\lambda(X,Y)[X, Y^l]\lambda^*(X,Y),$$

and

$$a[X^k, Y]X^m + b\lambda(X,Y)[X, Y^l]\lambda^*(X,Y),$$

where aX^k and bY^l are the leading terms of $f(X)$ and $g(Y)$, respectively. Now it is easy to see that

$$\Theta(a[X^k, Y]X^m) = ak \text{ and } \Theta(b\lambda(X,Y)[X, Y^l]\lambda^*(X,Y) = bl.$$

Hence, by Lemma 8 there exists a positive integer, n such that $n[x,y] = 0$ for all $x, y \in R$. Since R satisfies $\mathbf{Q}(d)$, we may assume that $(n,d) = 1$. If T is any factor subring of R, then T inherits the property that $n[x,y] = 0$ for all $x, y \in T$. Thus T satisfies $\mathbf{Q}(d)$.

Next, suppose that $R = M_\eta(\mathbf{K})$. Let $c = \begin{bmatrix} a & 0 \\ 0 & \eta(a) \end{bmatrix}$, $(\eta(a) \neq a)$, $e = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Then, by our assumption, we get $[f(c), e]c^m = -\lambda(c,e)[c, g(e)]\lambda^*(c,e) = 0$. But c is invertible, so we have $[f(c), e] = 0$. So $[f(c), 1+e]c^m = -\lambda(c, 1+e)[c, g(1+e)]\lambda^*(c, 1+e) = 0$. Therefore, $g'(1)[c,e] = [c, g(1+e)] = 0$. Now, $[f(c), c+e]c^m = -\lambda(c, c+e)[c, g(c+e)]\lambda^*(c, c+e)$ and both c and $c+e$ are invertibe, then we obtain $[c, g(c+e)] = 0$. We have

$$g(c+e) = \begin{bmatrix} g(a) & (\eta(g(a)) - g(a))(\eta(a) - a)^{-1} \\ 0 & \eta(g(a)) \end{bmatrix}.$$

Therefore, $[c, g(c+e)] = 0$ means that $\eta(g(a)) = g(a)$, and this implies that $[g(c), e] = 0$. Hence, it follows that

$$[f(1+e), c](1+e)^m = -\lambda(1+e, c)[1+e, g(c)]\lambda^*(1+e, c) = 0,$$

and hence $[e, c]f'(1) = [f(1+e), c] = 0$. This together with $[c, e]g'(1) = 0$ implies that $d[c, e] = 0$. By $\mathbf{Q}(d)$, we get $[c, e] = 0$. Thus we have a contradiction.

Finally, we suppose that R is of type (e)$'$. Choose $s, t \in S$ with $[s, t] \neq 0$. Then

$$[s,t]f'(1) = [f(1+s), t](1+s)^m = -\lambda(1+s, t)[1+s, g(t)]\lambda^*(1+s, t) = 0.$$

So $0 = [s,t]f'(1) = [f(1+s), 1+t](1+s)^m = -\lambda(1+s, 1+t)[1+s, g(1+t)]\lambda^*(1+s, 1+t) = -[s,t]g'(1)$. Hence $d[s,t] = 0$. By $\mathbf{Q}(d)$, we have $[s, t] = 0$ which is a contradiction. ◊

Corollary 3. *Let R be a right or left s-unital ring. Suppose that R satisfies the polynomial identity $[f(X), Y]X^m + [X, g(Y)]\lambda^*(X, Y) = 0$, where m is a non-negative integer, $\lambda^*(X, Y)$ is a monic monomial in $\mathbb{Z}<X, Y>$, $f(X)$, $g(X)$ are polynomials in $X\mathbb{Z}[X]$ with $f(1) = \pm 1$, $g(1) = \pm 1$, and every monomial of $g(Y)\lambda^*(X, Y)$ has degree ≥ 2 in Y. Suppose that $d = (f'(1), g'(1))$ is non-zero. If R satisfies $\mathbf{Q}(d)$, then R is commutative.*

Proof. As in the proof of Th. 3, we can see that R has no factor subrings of type (a)$_{\mathrm{i}}$ and R is s-unital. Therefore, R is commutative by Th. 4. ◊

Corollary 5. *Let R be a right or left s-unital ring. Suppose that R satisfies the polynomial identity $[X^k, Y]X^m - [X, Y^l]X^n = 0$, where $k > 0$, $l > 1$, $m \geq 0$, and $n \geq 0$. Let $d = (k, l)$. If R satisfies $\mathbf{Q}(d)$, then R is commutative.*

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PROPER SHAPE INVARIANTS: SMOOTHNESS AND CALMNESS

Zvonko **ČERIN**

41020 Zagreb, Kopernikova 7, Croatia

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Abstract: We study properly $(\mathcal{B},\mathcal{C})$-smooth and properly $\mathcal{C}$-calm spaces, where $\mathcal{B}$ and $\mathcal{C}$ denote classes of topological spaces. Both proper smoothness and proper calmness are invariants of a recently invented author's proper shape theory and are described by the use of proper multi-valued functions. The dual notions are also examined.

1. Introduction

The notions and results in this paper belong to the part of topology that could be described as proper shape theory. As shape theory is an improved homotopy theory designed to handle more successfully complicated spaces so is proper shape theory a modification of proper homotopy theory made with the same goal to provide us with a new insight into global properties even of those spaces for which the classical proper homotopy gives doubtful information.

In [7] the author has described proper shape category of all topological spaces using Sanjurjo's method of multi-valued functions from [12]. Our approach was formally very similar to the one taken by Ball and Sher [2]. Instead of proper fundamental nets we considered proper multi-nets. The other steps were identical. We defined a notion of a proper homotopy for proper multi-nets and took for the morphisms of the proper shape category $\mathcal{S}h_p$ proper homotopy classes of proper multi-nets.

In the present paper we shall introduce and investigate proper shape invariants called smoothness and calmness. It is useful to consider these notions in terms of arbitrary classes $\mathcal{B}$ and $\mathcal{C}$ of topological spaces. In other words, we shall define $M_p^{\mathcal{B},\mathcal{C}}$-smooth and $M_p^{\mathcal{C}}$-calm spaces and explore their properties. In our notation the letter "M" suggests the use of multi-valued functions while "p" replaces "proper" or "properly".

Let us describe the content of the paper in greater detail. In §2 we recall notions and results from [7] that are necessary in further developments. The next §3 studies $M_p^{\mathcal{B},\mathcal{C}}$-smooth spaces. The idea is that we require that small enough proper multi-valued functions from members of a class of spaces $\mathcal{B}$ into a given space X which are properly homotopic over members of another class $\mathcal{C}$ are already properly homotopic through sufficiently small proper multi-valued functions. This concept is related to the notion of n-types of Whitehead and it could be regarded as a substitute for it in the proper shape theory. We prove that this is an invariant in the category $\mathcal{S}h_p$, explore the role of classes $\mathcal{B}$ and $\mathcal{C}$, and study what kind of maps will preserve and inversely preserve $M_p^{\mathcal{B},\mathcal{C}}$-smooth spaces. The classes of proper $\mathcal{B}$-surjections and proper $\mathcal{B}$-injections from [8] are of key importance.

In the following §4 we consider $M_p^{\mathcal{B}}$-calm spaces. The calm spaces have proved useful in shape theory and geometric topology and are dual in many respects to the movable spaces of Borsuk [4] just as smooth spaces are dual to tame spaces which becomes clear when comparing this paper with [9]. For the first time we have now this concepts in the proper shape theory of arbitrary topological spaces.

Since the method of investigating properties of spaces by looking at maps from some objects into a space has an obvious dual approach where we utilize maps from a space into those objects, we also consider in §§5 and 6 so called $N_p^{\mathcal{B},\mathcal{C}}$-smooth, $P_p^{\mathcal{B},\mathcal{C}}$-smooth and $N_p^{\mathcal{B}}$-calm classes of spaces, where the change from the letter "M" to the letters "N" and "P" should reflect duality between these notions. As the reader will see this duality is striking.

Finally, in §7 we consider dependence of these notions on classes $\mathcal{B}$ and $\mathcal{C}$ under the assumption that they are connected with each other by morphisms from [8].

2. Preliminaries on proper shape theory

In this section we shall introduce notions and results from [7] that are required for our theory.

Let X and Y be topological spaces. By a *multi-valued function* or an *M-function* $F\colon X \to Y$ we mean a rule which associates a non-empty subset $F(x)$ of Y to every point x of the space X. An M-function $F\colon X \to Y$ is *própper* provided for every compact subset C of Y its small counterimage $F'(C) = \{x \in X \mid F(x) \subset C\}$ is a compact subset of X. On the other hand, F is *pròper* provided for every compact subset C of Y its big counterimage $F''(C) = \{x \in X \mid F(x) \cap C \neq \emptyset\}$ is a compact subset of X. We shall use the term *proper* to name either própper or pròper. However, in a given situation, once we make a selection between two different kinds of properness it is understood that it will be retained throughout. Instead of proper multi-valued function we shall use the shorter name *M_p-function*

Observe that for single-valued functions the two notions of properness coincide. Classes of própper and pròper M-functions are completely unrelated [7]. It follows that each of our notions and results on M_p-functions actually has two versions.

In this paper by a *cover* we mean an open normal cover [1]. Let $\mathrm{Cov}(Y)$ denote the collection of all covers of a topological space Y. With respect to the refinement relation $>$ the set $\mathrm{Cov}(Y)$ is a directed set. Two covers σ and τ of Y are equivalent provided $\sigma > \tau$ and $\tau > \sigma$. In order to simplify our notation we denote a cover and it's equivalence class by the same symbol. Consequently, $\mathrm{Cov}(Y)$ also stands for the associated quotient set.

If σ is a cover of a space Y, let σ^+ be the collection of all covers of Y which refine σ while $\sigma^\star$ denotes the set of all covers τ of Y such that the star $st(\tau)$ of τ refines σ. Similarly, for a natural number n, $\sigma^{\star n}$ denotes the set of all covers τ of Y such that the n-th star $st^n(\tau)$ of τ refines σ.

Let $\mathrm{Inc}(Y)$ denote the collection of all finite subsets c of $\mathrm{Cov}(Y)$ which have a unique (with respect to the refinement relation) maximal element which we denote by $[c]$. The notation $\mathrm{Inc}(Y)$ comes from "indices of covers". The set $\mathrm{Inc}(Y)$ will be used as indexing set for proper multi-nets into Y. We consider $\mathrm{Inc}(Y)$ ordered by the inclusion relation and regard $\mathrm{Cov}(Y)$ as a subset of single-element subsets of

Cov (Y). Notice that Inc (Y) is a cofinite directed set.

For our proper shape theory the following notion of size for M-functions will play the most important role. Let $F\colon X \to Y$ be an M_p-function and let $\alpha \in \mathrm{Cov}(X)$ and $\gamma \in \mathrm{Cov}(Y)$. We shall say that F is an *$M_p^{\alpha,\gamma}$-function* provided for every $A \in \alpha$ there is a $C_A \in \gamma$ with $F(A) \subset C_A$. On the other hand, F is *γ-small* or an *M_p^{γ}-function* provided there is an $\alpha \in \mathrm{Cov}(X)$ such that F is an $M_p^{\alpha,\gamma}$-function. For an M_p^{σ}-function $F\colon X \to Y$ we use $S(F, \sigma)$ to denote the family of all $\alpha \in \mathrm{Cov}(X)$ such that F is an $M_p^{\alpha,\sigma}$-function.

Next we introduce the notions which correspond to the equivalence relation of proper homotopy for proper maps. Let F and G be M_p-functions from a space X into a space Y and let γ be a cover of Y. We shall say that F and G are *properly γ-homotopic* or *M_p^{γ}-homotopic* and write $F \overset{\gamma}{\sim} G$ provided there is an M_p^{γ}-function H from the product $X \times I$ of X and the unit segment $I = [0, 1]$ into Y such that $F(x) = H(x, 0)$ and $G(x) = H(x, 1)$ for every $x \in X$. We shall say that H is a *proper γ-homotopy* or an *M_p^{γ}-homotopy* that *joins* F and G or that it *realizes* the relation $F \overset{\gamma}{\sim} G$.

The following lemma from [7] is crucial because it provides an adequate substitute for the transitivity of the relation of proper homotopy.

2.1 Lemma. *Let F, G, and H be M_p-functions from a space X into a space Y. Let $\sigma \in \mathrm{Cov}(Y)$ and $\tau \in \sigma^*$. If $F \overset{\tau}{\sim} G$ and $G \overset{\tau}{\sim} H$, then $F \overset{\sigma}{\sim} H$.*

The proof of Lemma 2.1 requires an interesting proposition from A. Dold's book [10, p. 358] on covers of the product $X \times I$ of a space X with the unit segment I. We assume that the reader is familiar with this result and the notion of a stacked covering of $X \times I$ over a cover of X. For a cover σ of $X \times I$, we shall use $D(X, \sigma)$ to denote the collection of all covers τ of X such that some stacked covering of $X \times I$ over τ refines σ. As a consequence of the above proposition, this collection is always non-empty.

The following two definitions correspond to Ball and Sher's definitions of proper fundamental net and proper homotopy for proper fundamental nets.

Let X and Y be topological spaces. By a *proper multi-net* or an M_p-net from X into Y we shall mean a collection $\varphi = \{F_c\}_{c \in \mathrm{Inc}(Y)}$ of M_p-functions $F_c\colon X \to Y$ such that for every $\gamma \in \mathrm{Cov}(Y)$ there is a $c \in \mathrm{Inc}(Y)$ with $F_d \overset{\gamma}{\sim} F_c$ for every $d > c$. We use functional notation $\varphi\colon X \to Y$ to indicate that φ is an M_p-net from X into Y. Let $MN_p(X, Y)$ denote all M_p-nets $\varphi\colon X \to Y$.

Two M_p-nets $\varphi = \{F_c\}$ and $\psi = \{G_c\}$ between topological spaces X and Y are *M_p-homotopic* and we write $\varphi \sim \psi$ provided for every $\gamma \in \mathrm{Cov}(Y)$ there is a $c \in \mathrm{Inc}(Y)$ such that $F_d \overset{\gamma}{\sim} G_d$ for every $d > c$. On the other hand, we write $\varphi \overset{\gamma}{\sim} \psi$ and call φ and ψ *M_p^γ-homotopic* provided there is a $c \in \mathrm{Inc}(Y)$ such that $F_d \overset{\gamma}{\sim} G_d$ for every $d > c$.

It follows from Lemma 2.1 that the relation of M_p-homotopy is an equivalence relation on the set $MN_p(X, Y)$. The M_p-homotopy class of an M_p-net φ is denoted by $[\varphi]$ and the set of all M_p-homotopy classes by $\mathcal{S}h_p(X, Y)$.

Our goal now is to define a composition for M_p-homotopy classes of M_p-nets. Let $\varphi = \{F_c\}\colon X \to Y$ be a M_p-net. Let $\varphi\colon \mathrm{Inc}(Y) \to \to \mathrm{Inc}(Y)$ be an increasing function such that for every $c \in \mathrm{Inc}(Y)$ the relation $d, e > \varphi(c)$ implies the relation $F_d \overset{[c]}{\sim} F_e$. Here we make an assumption that an increasing function φ from a partially ordered set P into itself always satisfies the condition that $\varphi(p) > p$ for every $p \in P$. Let $\mathcal{C} = \{(c, d, e) \mid c \in \mathrm{Inc}(Y),\ d, e > \varphi(c)\}$. Then $\mathcal{C}$ is a subset of $\mathrm{Inc}(Y) \times \mathrm{Inc}(Y) \times \mathrm{Inc}(Y)$ that becomes a cofinite directed set when we define that $(c, d, e) > (c', d', e')$ if and only if $c > c'$, $d > d'$, and $e > e'$. We shall use the same notation φ for an increasing function $\varphi\colon \mathcal{C} \to \mathrm{Cov}(X \times I)$ such that F_d and F_e are joined by a proper $(\varphi(c, d, e), [c])$-homotopy whenever $(c, d, e) \in \mathcal{C}$. Let $\bar{\varphi}\colon \mathcal{C} \to \mathrm{Inc}(X)$ be an increasing function such that $[\bar{\varphi}(c, d, e)] \in D(X, \varphi(c, d, e))$ for every $(c, d, e) \in \mathcal{C}$. In [7] it was proved that there is an increasing function $\varphi^*\colon \mathrm{Inc}(Y) \to \mathrm{Inc}(X)$ such that (1) $\varphi^*(c) > \bar{\varphi}(c, \varphi(c), \varphi(c))$ for every $c \in \mathrm{Inc}(Y)$, and (2) φ^* is cofinal in $\bar{\varphi}$, i. e., for every $(c, d, e) \in \in \mathcal{C}$ there is an $m \in \mathrm{Inc}(Y)$ with $\varphi^*(m) > \bar{\varphi}(c, d, e)$. With the help of functions φ and φ^* we shall define the composition of M_p-homotopy classes of M_p-nets as follows.

Let $\varphi = \{F_c\}\colon X \to Y$ and $\psi = \{G_s\}\colon Y \to Z$ be M_p-nets. Let $\chi = \{H_s\}$, where $H_s = G_{\psi(s)} \circ F_{\varphi(\psi^*(s))}$ for every $s \in \mathrm{Inc}(Z)$. Observe that each H_s is a M_p-function because the composition of two M_p-functions is an M_p-function. In [7] it was proved that the collection χ is an M_p-net from X into Z. We now define the composition of M_p-homotopy classes of M_p-nets by the rule $[\{G_s\}] \circ [\{F_c\}] =$

$= [\{G_{\psi(s)} \circ F_{\varphi(\psi^*(s))}\}]$. This composition of M_p-homotopy classes of M_p-nets is well-defined and associative.

For a space X, let $\iota^X = \{I_a\}\colon X \to X$ be the identity M_p-net defined by $I_a = \mathrm{id}_X$ for every $a \in \mathrm{Inc}(X)$. It is easy to show that for every M_p-net $\varphi\colon X \to Y$, the following relations hold: $[\varphi] \circ [\iota^X] = [\varphi] = [\iota^Y] \circ [\varphi]$.

We can summarize the above with the following main result from [7].

2.2 Theorem. *The topological spaces as objects together with the M_p-homotopy classes of M_p-nets as morphisms and the composition of M_p-homotopy classes form the proper shape category $\mathcal{S}h_p$.*

The above constructions may be done without any reference to própet and pròper M-functions. In this way we shall get the shape category $\mathcal{S}h$. On the other hand, in both cases, we may require that all M-functions belong to a class of M-functions which is closed with respect to pastings from the proof of Lemma 2.1 in [7] and compositions. In particular, we may assume that they are either upper semi-continuous or lower semi-continuous.

3. $M_p^{\mathcal{B},\mathcal{C}}$-smooth spaces

In this section we shall explore the following interesting notion which in the case of compacta reduces to the author's $(\mathcal{B}, \mathcal{C})$-smoothness from [5] and [6].

Let $\mathcal{D}$ be a class of spaces, let F and G be M_p-functions from a space X into a space Y, and let σ be a cover of Y. We shall say that F and G are *properly σ-homotopic over $\mathcal{D}$* and write $F \overset{\tau}{\sim}_{\mathcal{D}} G$ provided there is a cover τ of X such that for every M_p^{τ}-function H from a member of $\mathcal{D}$ into X the compositions $F \circ H$ and $G \circ H$ are M_p^{σ}-homotopic.

Let $\mathcal{B}$ and $\mathcal{C}$ be classes of topological spaces. A space X is $M_p^{\mathcal{B},\mathcal{C}}$-*smooth* provided for every cover σ of X there is a cover τ of X with the property that for M_p^{τ}-functions F and G from a member of $\mathcal{B}$ into X the relation $F \overset{\tau}{\sim}_{\mathcal{C}} G$ implies the relation $F \overset{\sigma}{\sim} G$. A class of spaces is $M_p^{\mathcal{B},\mathcal{C}}$-smooth provided each member of it is $M_p^{\mathcal{B},\mathcal{C}}$-smooth.

We shall first show that the property of being $M_p^{\mathcal{B},\mathcal{C}}$-smooth is a proper shape invariant, i. e., that if X and Y are equivalent objects of the category $\mathcal{S}h_p$ and X is $M_p^{\mathcal{B},\mathcal{C}}$-smooth then Y is also $M_p^{\mathcal{B},\mathcal{C}}$-smooth.

In fact, a much better result is true. The $M_p^{\mathcal{B},\mathcal{C}}$-smooth spaces are preserved under the following weak form of domination.

A class of spaces $\mathcal{B}$ is *M_p-dominated* by a class of spaces $\mathcal{A}$ provided for every $B \in \mathcal{B}$ and every $\beta \in \mathrm{Cov}(B)$ there is an $A \in \mathcal{A}$ and an M_p^β-function $G\colon A \to B$ such that for every $\alpha \in \mathrm{Cov}(A)$ we can find an M_p^α-function $F\colon B \to A$ with $G \circ F \overset{\beta}{\sim} \mathrm{id}_B$.

3.1 Theorem *A space X is $M_p^{\mathcal{B},\mathcal{C}}$-smooth if and only if it is M_p-dominated by a class of $M_p^{\mathcal{B},\mathcal{C}}$-smooth spaces.*

Proof. Since every space M_p-dominates itself, it remains to prove the "if" part. Let a cover σ of X be given. Let $\eta \in \sigma^*$. By assumption, there is an $M_p^{\mathcal{B},\mathcal{C}}$-smooth space Y and an M_p^η-function $D\colon Y \to X$ such that for every $\varepsilon \in \mathrm{Cov}(Y)$ there is an M_p^ε-function $U\colon X \to Y$ with $G \circ F \overset{\eta}{\sim} \mathrm{id}_X$.

Let $\delta \in S(D, \eta)$. Since Y is $M_p^{\mathcal{B},\mathcal{C}}$-smooth, there is an $\varepsilon \in \mathrm{Cov}(Y)$ such that for every M_p^ε-functions K and L from a member of $\mathcal{B}$ into Y the relation $K \overset{\varepsilon}{\sim}_{\mathcal{C}} L$ implies the relation $K \overset{\delta}{\sim} L$.

Pick a U as above. Let W be an M_p^η-homotopy joining id_X and $D \circ U$. Let $\tau \in \mathrm{Cov}(X)$ belong to $D(W, \eta)$ and $S(F, \varepsilon)$. Then τ is the required cover of X. To verify this, consider a member B of $\mathcal{B}$ and M_p^τ-functions $F, G\colon B \to X$ with $F \overset{\tau}{\sim}_{\mathcal{C}} G$. Let K and L be $U \circ F$ and $U \circ G$. Then K and L are M_p^ε-functions from B into Y with $K \overset{\varepsilon}{\sim}_{\mathcal{C}} L$. It follows that $K \overset{\delta}{\sim} L$ so that after composing with D we obtain $F \overset{\eta}{\sim} D \circ U \circ F = D \circ K \overset{\eta}{\sim} D \circ L = D \circ U \circ G \overset{\eta}{\sim} G$. Hence, $F \overset{\sigma}{\sim} G$. ◊

The M_p-domination is weaker than the quasi $\mathcal{S}h_p$-domination and thus also weaker than $\mathcal{S}h_p$-domination [8]. Recall that a class of spaces $\mathcal{A}$ is *$\mathcal{S}h_p$-dominated* by a class of spaces $\mathcal{B}$ provided for every $X \in \mathcal{A}$ there is a $Y \in \mathcal{B}$ and M_p-nets $\varphi\colon X \to Y$ and $\psi\colon Y \to X$ with the composition $\psi \circ \varphi$ M_p-homotopic to the identity M_p-net ι^X on X. On the other hand, $\mathcal{A}$ is *quasi $\mathcal{S}h_p$-dominated* by $\mathcal{B}$ provided for every $X \in \mathcal{A}$ and every $\sigma \in \mathrm{Cov}(X)$ there is a $Y \in \mathcal{B}$ and M_p-nets $\varphi\colon X \to Y$ and $\psi\colon Y \to X$ with the composition $\psi \circ \varphi$ M_p^σ-homotopic ι^X. The notion of quasi $\mathcal{S}h_p$-domination is similar to the notion of quasi-domination in [3].

3.2 Corollary. *A space is $M_p^{\mathcal{B},\mathcal{C}}$-smooth if and only if it is either $\mathcal{S}h_p$-dominated or quasi $\mathcal{S}h_p$-dominated by a class of $M_p^{\mathcal{B},\mathcal{C}}$-smooth spaces.*

Another example of M_p-domination provides the notion of being properly $\mathcal{B}$-like. Recall that a space X is *properly $\mathcal{B}$-like*, where $\mathcal{B}$ is a class of spaces, provided for every $\sigma \in \operatorname{Cov}(X)$ there is a member Y of $\mathcal{B}$ and a proper map $f: X \to Y$ such that the inverse $f^{-1}: Y \to X$ is an M_p^σ-function. In [8] we showed that if a space X is properly $\mathcal{B}$-like, then X is M_p-dominated by $\mathcal{B}$. Hence, we get the following conclusion.

3.3 Corollary. *A space X is $M_p^{\mathcal{B},\mathcal{C}}$-smooth if and only if it is properly $\mathcal{D}$-like, where $\mathcal{D}$ is a class of $M_p^{\mathcal{B},\mathcal{C}}$-smooth spaces.*

In the following two theorems we explore in which way does the definition of $M_p^{\mathcal{B},\mathcal{C}}$-smooth spaces depend on classes $\mathcal{B}$ and $\mathcal{C}$. The first result uses the following notion from [9].

Let $\mathcal{B}$ and $\mathcal{C}$ be classes of spaces. A space X is *$M_p^{\mathcal{B},\mathcal{C}}$-tame* provided for every $\sigma \in \operatorname{Cov}(X)$ there is a $\tau \in \operatorname{Cov}(X)$ such that for every $B \in \mathcal{B}$ and every M_p^τ-function $F: B \to X$ there is a $C \in \mathcal{C}$ and an M_p^σ-function $H: C \to X$ with the property that for every $\alpha \in \operatorname{Cov}(C)$ there is an M_p^α-function $G: B \to C$ with $F \overset{\sigma}{\sim} H \circ G$. A class of spaces is $M_p^{\mathcal{B},\mathcal{C}}$-tame provided each member of it is $M_p^{\mathcal{B},\mathcal{C}}$-tame.

3.4 Theorem. *Let $\mathcal{A}$ and $\mathcal{C}$ be classes of topological spaces and let $\mathcal{B}$ be a class of $M_p^{\mathcal{A},\mathcal{C}}$-tame spaces. Then every $M_p^{\mathcal{B},\mathcal{A}}$-smooth space X is also $M_p^{\mathcal{B},\mathcal{C}}$-smooth.*

Proof. Let a cover σ of X be given. Let $\mu \in \sigma^*$. Pick a $\mu \in \operatorname{Cov}(X)$ such that for M_p^μ-functions F and G from a member of $\mathcal{B}$ into X the relation $F \overset{\mu}{\sim}_{\mathcal{A}} G$ implies the relation $F \overset{\sigma}{\sim} G$. Let $\tau \in \mu^*$.

Consider M_p^τ-functions $F, G: B \to X$ such that $B \in \mathcal{B}$ and $F \overset{\tau}{\sim}_{\mathcal{C}} G$. Then there is a $\beta \in \operatorname{Cov}(B)$ with the property that for every M_p^β-function K from a member of $\mathcal{C}$ into B the compositions $F \circ K$ and $G \circ K$ are M_p^τ-homotopic. Let a $\gamma \in \beta^+$ belong to both $S(F, \tau)$ and $S(G, \tau)$. Since B is $M_p^{\mathcal{A},\mathcal{C}}$-tame, there is a $\delta \in \operatorname{Cov}(B)$ such that for every $A \in \mathcal{A}$ and every M_p^δ-function $L: A \to B$ there is a $C \in \mathcal{C}$ and an M_p^γ-function $K: C \to B$ so that for every $\alpha \in \operatorname{Cov}(C)$ there is an M_p^α-function $J: A \to C$ with $K \overset{\gamma}{\sim} K \circ J$.

Let $A \in \mathcal{A}$ and let $L: A \to B$ be an M_p^δ-function. Pick a C and a K as above. Our choices imply $F \circ K \overset{\tau}{\sim} G \circ K$. Let N be a M_p^τ-homotopy which realizes this relation. Let $\alpha \in D(N, \tau)$. Choose a J as above. Then we obtain $F \circ L \overset{\tau}{\sim} F \circ K \circ J \overset{\tau}{\sim} G \circ K \circ J \overset{\tau}{\sim} G \circ L$. It follows that $F \overset{\mu}{\sim}_{\mathcal{A}} G$ and therefore $F \overset{\sigma}{\sim} G$. $\Diamond$

3.5 Theorem. *Let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, and $\mathcal{D}$ be classes of spaces such that $\mathcal{B}$ and $\mathcal{D}$ are M_p-dominated by $\mathcal{A}$ and $\mathcal{C}$, respectively. If a space X is $M_p^{\mathcal{A},\mathcal{D}}$-smooth, then it is also $M_p^{\mathcal{B},\mathcal{C}}$-smooth.*

Proof. Let a cover σ of X be given. Let $\pi \in \sigma^*$. Since X is $M_p^{\mathcal{A},\mathcal{D}}$-smooth, there is a $\varrho \in \pi^+$ such that for M_p^{ϱ}-functions K and L from a member of $\mathcal{A}$ into X the relation $K \overset{\varrho}{\sim}_{\mathcal{D}} L$ implies the relation $K \overset{\pi}{\sim} L$. Let $\tau \in \varrho^*$. Then τ is the required cover of X. Indeed, consider a member B of $\mathcal{B}$ and M_p^{τ}-functions $F, G \colon B \to X$ such that $F \overset{\tau}{\sim}_{\mathcal{C}} G$. Let $\beta \in \operatorname{Cov}(B)$ belong to both $S(F, \tau)$ and $S(G, \tau)$ and be such that for every M_p^{β}-function R from a member of $\mathcal{C}$ into B we have $F \circ R \overset{\tau}{\sim} G \circ R$. Since the class $\mathcal{B}$ is M_p-dominated by the class $\mathcal{A}$, there is an $A \in \mathcal{A}$ and an M_p^{β}-function $J \colon A \to B$ such that for every $\alpha \in \operatorname{Cov}(A)$ there is an M_p^{α}-function $E \colon B \to A$ with $J \circ E \overset{\beta}{\sim} \mathrm{id}_B$.

Let K and L be the compositions $F \circ J$ and $G \circ J$, respectively. Then K and L are M_p^{ϱ}-functions from A into X. We claim that $K \overset{\varrho}{\sim}_{\mathcal{D}} L$.

In order to verify this, let $\alpha \in S(J, \beta)$. Suppose that $D \in \mathcal{D}$ and $T \colon D \to A$ is an M_p^{α}-function. Let $\delta \in S(T, \alpha)$. We utilize now the assumption that the class $\mathcal{D}$ is M_p-dominated by the class $\mathcal{C}$ to select a $C \in \mathcal{C}$ and an M_p^{δ}-function $W \colon C \to D$ with the property that for every cover γ of C there is an M_p^{γ}-function $V \colon D \to C$ with $W \circ V \overset{\delta}{\sim} \mathrm{id}_D$. The composition $J \circ T \circ W$ is an M_p^{β}-function from C into B. It follows that there is an M_p^{τ}-homotopy $P \colon C \times I \to X$ joining $F \circ J \circ T \circ W$ and $G \circ J \circ T \circ W$. Let $\gamma \in D(P, \tau)$. Choose a V as above. Then we have $K \circ T = F \circ J \circ T \overset{\tau}{\sim} F \circ J \circ T \circ W \circ$ $\circ V \overset{\tau}{\sim} G \circ J \circ T \circ W \circ V \overset{\tau}{\sim} G \circ J \circ T = L \circ T$. Hence, $K \circ T \overset{\varrho}{\sim} L \circ T$ and the claim has been verified.

Now our assumption implies existence of an M_p^{π}-homotopy $Q \colon A \times I \to X$ joining K and L. Let $\alpha \in D(Q, \pi)$. Pick an E as above. Then we obtain that $F \overset{\pi}{\sim} F \circ J \circ E = K \circ E \overset{\pi}{\sim} L \circ E =$ $= G \circ J \circ E \overset{\pi}{\sim} G$. Hence, $F \overset{\sigma}{\sim} G$. ◊

3.6 Corollary. *Let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, and $\mathcal{D}$ be classes of spaces such that $\mathcal{B}$ and $\mathcal{D}$ are (quasi) Sh_p-dominated by $\mathcal{A}$ and $\mathcal{C}$, respectively. If a space X is $M_p^{\mathcal{A},\mathcal{D}}$-smooth, then it is also $M_p^{\mathcal{B},\mathcal{C}}$-smooth.*

The following weak form of the notion of being properly $\mathcal{B}$-like is more in line with our point of view because it is based on M_p-functions. It offers us the possibility to improve Cor. 3.3 in Th. 3.7.

Let $\mathcal{C}$ be a class of spaces. A space X is $M_p^{\mathcal{C}}$-*like* provided for every $\sigma \in \mathrm{Cov}(X)$ there is a member Y of $\mathcal{C}$ and a cover α of Y such that for every $\beta \in \mathrm{Cov}(Y)$ there is an M_p^β-function $F\colon X \to Y$ such that F^{-1} is an $M_p^{\alpha,\sigma}$-function.

3.7 Theorem *A space X is $M_p^{\mathcal{B},\mathcal{C}}$-smooth if and only if it is $M_p^{\mathcal{D}}$-like, where $\mathcal{D}$ is a class of $M_p^{\mathcal{B},\mathcal{C}}$-smooth spaces.*

Proof. Let a cover σ of X be given. Let $\mu \in \sigma^*$. Since X is $M_p^{\mathcal{D}}$-like, there is a $Y \in \mathcal{D}$ and a cover α of Y such that for every $\beta \in \mathrm{Cov}(Y)$ there is an M_p^β-function $R\colon X \to Y$ such that R^{-1} is an $M_p^{\alpha,\mu}$-function. Since Y is $M_p^{\mathcal{B},\mathcal{C}}$-smooth, there is a $\beta \in \mathrm{Cov}(Y)$ such that for M_p^β-functions K and L from a member of $\mathcal{B}$ into Y the relation $K \overset{\beta}{\sim}_{\mathcal{C}} L$ implies the relation $K \overset{\alpha}{\sim} L$. Choose an R as above and let $\tau \in S(R, \beta)$. The cover τ is the one we were looking for. In fact, let $B \in \mathcal{B}$ and assume that $F, G\colon B \to X$ are M_p^τ-functions with $F \overset{\tau}{\sim}_{\mathcal{C}} G$. Let K and L be $R \circ F$ and $R \circ G$. Then K and L are M_p^β-functions from B into Y and $K \overset{\beta}{\sim}_{\mathcal{C}} L$. As in the proof of Theorem 3.4 in [9], it follows that $F \overset{\mu}{\sim} R^{-1} \circ R \circ F \overset{\mu}{\sim} R^{-1} \circ R \circ G \overset{\mu}{\sim} G$. Hence, $F \overset{\sigma}{\sim} G$. ◊

In the rest of this section we shall address the question of identifying those proper maps which will preserve or inversely preserve $M_p^{\mathcal{B},\mathcal{C}}$-smooth spaces. The answer provide proper maps studied in [8] whose definitions we now recall.

Let $\mathcal{B}$ be a class of spaces. A proper map $f\colon X \to Y$ is called an *$M_p^{\mathcal{B}}$-injection* provided for every $\sigma \in \mathrm{Cov}(X)$ there is a $\tau \in \mathrm{Cov}(X)$ and a $\xi \in \mathrm{Cov}(Y)$ such that for M_p^τ-functions F and G from a member B of $\mathcal{B}$ into X the relation $f \circ F \overset{\xi}{\sim} f \circ G$ implies the relation $F \overset{\sigma}{\sim} G$. A proper map $f\colon X \to Y$ is *M_p^l-placid* provided for every $\sigma \in \mathrm{Cov}(X)$ there is an M_p^σ-function $J\colon Y \to X$ such that $J \circ f \overset{\sigma}{\sim} \mathrm{id}_X$.

Observe that every proper map $f\colon X \to Y$ which has a left proper homotopy inverse (i. e., for which there is a proper map $g\colon Y \to X$ with the composition $g \circ f$ properly homotopic to id_X) is M_p^l-placid. The same is true if the map f has a left $\mathcal{S}h_p$-inverse. Moreover, an M_p^l-placid proper map is an $M_p^{\mathcal{S}}$-injection, where $\mathcal{S}$ denotes the class of all topological spaces.

The following result shows that $M_p^{\mathcal{B}}$-injections inversely preserve $M_p^{\mathcal{B},\mathcal{C}}$-smooth spaces.

3.8 Theorem. *If $f\colon X \to Y$ is an $M_p^{\mathcal{B}}$-injection and Y is $M_p^{\mathcal{B},\mathcal{C}}$-smooth, then X is also $M_p^{\mathcal{B},\mathcal{C}}$-smooth.*

Proof. Let a cover σ of X be given. Since f is an $M_p^{\mathcal{B}}$-injection, there is an $\alpha \in \mathrm{Cov}(X)$ and $\beta \in \mathrm{Cov}(Y)$ such that for M_p^{α}-functions F and G from a member of $\mathcal{B}$ into X the relation $f \circ F \overset{\beta}{\sim} f \circ G$ implies the relation $F \overset{\sigma}{\sim} G$. We utilize now the assumption that Y is $M_p^{\mathcal{B},\mathcal{C}}$-smooth to select a $\gamma \in \mathrm{Cov}(Y)$ with the property that for M_p^{γ}-functions K and L from a member of $\mathcal{B}$ into Y the relation $K \overset{\gamma}{\sim}_{\mathcal{C}} L$ implies the relation $K \overset{\beta}{\sim} L$. Let $\tau \in \mathrm{Cov}(X)$ be a common refinement of α and $f^{-1}(\gamma)$. Then τ is the required cover. In fact, let F and G be M_p^{τ}-functions from a member B of $\mathcal{B}$ into X and assume that $F \overset{\tau}{\sim}_{\mathcal{C}} G$. Let K and L be $f \circ F$ and $f \circ G$. Then K and L are M_p^{β}-functions from B into Y. The last relation implies $K \overset{\gamma}{\sim}_{\mathcal{C}} L$ so that $K \overset{\beta}{\sim} L$. It follows that $F \overset{\sigma}{\sim} G$. ◊

An example of M_p^l-placid maps provide inclusions $i_{A,X}$ of the M_p-retracts A of a space X. Here, we will say that a closed subset A of a space X is an *M_p-retract* of X provided for every cover σ of A there is an M_p^{σ}-function $R\colon X \to A$ such that $a \in R(a)$ for every $a \in A$. Hence, the following is a consequence of Th. 3.8.

3.9 Corollary. *An M_p-retract of an $M_p^{\mathcal{B},\mathcal{C}}$-smooth space is $M_p^{\mathcal{B},\mathcal{C}}$-smooth.*

For the preservation of $M_p^{\mathcal{B},\mathcal{C}}$-smooth spaces from the domain to the codomain we must assume that the map f is either M_p^r-placid or that it is an $M_p^{\mathcal{B},\mathcal{C}}$-bijection. Let us recall the definitions of these notions from [8].

Let $\mathcal{B}$ be a class of spaces. A proper map $f\colon X \to Y$ is an *$M_p^{\mathcal{B}}$-surjection* provided for every $\sigma \in \mathrm{Cov}(X)$ and every $\tau \in \mathrm{Cov}(Y)$ there is a $\varrho \in \mathrm{Cov}(Y)$ such that for every M_p^{ϱ}-function F from a member of $\mathcal{B}$ into Y there is an M_p^{σ}-function G with $F \overset{\tau}{\sim} f \circ G$. A special case of $M_p^{\mathcal{B}}$-surjections are *M_p^r-placid* maps, i. e. proper maps $f\colon X \to Y$ such that for every $\sigma \in \mathrm{Cov}(X)$ and every $\tau \in \mathrm{Cov}(Y)$ there is an M_p^{σ}-function $J\colon Y \to X$ with $f \circ J \overset{\tau}{\sim} \mathrm{id}_Y$. In fact, every M_p^r-placid map is an $M_p^{\mathcal{S}}$-surjection, where $\mathcal{S}$ denotes the class of all topological spaces.

Observe that a proper map $f\colon X \to Y$ which has a right proper homotopy inverse (i. e., for which there is a proper map $g\colon Y \to X$ with $f \circ g$ properly homotopic to id_Y) is M_p^r-placid. The same is true if the proper map has a right $\mathcal{S}h_p$-inverse.

At last, for classes $\mathcal{B}$ and $\mathcal{C}$ of spaces, a proper map is an $M_p^{\mathcal{B},\mathcal{C}}$-*bijection* if it is both an $M_p^{\mathcal{B}}$-injection and an $M_p^{\mathcal{C}}$-surjection. We shall use a shorter name $M_p^{\mathcal{B}}$-*bijection* for an $M_p^{\mathcal{B},\mathcal{B}}$-bijection.

3.10 Theorem. *If a map $f\colon X \to Y$ is M_p^{τ}-placid and X is $M_p^{\mathcal{B},\mathcal{C}}$-smooth, then Y is also $M_p^{\mathcal{B},\mathcal{C}}$-smooth.*

Proof. Let a cover σ of Y be given. Let $\pi \in \sigma^*$ and $\alpha = f^{-1}(\pi)$. Since X is $M_p^{\mathcal{B},\mathcal{C}}$-smooth, there is a $\beta \in \mathrm{Cov}(X)$ such that for M_p^{β}-functions K and L from a member C of $\mathcal{C}$ into X the relation $K \overset{\beta}{\sim}_C L$ implies the relation $K \overset{\alpha}{\sim} L$. Now we utilize the fact that f is M_p^{τ}-placid to select an M_p^{β}-function $H\colon Y \to X$ with $\mathrm{id}_Y \overset{\pi}{\sim} f \circ H$. Let $M\colon Y \times I \to Y$ be an M_P^{π}-homotopy that realizes this relation and let $\zeta \in D(M, \pi)$. Let $\gamma \in S(H, \beta)$. Let a $\tau \in \mathrm{Cov}(Y)$ be a common refinement of ζ and γ. Then τ is the required cover of Y. Indeed, consider M_p^{τ}-functions F and G from a member B of $\mathcal{B}$ into Y and assume that $F \overset{\tau}{\sim}_C G$. Let K and L be $H \circ F$ and $H \circ G$. Then K and L are M_p^{β}-functions from B into X. From the last relation it follows that $K \overset{\beta}{\sim}_C L$ so that $K \overset{\alpha}{\sim} L$. Composing this relation with the map f we obtain $f \circ K \overset{\pi}{\sim} f \circ L$. Our choices imply the following chain of relations $F \overset{\pi}{\sim} f \circ H \circ F =$ $= f \circ K \overset{\pi}{\sim} f \circ L = f \circ H \circ G \overset{\pi}{\sim} G$. Hence, $F \overset{\sigma}{\sim} G$. ◊

It has been shown in [8, (3.1)] that another important example of M_p^{τ}-placid maps provide properly refinable maps. We call an onto proper map $f\colon X \to Y$ between spaces *properly refinable* provided for every cover τ of Y and every cover σ of X there is an onto proper map $g\colon X \to Y$ such that f and g are τ-close and g^{-1} is an M_p^{σ}-function. We call g a proper (σ, τ)-*refinement* of the map f. The notion of a refinable map between compact metric spaces was first defined by Jo Ford and James Rogers Jr.. The above extension to arbitrary spaces is particularly suitable for our theory.

The existence of a properly refinable map from a space X onto a space Y clearly implies that X is $M_p^{\{Y\}}$-like. Hence, as a consequence of Ths. 3.7 and 3.10 we obtain the following analogue of Th. 1.8 in [11] for $M_p^{\mathcal{B},\mathcal{C}}$-smooth spaces.

3.11 Corollary. *Let $f\colon X \to Y$ be a properly refinable map. Then the space X is $M_p^{\mathcal{B},\mathcal{C}}$-smooth if and only if Y is $M_p^{\mathcal{B},\mathcal{C}}$-smooth.*

3.12 Theorem. *If a map $f\colon X \to Y$ is an $M_p^{\mathcal{C},\mathcal{B}}$-bijection and the domain X is $M_p^{\mathcal{B},\mathcal{C}}$-smooth, then the codomain Y is also $M_p^{\mathcal{B},\mathcal{C}}$-smooth.*

Proof. Let a cover σ of Y be given. Let $\varrho \in \sigma^*$ and $\alpha = f^{-1}(\varrho)$. Since X is $M_p^{\mathcal{B},\mathcal{C}}$-smooth, there is a $\beta \in \mathrm{Cov}(X)$ such that for M_p^{β}-functions K and L from a member of $\mathcal{B}$ into Y the relation $K \overset{\beta}{\sim}_{\mathcal{C}} L$ implies the relation $K \overset{\alpha}{\sim} L$. We now use the assumption that f is an $M_p^{\mathcal{C}}$-injection to select a $\gamma \in \beta^+$ and a $\lambda \in \varrho^+$ such that for M_p^{γ}-functions P and Q from a member of $\mathcal{C}$ into X the relation $f \circ P \overset{\lambda}{\sim} f \circ Q$ implies the relation $P \overset{\beta}{\sim} Q$. Let $\mu \in \lambda^*$. At last, since f is also an $M_p^{\mathcal{B}}$-surjection, there is a $\tau \in \mu^+$ such that for every M_p^{τ}-function F from a member of $\mathcal{B}$ into Y there is an M_p^{γ}-function K with $F \overset{\mu}{\sim} f \circ K$. Then τ is the required cover of Y.

In order to verify this claim, assume that B is a member of $\mathcal{B}$ and $F, G\colon B \to Y$ are M_p^{τ}-functions with $F \overset{\tau}{\sim}_{\mathcal{C}} G$. In other words, suppose that there is a cover ξ of B such that $F \circ H \overset{\tau}{\sim} G \circ H$ for every M_p^{ξ}-function H from a member of $\mathcal{C}$ into B. Choose M_p^{γ}-functions $K, L\colon B \to X$ and M_p^{μ}-homotopies $V, W\colon B \times I \to Y$ such that $V_0 = F$, $W_0 = G$, $V_1 = f \circ K$, and $W_1 = f \circ L$. Let $\theta \in \xi^+$ be from the intersection of sets $S(K, \gamma)$, $S(L, \gamma)$, $D(V, \mu)$, and $D(W, \mu)$. Let C be a member of $\mathcal{C}$ and let $H\colon C \to B$ be an M_p^{θ}-function. Our choices imply the following extended chain of relations $f \circ K \circ H \overset{\mu}{\sim} F \circ H \overset{\tau}{\sim} G \circ H \overset{\mu}{\sim} f \circ L \circ H$. It follows that $f \circ K \circ H \overset{\lambda}{\sim} f \circ L \circ H$. Since $K \circ H$ and $L \circ H$ are the M_p^{γ}-functions from C into X, we get $K \circ H \overset{\beta}{\sim} L \circ H$. Thus, we have checked that $K \overset{\beta}{\sim}_{\mathcal{C}} L$. The way in which we selected the cover β implies that $K \overset{\alpha}{\sim} L$. Therefore, $F \overset{\mu}{\sim} f \circ K \overset{\varrho}{\sim} f \circ L \overset{\mu}{\sim} G$. Hence, $F \overset{\sigma}{\sim} G$. ◊

4. $M_p^{\mathcal{B}}$-calm spaces

In the present section we shall transfer from shape theory into proper shape theory the important invariant of calmness. This concept was invented by the author [6] for compact metric spaces. We shall define $M_p^{\mathcal{B}}$-calm spaces with respect to a class $\mathcal{B}$ of spaces in order to cover all possible variations of calmness (see [6]).

Let $\mathcal{B}$ be a class of spaces. A space X is $M_p^{\mathcal{B}}$-*calm* provided there is a cover σ of X with the property that for every cover τ of X we can find a cover ϱ of X such that M_p^{ϱ}-functions F and G from a member C

of $\mathcal{C}$ into X which are M_p^σ-homotopic are also M_p^τ-homotopic.

We shall first consider how this definition depends on the class $\mathcal{B}$. Once again the M_p-domination offers an answer.

4.1 Theorem. *If a class of spaces $\mathcal{B}$ is M_p-dominated by another such class $\mathcal{C}$ and a space X is $M_p^{\mathcal{C}}$-calm, then X is also $M_p^{\mathcal{B}}$-calm.*

Proof. Since X is $M_p^{\mathcal{C}}$-calm, there is a cover σ of X such that for every $\nu \in \mathrm{Cov}(X)$ there is a $\varrho \in \mathrm{Cov}(X)$ so that for M_p^ϱ-functions K and L from a member C of $\mathcal{C}$ into X the relation $K \overset{\sigma}{\sim} L$ implies the relation $K \overset{\nu}{\sim} L$. Then σ is the required cover. Indeed, let τ be an arbitrary cover of X. Let $\nu \in \tau^*$. Choose a cover ϱ as above. Let a $B \in \mathcal{B}$ and M_p^ϱ-functions F and G from B into X be given and assume that $F \overset{\sigma}{\sim} G$. Let H be an M_p^σ-homotopy joining F and G. Let $\beta \in D(H, \sigma)$. We can assume that β is so fine that both F and G are $M_p^{\beta,\,\varrho}$-functions. Since the class $\mathcal{B}$ is M_p-dominated by the class $\mathcal{C}$, there is a $C \in \mathcal{C}$ and an M_p^β-function $D \colon B \to C$ such that for every $\gamma \in \mathrm{Cov}(C)$ there is an M_p^γ-function $U \colon C \to B$ with $\mathrm{id}_B \overset{\beta}{\sim} D \circ U$. Let K and L be the compositions $F \circ D$ and $G \circ D$, respectively. Then K and L are M_p^ϱ-functions from C into X with $K \overset{\sigma}{\sim} L$. Our choices imply $K \overset{\nu}{\sim} L$. Let E be an M_p^ν-homotopy joining K and L. Let $\gamma \in$ $\in D(E, \nu)$. Choose a U as above. Then we have the following chain of relations $F \overset{\nu}{\sim} F \circ D \circ U = K \circ U \overset{\nu}{\sim} L \circ U = G \circ D \circ U \overset{\nu}{\sim} G$. Hence, $F \overset{\tau}{\sim} G$. ◊

Our goal now is to show that $M_p^{\mathcal{B}}$-calmness is indeed a proper shape invariant. We can prove a far better result, namely that it is preserved under Sh_p-domination.

4.2 Theorem. *A space is $M_p^{\mathcal{C}}$-calm if and only if it is Sh_p-dominated by an $M_p^{\mathcal{C}}$-calm space.*

Proof. Let X be a space, let Y be an $M_p^{\mathcal{C}}$-calm space, and assume that $\varphi \colon X \to Y$ and $\psi \colon Y \to X$ are M_p-nets such that the composition $\psi \circ \varphi$ is M_p-homotopic to the identity M_p-net ι^X on X.

Since Y is $M_p^{\mathcal{C}}$-calm, there is a cover α of Y with the property that for every cover β of Y there is a $\gamma \in \mathrm{Cov}(Y)$ such that M_p^γ-functions K and L from a member C of $\mathcal{C}$ into Y which are M_p^α-homotopic are already M_p^β-homotopic. Let $\mu \in \alpha^*$.

Since φ is an M_p-net, there is an index $c \in \mathrm{Inc}(Y)$ such that $F_d \overset{\mu}{\sim} F_e$ for all $d, e > c$. Choose a $d > c$ and a cover σ of X so that

F_d is an $M_p^{\sigma,\mu}$-function. Then σ is the required cover. Indeed, let a cover τ of X be given. Let $\nu \in \tau^*$. By assumption, there is an index $a \in \operatorname{Inc}(X)$ such that $a > \{\nu\}$ and $G_x \circ F_z \overset{\nu}{\sim} \mathrm{id}_X$, where $x = \psi(a)$, $\delta = \psi^*(a)$, $y = \{\delta\}$, and $z = \varphi(y)$. Notice that G_x is an $M_p^{\delta,\nu}$-function. Let $M\colon X \times I \to X$ be an M_p^{ν}-homotopy that realizes the last relation. Let $\varepsilon \in D(M, \nu)$ and $\beta \in \delta^*$. Select an index $w > z$ such that $F_b \overset{\beta}{\sim} F_w$ for every $b > w$. Observe that the condition $w > z$ implies that F_w and F_z are joined by an M_p^{δ}-homotopy $N\colon X \times I \to Y$. Let $\xi \in D(N, \delta)$. Pick a cover γ of Y with respect to α and β as above. Finally, we select an index $b > w$ and a $\pi \in \xi^+$ such that F_b is an $M_p^{\pi,\gamma}$-function.

Let $P\colon X \times I \to Y$ be an M_p^{β}-homotopy joining F_b and F_w and let $R\colon X \times I \to Y$ be an M_p^{μ}-homotopy joining F_b and F_d. Let ϱ be from the intersection of sets $D(P, \beta)$ and $D(R, \mu)$. Consider M_p^{ϱ}-functions F and G from a member C of $\mathcal{C}$ into X and assume that $F \overset{\sigma}{\sim} G$. Let K and L denote compositions $F_b \circ F$ and $F_b \circ G$, respectively. These are M_p^{γ}-functions and $K = F_b \circ F \overset{\mu}{\sim} F_d \circ F \overset{\mu}{\sim} \overset{\mu}{\sim} F_d \circ G = L$, i.e., $K \overset{\alpha}{\sim} L$. By assumption, it follows that $K \overset{\beta}{\sim} L$. This relation implies the following chain $F_w \circ F \overset{\beta}{\sim} F_b \circ F = = K \overset{\beta}{\sim} L = F_b \circ G \overset{\beta}{\sim} F_w \circ G$. Hence, $F_w \circ F \overset{\delta}{\sim} F_w \circ G$ so that we get $G_x \circ F_w \circ F \overset{\nu}{\sim} G_x \circ F_w \circ G$. But, we also have relations $G_x \circ F_z \circ F \overset{\nu}{\sim} \overset{\nu}{\sim} G_x \circ F_w \circ F$, $F \overset{\nu}{\sim} G_x \circ F_z \circ F$, $G \overset{\nu}{\sim} G_x \circ F_z \circ G$, and $G_x \circ F_w \circ G \overset{\nu}{\sim} \overset{\nu}{\sim} G_x \circ F_z \circ G$. Together these relations imply the desired conclusion $F \overset{\tau}{\sim} G$. ◊

The next result is typical for shape theory. It shows the role of $M_p^{\mathcal{B},\mathcal{C}}$-smooth spaces and is similar to the author's theorem that a $(\mathcal{B}, \mathcal{C})$-smooth and $\mathcal{C}$-calm compactum is $\mathcal{B}$-calm [6].

4.3 Theorem. *Let $\mathcal{B}$ and $\mathcal{C}$ be classes of topological spaces. If a space X is both $M_p^{\mathcal{B},\mathcal{C}}$-smooth and $M_p^{\mathcal{C}}$-calm, then it is also $M_p^{\mathcal{B}}$-calm.*

Proof. Since X is $M_p^{\mathcal{C}}$-calm, there is a cover σ of X such that for every $\tau \in \operatorname{Cov}(X)$ there is a $\varrho \in \operatorname{Cov}(X)$ with the property that for M_p^{ϱ}-functions K and L from a member of $\mathcal{C}$ into X the relation $K \overset{\sigma}{\sim} L$ implies the relation $K \overset{\tau}{\sim} L$.

Let $\beta \in \operatorname{Cov}(X)$. We utilize the assumption that X is $M_p^{\mathcal{B},\mathcal{C}}$-smooth to select a $\tau \in \operatorname{Cov}(X)$ such that for M_p^{τ}-functions F and G from a member $\mathcal{C}$ into X the relation $F \overset{\tau}{\sim}_{\mathcal{C}} G$ implies the relation $F \overset{\beta}{\sim} G$. Finally, choose a cover $\varrho \in \tau^+$ as above.

Consider M_p^ϱ-functions F and G from a member B of $\mathcal{B}$ into X and assume that $F \overset{\sigma}{\sim} G$. Let $K\colon B \times I \to X$ be an M_p^σ-homotopy joining F and G. Let $\gamma \in D(H, \sigma)$. For every M_p^γ-function H from a member C of $\mathcal{C}$ into B the compositions K and L of H and F and H and G, respectively, satisfy $K \overset{\sigma}{\sim} L$. It follows that $K \overset{\tau}{\sim} L$. Hence, $F \overset{\tau}{\sim}_{\mathcal{C}} G$, and we get the desired conclusion $F \overset{\beta}{\sim} G$. ◊

In the rest of this section we shall consider the question of identifying those proper maps which will preserve or inversely preserve $M_p^{\mathcal{B}}$-calm spaces. The answer provide proper maps studied in [8] whose definitions have been recalled in §3. The following result resembles Th. 3.8.

4.4 Theorem. *If $f\colon X \to Y$ is an $M_p^{\mathcal{C}}$-injection and Y is $M_p^{\mathcal{C}}$-calm, then X is also $M_p^{\mathcal{C}}$-calm.*

Proof. Since Y is $M_p^{\mathcal{C}}$-calm, there is a cover α of Y such that for every $\beta \in \operatorname{Cov}(Y)$ there is a $\gamma \in \operatorname{Cov}(Y)$ with the property that for every M_p^γ-functions K and L from a member of $\mathcal{C}$ into Y the relation $K \overset{\alpha}{\sim} L$ implies the relation $K \overset{\beta}{\sim} L$. Let $\sigma = f^{-1}(\alpha)$. Then σ is the required cover of X.

In order to check this, assume that τ is a cover of X. Since f is an $M_p^{\mathcal{C}}$-injection, there is a $\pi \in \operatorname{Cov}(X)$ and a $\beta \in \operatorname{Cov}(Y)$ such that for M_p^π-functions F and G from a member of $\mathcal{C}$ into X the relation $f \circ F \overset{\beta}{\sim} f \circ G$ implies the relation $F \overset{\tau}{\sim} G$. Pick a γ as above. Let $\varrho \in \operatorname{Cov}(X)$ be a common refinement of π and $f^{-1}(\gamma)$.

Let $C \in \mathcal{C}$ and assume that M_p^ϱ-functions $F, G\colon C \to X$ satisfy $F \overset{\sigma}{\sim} G$. Let K and L be the compositions $f \circ F$ and $f \circ G$, respectively. Then K and L are M_p^γ-functions from C into Y and we have $K \overset{\alpha}{\sim} L$. It follows that $f \circ F \overset{\beta}{\sim} f \circ G$ and therefore that $F \overset{\tau}{\sim} G$. ◊

The following result gives a partial converse to Theorem 4.4.

4.5 Theorem. *If a proper map $f\colon X \to Y$ is properly refinable and the codomain Y is $M_p^{\mathcal{C}}$-calm, then f is an $M_p^{\mathcal{C}}$-injection.*

Proof. Let a cover α of X be given. Since Y is $M_p^{\mathcal{C}}$-calm, there is a $\sigma \in \operatorname{Cov}(Y)$ with the property that for every $\tau \in \operatorname{Cov}(Y)$ we can find a $\varrho \in \tau^+$ such that for M_p^ϱ-functions K and L from a member of $\mathcal{C}$ into Y the relation $K \overset{\sigma}{\sim} L$ implies the relation $K \overset{\tau}{\sim} L$. Let $\xi \in \sigma^{*2}$ and $\beta \in \alpha^*$. Since f is properly refinable, there is a proper map g from X onto Y such that f and g are ξ-close and g^{-1} is an M_p^β-function. Let

$\tau \in S(g^{-1}, \beta)$. Next, we select a ϱ as above and let η be a common refinement of β and $g^{-1}(\varrho)$.

Consider M_p^η-functions F and G from a member C of $\mathcal{C}$ into X and assume that $f \circ F \overset{\xi}{\sim} f \circ G$. Let K and L be the compositions $g \circ F$ and $g \circ G$, respectively. Then K and L are M_p^ϱ-functions from C into Y. From the previous selections we get $K = g \circ F \overset{st(\xi)}{\sim} f \circ F \overset{\xi}{\sim} f \circ G \overset{st(\xi)}{\sim} g \circ G = L$ and thus $K \overset{\sigma}{\sim} L$. Our choices now imply that $K \overset{\tau}{\sim} L$. It follows that $F \overset{\beta}{\sim} g^{-1} \circ g \circ F = g^{-1} \circ K \overset{\beta}{\sim} g^{-1} \circ L = = g^{-1} \circ g \circ G \overset{\beta}{\sim} G$. Hence, $F \overset{\alpha}{\sim} G$. ◊

4.6 Corollary. *The image Y of an $M_p^{\mathcal{C}}$-calm space X under a properly refinable proper map $f \colon X \to Y$ is $M_p^{\mathcal{C}}$-calm if and only if the map f is an $M_p^{\mathcal{C}}$-injection.*

In an attempt to prove an analogue of Theorem 3.10 for $M_p^{\mathcal{B}}$-calm spaces instead of M_p^r-placid maps we must use the following stronger form of this notion. A proper map $f \colon X \to Y$ between spaces is M_q^r-*placid* provided for every cover σ of X there is a cover α of Y such that for every cover ϱ of X and every cover β of Y there is an M-function $J \colon Y \to X$ which is both an M_p^ϱ-function and an $M_p^{\alpha,\sigma}$-function and $f \circ J$ and id_Y are M_p^β-homotopic.

4.7 Theorem. *If a proper map $f \colon X \to Y$ is M_q^r-placid and the domain X is $M_p^{\mathcal{C}}$-calm, then the codomain Y is also $M_p^{\mathcal{C}}$-calm.*

Proof. Since X is $M_p^{\mathcal{C}}$-calm, there is an $\sigma \in \mathrm{Cov}(X)$ such that for every $\tau \in \mathrm{Cov}(X)$ we can find a $\varrho \in \mathrm{Cov}(X)$ with the property that for M_p^ϱ-functions K and L from a member of $\mathcal{C}$ into X the relation $K \overset{\sigma}{\sim} L$ implies the relation $K \overset{\tau}{\sim} L$. Since f is M_q^r-placid there is a cover α of Y such that for every cover ϱ of X and every cover δ of Y there is an M-function $J \colon Y \to X$ which is both an M_p^ϱ-function and an $M_p^{\alpha,\sigma}$-function and there is an M_p^δ-homotopy H joining $f \circ J$ and id_Y. Then α is the required cover of Y.

To check this, let a cover β of Y be given. Let $\delta \in \beta^*$ and let $\tau = f^{-1}(\delta)$. Pick a ϱ and a J as above. Let $\gamma \in S(J, \varrho)$. Then γ has the required property. Indeed, let D and E be M_p^γ-functions from a member C of $\mathcal{C}$ into Y and assume that $D \overset{\alpha}{\sim} E$. Let K and L be the compositions $J \circ D$ and $J \circ E$, respectively. Then K and L are M_p^γ-functions from C into Y and since J is an $M_p^{\alpha,\sigma}$-function we obtain that $K \overset{\alpha}{\sim} L$. It follows from our selections that $K \overset{\tau}{\sim} L$ so that after composing with f we get $f \circ K \overset{\mu}{\sim} f \circ L$. Thus, we have the following chain of relations $D \overset{\delta}{\sim} f \circ J \circ D = f \circ K \overset{\delta}{\sim} f \circ L = f \circ J \circ E \overset{\delta}{\sim} E$. Hence, $D \overset{\beta}{\sim} E$. ◊

4.8 Theorem. *Let $\mathcal{C}$ be a class of spaces. Let X be an $M_p^{\mathcal{C}}$-calm space. If a map $f\colon X \to Y$ is an $M_p^{\mathcal{C}}$-bijection, then the space Y is also $M_p^{\mathcal{C}}$-calm.*

Proof. Since X is $M_p^{\mathcal{C}}$-calm, there is an $\alpha \in \mathrm{Cov}(X)$ such that for every $\beta \in \mathrm{Cov}(X)$ we can find a $\gamma \in \mathrm{Cov}(X)$ with the property that for M_p^{γ}-functions K and L from a member of $\mathcal{C}$ into X the relation $K \overset{\alpha}{\sim} L$ implies the relation $K \overset{\beta}{\sim} L$.

Since f is an $M_p^{\mathcal{C}}$-injection, there is a $\xi \in \mathrm{Cov}(Y)$ and an $\eta \in \mathrm{Cov}(X)$ such that for M_p^{η}-functions K and L from a member of $\mathcal{C}$ into X the relation $f \circ K \overset{\xi}{\sim} f \circ L$ implies the relation $K \overset{\alpha}{\sim} L$. Let $\sigma \in \xi^*$. Then σ is the required cover of Y.

In order to check this, let a $\tau \in \mathrm{Cov}(Y)$ be given. Let a $\mu \in \tau^*$ refines σ. Put $\beta = f^{-1}(\mu)$. Choose a cover γ as above. Since f is an $M_p^{\mathcal{C}}$-surjection, there is a $\varrho \in \eta^+$ such that for every M_p^{ϱ}-function F from a member C of $\mathcal{C}$ into Y there is an M_p^{γ}-function $K\colon C \to X$ with $F \overset{\mu}{\sim} f \circ K$.

Consider M_p^{ϱ}-functions F and G from a member C of $\mathcal{C}$ into Y and assume that $F \overset{\sigma}{\sim} G$. Choose M_p^{γ}-functions K and L from C into X such that $F \overset{\mu}{\sim} f \circ K$ and $G \overset{\mu}{\sim} f \circ L$. From the previous two relations we obtain $f \circ K \overset{\xi}{\sim} f \circ L$. It follows that $K \overset{\alpha}{\sim} L$ and therefore $K \overset{\beta}{\sim} L$ and $f \circ K \overset{\mu}{\sim} f \circ L$. Combining the last two relations, this time we shall get the conclusion $F \overset{\tau}{\sim} G$. ◊

5. $N_p^{\mathcal{B},\mathcal{C}}$-smooth and $P_p^{\mathcal{B},\mathcal{C}}$-smooth classes

The notion of an $M_p^{\mathcal{B},\mathcal{C}}$-smooth class of spaces allow us to obtain two new properties that are preserved under M_p-domination. They could be considered as dual to the notion of an $M_p^{\mathcal{B},\mathcal{C}}$-smooth space. While in the previous three sections we investigated a space X by looking at small proper multi-valued functions from members of a given class of spaces $\mathcal{B}$ into X, we now change our point of view by concentrating on small proper multi-valued functions from X into members of $\mathcal{B}$.

Let $\mathcal{B}$ and $\mathcal{C}$ be classes of spaces. A class of spaces $\mathcal{X}$ is (1) $N_p^{\mathcal{B},\mathcal{C}}$-*smooth* and (2) $P_p^{\mathcal{B},\mathcal{C}}$-*smooth* provided the class $\mathcal{B}$ is (1) $M_p^{\mathcal{X},\mathcal{C}}$-smooth and (2) $M_p^{\mathcal{C},\mathcal{X}}$-smooth, respectively. In other words, provided that

(1) for every $B \in \mathcal{B}$ and every $\sigma \in \mathrm{Cov}(B)$ there is a $\tau \in \mathrm{Cov}(B)$ with the property that for M_p^τ-functions F and G from a member of $\mathcal{X}$ into B the relation $F \overset{\tau}{\sim}_{\mathcal{C}} G$ implies the relation $F \overset{\sigma}{\sim} G$;

(2) for every $B \in \mathcal{B}$ and every $\sigma \in \mathrm{Cov}(B)$ there is a $\tau \in \mathrm{Cov}(B)$ with the property that for M_p^τ-functions F and G from a member of $\mathcal{C}$ into B the relation $F \overset{\tau}{\sim}_{\mathcal{X}} G$ implies the relation $F \overset{\sigma}{\sim} G$.

We shall say that a space X has one of the above properties provided the class $\{X\}$ consisting just of a space X has this property.

The three versions of proper smoothness share many properties. We shall now state and prove the N and the P versions of most results from §3.

5.1 Theorem. *A class $\mathcal{X}$ of topological spaces is $N_p^{\mathcal{B},\mathcal{C}}$-smooth if and only if it is M_p-dominated by an $N_p^{\mathcal{B},\mathcal{C}}$-smooth class of spaces.*

Proof. Suppose that $\mathcal{X}$ is M_p-dominated by an $N_p^{\mathcal{B},\mathcal{C}}$-smooth class $\mathcal{Y}$. Then the class $\mathcal{B}$ is $M_p^{\mathcal{Y},\mathcal{C}}$-smooth so that $\mathcal{B}$ is $M_p^{\mathcal{X},\mathcal{C}}$-smooth by Theorem 3.5. Hence, $\mathcal{X}$ is $N_p^{\mathcal{B},\mathcal{C}}$-smooth. ◊

5.2 Theorem. *A class $\mathcal{X}$ of spaces is $P_p^{\mathcal{B},\mathcal{C}}$-smooth if and only if it M_p-dominates a $P_p^{\mathcal{B},\mathcal{C}}$-smooth class of spaces.*

Proof. Similar to the proof of Th. 5.1. ◊

5.3 Theorem. *Let $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ be classes of spaces. If a class $\mathcal{X}$ of spaces is both $N_p^{\mathcal{B},\mathcal{A}}$-smooth and $M_p^{\mathcal{A},\mathcal{C}}$-tame, then $\mathcal{X}$ is also $N_p^{\mathcal{B},\mathcal{C}}$-smooth.*

Proof. Let a member B of $\mathcal{B}$ and a cover σ of B be given. Since X is $N_p^{\mathcal{B},\mathcal{A}}$-smooth, there is a $\pi \in \mathrm{Cov}(B)$ such that for M_p^π-functions F and G from a member of $\mathcal{X}$ into B the relation $F \overset{\pi}{\sim}_{\mathcal{A}} G$ implies the relation $F \overset{\sigma}{\sim} G$. Let $\tau \in \pi^*$. Then τ is the cover we have been looking for.

Indeed, let $X \in \mathcal{X}$ and let $F, G\colon X \to B$ be M_p^τ-functions such that $F \overset{\tau}{\sim}_{\mathcal{C}} G$. By definition, this means that there is a cover $\alpha \in \mathrm{Cov}(X)$ such that α belongs to both $S(F, \tau)$ and $S(G, \tau)$ and the compositions $F \circ K$ and $G \circ K$ are M_p^τ-homotopic for every M_p^α-function $K\colon C \to X$ from a member of $\mathcal{C}$ into X. Now we utilize the fact that $\mathcal{X}$ is also $M_p^{\mathcal{A},\mathcal{C}}$-tame to select a cover β of X such that for every $A \in \mathcal{A}$ and every M_p^{β}-function $H: A \to X$ there is a $C \in \mathcal{C}$ and an M_p^{α}-function $K: C \to X$ so that for every $\gamma \in \mathrm{Cov}(C)$ there is an M_p^{γ}-function $D: A \to C$ with $H \overset{\alpha}{\sim} K \circ D$.

Consider an $A \in \mathcal{A}$ and an M_p^{β}-function $H: A \to X$. Choose a C and then a K as above. By assumption, the compositions $F \circ K$ and $G \circ K$ are joined by an M_p^{τ}-homotopy W. Let $\delta \in S(W, \tau)$ and $\gamma \in D(C, \delta)$. Pick an M_p^{γ}-function D as above. Then we obtain the following chain of relations $F \circ H \overset{\tau}{\sim} F \circ K \circ D \overset{\mu}{\sim} G \circ K \circ D \overset{\tau}{\sim} G \circ H$. It follows that $F \overset{\pi}{\sim}_{\mathcal{A}} G$. Hence, $F \overset{\sigma}{\sim} G$. ◊

5.4 Theorem. *Let $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ be classes of spaces. If a class $\mathcal{X}$ of spaces is $P_p^{\mathcal{B},\mathcal{A}}$-smooth and the class $\mathcal{C}$ is $M_p^{\mathcal{C},\mathcal{A}}$-smooth, then $\mathcal{X}$ is also $P_p^{\mathcal{B},\mathcal{C}}$-smooth.*

Proof. Similar to the proof of Th. 5.3. ◊

5.5 Theorem. *Let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, and $\mathcal{D}$ be classes of spaces such that $\mathcal{B}$ and $\mathcal{D}$ are M_p-dominated by $\mathcal{A}$ and $\mathcal{C}$, respectively. If a class $\mathcal{X}$ of spaces is $N_p^{\mathcal{A},\mathcal{D}}$-smooth, then it is also $N_p^{\mathcal{B},\mathcal{C}}$-smooth.*

Proof. The assumption that $\mathcal{X}$ is $N_p^{\mathcal{A},\mathcal{D}}$-smooth means that $\mathcal{A}$ is $M_p^{\mathcal{X},\mathcal{D}}$-smooth. Since $\mathcal{B}$ is M_p-dominated by $\mathcal{A}$, it follows from (3.1) that $\mathcal{B}$ is $M_p^{\mathcal{X},\mathcal{D}}$-smooth. But, since $\mathcal{D}$ is M_p-dominated by $\mathcal{C}$, we get that $\mathcal{B}$ is $M_p^{\mathcal{X},\mathcal{C}}$-smooth and therefore that $\mathcal{X}$ is $N_p^{\mathcal{B},\mathcal{C}}$-smooth. ◊

5.6 Theorem. *Let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, and $\mathcal{D}$ be classes of spaces such that $\mathcal{B}$ and $\mathcal{C}$ are M_p-dominated by $\mathcal{A}$ and $\mathcal{D}$, respectively. If a class $\mathcal{X}$ of spaces is $P_p^{\mathcal{A},\mathcal{D}}$-smooth, then it is also $P_p^{\mathcal{B},\mathcal{C}}$-smooth.*

Proof. See the proof of Th. 5.5. ◊

There seems to be no analogue of (3.7) for $N_p^{\mathcal{B},\mathcal{C}}$-smooth and $P_p^{\mathcal{B},\mathcal{C}}$-smooth spaces. In order to state versions of (3.8) we need the following dual form of the notion of an $M_p^{\mathcal{B}}$-injection.

Let $\mathcal{B}$ be a class of spaces. A class $\mathcal{F}$ of proper maps is $N_p^{\mathcal{B}}$-*injective* provided for every $B \in \mathcal{B}$ and every $\sigma \in \mathrm{Cov}(B)$ there is a $\tau \in \mathrm{Cov}(B)$ such that for every $f: X \to Y$ from $\mathcal{F}$ and for every M_p^{τ}-functions F and G from Y into B the relation $F \circ f \overset{\tau}{\sim} G \circ f$ implies the relation $F \overset{\sigma}{\sim} G$. A proper map $f: X \to Y$ is an $N_p^{\mathcal{B}}$-*injection* provided the class $\{f\}$ is $N_p^{\mathcal{B}}$-injective.

For a class $\mathcal{F}$ of maps let $\mathcal{F}'$ and $\mathcal{F}''$ denote collections of all domains and of all codomains of members of $\mathcal{F}$, respectively.

5.7 Theorem. *If $\mathcal{F}$ is an $N_p^{\mathcal{B}}$-injective class of proper maps and the* *class $\mathcal{F}'$ is $N_p^{\mathcal{B},\mathcal{C}}$-smooth, then the class $\mathcal{F}''$ is also $N_p^{\mathcal{B},\mathcal{C}}$-smooth.*

Proof. Let a member B of $\mathcal{B}$ and a cover σ of B be given. Since the class $\mathcal{F}$ is $M_p^{\mathcal{B}}$-injective, there is a $\mu \in \mathrm{Cov}(B)$ such that for every proper map $f\colon X \to Y$ from $\mathcal{F}$ and all M_p^{μ}-functions $F, G\colon Y \to B$ the relation $F \circ f \overset{\mu}{\sim} G \circ f$ implies the relation $F \overset{\sigma}{\sim} G$. We utilize now the assumption that the class $\mathcal{F}'$ is $M_p^{\mathcal{B},\mathcal{C}}$-smooth to select the required cover τ of B such that for every member X of $\mathcal{F}'$ and all M_p^{τ}-functions $P, Q\colon X \to B$ the relation $P \overset{\tau}{\sim}_{\mathcal{C}} Q$ implies the relation $P \overset{\mu}{\sim} Q$.

Let Y be a member of the class $\mathcal{F}''$ and let $F, G\colon Y \to B$ be M_p^{τ}-functions and assume that $F \overset{\tau}{\sim}_{\mathcal{C}} G$. Let $f\colon X \to Y$ be from the class $\mathcal{F}$. Let P and Q be the compositions $F \circ f$ and $G \circ f$. It is easy to check that $P \overset{\tau}{\sim}_{\mathcal{C}} Q$. It follows that $P \overset{\mu}{\sim} Q$ and therefore that $F \overset{\sigma}{\sim} G$. ◊

In a similar way one can prove the following dual result for the $P_p^{\mathcal{B},\mathcal{C}}$-smooth classes of spaces.

5.8 Theorem. *If $\mathcal{F}$ is an $N_p^{\mathcal{B}}$-injective class of proper maps and the class $\mathcal{F}''$ is $P_p^{\mathcal{B},\mathcal{C}}$-smooth, then the class $\mathcal{F}'$ is also $P_p^{\mathcal{B},\mathcal{C}}$-smooth.*

5.9 Theorem. *If $\mathcal{F}$ is a class of M_p^{l}-placid proper maps and the class $\mathcal{F}''$ is $N_p^{\mathcal{B},\mathcal{C}}$-smooth, then the class $\mathcal{F}'$ is also $N_p^{\mathcal{B},\mathcal{C}}$-smooth*

Proof. Let a member B of $\mathcal{B}$ and a cover σ of B be given. Let $\mu \in \sigma^*$. Since the class $\mathcal{F}''$ is $N_p^{\mathcal{B},\mathcal{C}}$-smooth, there is a $\tau \in \mu^+$ such that for every member Y of $\mathcal{F}''$ and all M_p^{τ}-functions $P, Q\colon Y \to B$ the relation $P \overset{\tau}{\sim}_{\mathcal{C}} Q$ implies the relation $P \overset{\mu}{\sim} Q$. Then τ is the required cover. Indeed, let X be from the class $\mathcal{F}'$ and let $F, G\colon X \to B$ be M_p^{τ}-functions and assume that $F \overset{\tau}{\sim}_{\mathcal{C}} G$. Let $f\colon X \to Y$ be a proper map from the class $\mathcal{F}$. Let $\theta \in \mathrm{Cov}(X)$ be from the intersection of sets $S(F, \tau)$ and $S(G, \tau)$ and have the property that $F \circ H \overset{\tau}{\sim} G \circ H$ for every M_p^{θ}-function H from a member of the class $\mathcal{C}$ into X. Since f is M_p^{l}-placid, there is an M_p^{θ}-function $J\colon Y \to X$ with $J \circ f \overset{\theta}{\sim} \mathrm{id}_X$. Let P and Q be the compositions $F \circ J$ and $G \circ J$. Then P and Q are M_p^{τ}-functions and $P \overset{\tau}{\sim}_{\mathcal{C}} Q$. It follows that $P \overset{\mu}{\sim} Q$. Our choices imply $F \overset{\tau}{\sim} F \circ J \circ f = P \circ f \overset{\mu}{\sim} Q \circ f = G \circ J \circ f \overset{\tau}{\sim} G$. Hence, $F \overset{\sigma}{\sim} G$. ◊

5.10 Corollary. *An M_p-retract of an $N_p^{\mathcal{B},\mathcal{C}}$-smooth space is itself $N_p^{\mathcal{B},\mathcal{C}}$-smooth.*

5.11 Theorem. *If $\mathcal{F}$ is a class of M_p^{r}-placid proper maps and the class $\mathcal{F}''$ is $N_p^{\mathcal{B},\mathcal{C}}$-smooth, then the class $\mathcal{F}'$ is also properly $N_p^{\mathcal{B},\mathcal{C}}$-smooth.*

Proof. The proof is similar to the proof of Th. 5.9. ◊

In the next result that corresponds to Th. 3.12 we shall use a notion of $N_p^{\mathcal{B}}$-surjective class of proper maps from [8] whose definition we now recall. Let $\mathcal{B}$ be a class of spaces. A class $\mathcal{F}$ of proper maps is $N_p^{\mathcal{B}}$*-surjective* provided for every $B \in \mathcal{B}$ and every $\sigma \in \mathrm{Cov}(B)$ there is a $\tau \in \mathrm{Cov}(B)$ such that for every $f\colon X \to Y$ from $\mathcal{F}$ and every M_p^{τ}-function $F\colon X \to B$ there is an M_p^{σ}-function $G\colon Y \to B$ with $F \overset{\sigma}{\sim} G \circ f$. A proper map $f\colon X \to Y$ is an $N_p^{\mathcal{B}}$*-surjection* provided the class $\{f\}$ is $N_p^{\mathcal{B}}$-surjective. Also, a class of proper maps which is both $N_p^{\mathcal{B}}$-injective and $N_p^{\mathcal{C}}$-surjective is called $N_p^{\mathcal{B},\mathcal{C}}$*-bijective.* We shall use $N_p^{\mathcal{B}}$*-bijective* for an $N_p^{\mathcal{B},\mathcal{B}}$-bijective class of proper maps. A proper map f is an $N_p^{\mathcal{B},\mathcal{C}}$*-bijection* provided the class $\{f\}$ made up of f alone is $N_p^{\mathcal{B},\mathcal{C}}$-bijective. An $N_p^{\mathcal{B}}$*-bijection* is defined analogously.

5.12 Theorem. *If $\mathcal{F}$ is an $N_p^{\mathcal{B}}$-surjective class of $M_p^{\mathcal{C}}$-surjections and the class $\mathcal{F}''$ is $N_p^{\mathcal{B},\mathcal{C}}$-smooth, then the class $\mathcal{F}'$ will be also $N_p^{\mathcal{B},\mathcal{C}}$-smooth.*

Proof. Let a member B of $\mathcal{B}$ and a cover σ be given. Let $\lambda \in \sigma^*$. Since the class $\mathcal{F}''$ is $N_p^{\mathcal{B},\mathcal{C}}$-smooth, there is a $\nu \in \mathrm{Cov}(B)$ such that for every member Y of $\mathcal{F}''$ and all M_p^{ν}-functions K, $L\colon Y \to B$ the relation $K \overset{\nu}{\sim}_{\mathcal{C}} L$ implies the relation $K \overset{\lambda}{\sim} L$. Let $\mu \in \nu$. We utilize now the assumption that the class $\mathcal{F}$ is $M_p^{\mathcal{B}}$-surjective to select the required cover $\tau \in \mu^+$ of B such that for every map $f\colon X \to Y$ from the class $\mathcal{F}$ and every M_p^{τ}-function $F\colon X \to B$ there is an M_p^{μ}-function $K\colon Y \to B$ with $F \overset{\mu}{\sim} K \circ f$.

Consider a member X of $\mathcal{F}'$ and M_p^{τ}-functions F, $G\colon X \to B$ and assume that $F \overset{\tau}{\sim}_{\mathcal{C}} G$. In other words, assume that there is a cover $\theta \in$ $\in \mathrm{Cov}(X)$ such that $F \circ H \overset{\tau}{\sim} G \circ H$ for every M_p^{θ}-function H from a member of the class $\mathcal{C}$ into X. Let $f\colon X \to Y$ be from the class $\mathcal{F}$. Pick M_p^{μ}-functions K, $L\colon Y \to B$ such that $F \overset{\mu}{\sim} K \circ f$ and $G \overset{\mu}{\sim} L \circ f$. Let V and W be M_p^{μ}-homotopies which realize the last two relations. Let $\alpha \in \theta^+$ be from the intersection of sets $D(V, \mu)$ and $D(W, \mu)$ and let $\xi \in \mathrm{Cov}(Y)$ be from the intersection of sets $S(K, \mu)$ and $S(L, \mu)$. Since f is an $M_p^{\mathcal{C}}$-surjection, there is a $\zeta \in \mathrm{Cov}(Y)$ with the property that for every $C \in \mathcal{C}$ and every M_p^{ζ}-function $M\colon C \to Y$ there is an M_p^{α}-function $H\colon C \to X$ with $M \overset{\xi}{\sim} f \circ H$.

Let C be a member of the class $\mathcal{C}$ and let $M\colon C \to Y$ be an M_p^{ζ}-function. Choose an H as above. Then we obtain the following chain of relations: $K \circ M \overset{\mu}{\sim} K \circ f \circ H \overset{\mu}{\sim} F \circ H \overset{\tau}{\sim} G \circ H \overset{\mu}{\sim} L \circ f \circ H \overset{\mu}{\sim} L \circ M$. It follows that $K \circ M \overset{\nu}{\sim} L \circ M$. In other words, we checked that $K \overset{\nu}{\sim}_{\mathcal{C}} L$. Now, we conclude that $K \overset{\lambda}{\sim} L$. This time we have $F \overset{\mu}{\sim} K \circ f \overset{\lambda}{\sim} L \circ f \overset{\mu}{\sim} G$. Hence, $F \overset{\sigma}{\sim} L$. ◊

The situation with $P_p^{\mathcal{B},\mathcal{C}}$-smooth classes of spaces is much simpler as the following theorem shows. The proof of it is left to the reader.

5.13 Theorem. *If $\mathcal{F}$ is an $N_p^{\mathcal{B}}$-surjective class of proper maps and the class $\mathcal{F}'$ is $P_p^{\mathcal{B},\mathcal{C}}$-smooth, then the class $\mathcal{F}''$ is also $P_p^{\mathcal{B},\mathcal{C}}$-smooth.*

6. $N_p^{\mathcal{B}}$-calm classes

In this section we shall do for $M_p^{\mathcal{C}}$-calm spaces what we have done in §5 for $M_p^{\mathcal{B},\mathcal{C}}$-smooth spaces. In other words, we shall introduce a dual notion called $N_p^{\mathcal{B}}$-calmness. It applies to classes of spaces and it satisfies five theorems which are analogues of results in §4.

Let $\mathcal{B}$ and $\mathcal{X}$ be classes of spaces. The class $\mathcal{X}$ is *$N_p^{\mathcal{B}}$-calm* provided the class $\mathcal{B}$ is $M_p^{\mathcal{X}}$-calm, i. e., provided for every $B \in \mathcal{B}$ there is a cover σ of B with the property that for every cover τ of B we can find a cover ϱ of B such that for every member X of $\mathcal{X}$ and M_p^{ϱ}-functions $F, G \colon X \to B$ the relation $F \overset{\sigma}{\sim} G$ implies the relation $F \overset{\tau}{\sim} G$. A space X is *$N_p^{\mathcal{B}}$-calm* provided the class $\{X\}$ consisting of X alone is $N_p^{\mathcal{B}}$-calm.

The following two theorems are easy consequences of Ths. 4.2 and 4.1, respectively.

6.1 Theorem. *If a class of spaces $\mathcal{B}$ is Sh_p-dominated by another such class $\mathcal{C}$ and a class of spaces $\mathcal{X}$ is $N_p^{\mathcal{C}}$-calm, then $\mathcal{X}$ is also $N_p^{\mathcal{B}}$-calm.*

6.2 Theorem. *A class of spaces $\mathcal{X}$ is $N_p^{\mathcal{B}}$-calm if and only if it is M_p-dominated by an $N_p^{\mathcal{B}}$-calm class of spaces $\mathcal{Y}$.*

6.3 Theorem. *If a class of proper maps $\mathcal{F}$ is $N_p^{\mathcal{B}}$-injective and the class $\mathcal{F}'$ is $N_p^{\mathcal{B}}$-calm, then the class $\mathcal{F}''$ is also $N_p^{\mathcal{B}}$-calm.*

Proof. Let a member B of $\mathcal{B}$ be given. Since the class $\mathcal{F}'$ is $N_p^{\mathcal{B}}$-calm, there is a cover σ of B such that for every $\theta \in \operatorname{Cov}(B)$ there is a $\varrho \in$ $\in \operatorname{Cov}(B)$ with the property that for every M_p^{ϱ}-function K and L from a member X of $\mathcal{F}'$ into B the relation $K \overset{\sigma}{\sim} L$ implies the relation $K \overset{\theta}{\sim} L$. Then σ is the required cover of B.

In order to check this, assume that τ is a cover of B. Since the class $\mathcal{F}$ is $N_p^{\mathcal{B}}$-injective, there is a $\theta \in \operatorname{Cov}(B)$ such that for every proper map $f\colon X \to Y$ from $\mathcal{F}$ and all M_p^θ-functions F and G from Y into B the relation $F \circ f \overset{\theta}{\sim} G \circ f$ implies the relation $F \overset{\tau}{\sim} G$. Pick a ϱ as above. We can assume that ϱ refines θ.

Let Y be a member of the class $\mathcal{F}''$ and let M_p^ϱ-functions $F, G\colon Y \to \to B$ satisfy $F \overset{\sigma}{\sim} G$. Let $f\colon X \to Y$ be from the class $\mathcal{F}$. Let K and L be the compositions $F \circ f$ and $G \circ f$, respectively. Then K and L are M_p^ϱ-functions from X into B and we have $K \overset{\sigma}{\sim} L$. It follows that $F \circ f \overset{\theta}{\sim} G \circ f$ and therefore that $F \overset{\tau}{\sim} G$. ◊

The $N_p^\mathcal{B}$-calm classes of spaces are inversely preserved under M_p^l-placid maps.

6.4 Theorem. *If $\mathcal{F}$ is a class of M_p^l-placid maps and the class $\mathcal{F}''$ is $N_p^\mathcal{B}$-calm, then the class $\mathcal{F}'$ is also $N_p^\mathcal{B}$-calm.*

Proof. Let a member B of $\mathcal{B}$ be given. Since the class $\mathcal{F}''$ is $N_p^\mathcal{B}$-calm, there is a $\sigma \in \operatorname{Cov}(B)$ such that for every $\mu \in \operatorname{Cov}(B)$ we can find a $\varrho \in \operatorname{Cov}(B)$ with the property that for M_p^ϱ-functions K and L from a member Y of $\mathcal{F}''$ into B the relation $K \overset{\sigma}{\sim} L$ implies the relation $K \overset{\mu}{\sim} L$. Then σ is the required cover of B.

To check this, let a cover τ of B be given. Let $\mu \in \tau^*$. Pick a ϱ as above. We can assume that ϱ refines μ. Let F and G be M_p^ϱ-functions from a member X of $\mathcal{F}'$ into B and assume that $F \overset{\sigma}{\sim} G$. Let W be an M_p^σ-homotopy joining F and G. Let $\theta \in \operatorname{Cov}(X)$ be from the intersection of sets $D(W, \sigma)$, $S(F, \varrho)$, and $S(G, \varrho)$.

Let $f\colon X \to Y$ be a map from $\mathcal{F}$. Since f is M_p^l-placid, there is an M_p^θ-function $J\colon Y \to X$ such that $\mathrm{id}_X \overset{\mu}{\sim} J \circ f$. Let K and L be $F \circ J$ and $G \circ J$. Then K and L are M_p^ϱ-functions from Y into B and we have $K \overset{\sigma}{\sim} L$. It follows from our selections that $K \overset{\mu}{\sim} L$ so that we have the following chain of relations. $F \overset{\varrho}{\sim} F \circ J \circ f = K \circ f \overset{\mu}{\sim} L \circ f = = G \circ J \circ f \overset{\varrho}{\sim} G$. Hence, $F \overset{\tau}{\sim} G$. ◊

6.5 Corollary. *An M_p-retract of an $N_p^\mathcal{B}$-calm space is itself $N_p^\mathcal{B}$-calm.*

6.6 Theorem. *Let $\mathcal{B}$ be a class of spaces. If $\mathcal{F}$ is an $N_p^\mathcal{B}$-bijective class of proper maps and the class $\mathcal{F}''$ is $N_p^\mathcal{B}$-calm, then the class $\mathcal{F}'$ is also $N_p^\mathcal{B}$-calm.*

Proof. Let a member B of $\mathcal{B}$ be given. Since the class $\mathcal{F}''$ is $N_p^\mathcal{B}$-calm, there is an $\alpha \in \operatorname{Cov}(B)$ such that for every $\mu \in \operatorname{Cov}(B)$ we can find a $\pi \in \operatorname{Cov}(X)$ with the property that for M_p^π-functions K and L from a member Y of $\mathcal{F}''$ into B the relation $K \overset{\alpha}{\sim} L$ implies the relation $K \overset{\mu}{\sim} L$.

Since the class $\mathcal{F}$ is $N_p^{\mathcal{B}}$-injective, there is a $\lambda \in \mathrm{Cov}(B)$ such that for every proper map $f\colon X \to Y$ from $\mathcal{F}$ and all M_p^{λ}-functions K and L from Y into B the relation $K \circ f \overset{\lambda}{\sim} L \circ f$ implies the relation $K \overset{\alpha}{\sim} L$. Let $\sigma \in \lambda^*$. Then σ is the required cover of B.

In order to check this, let a $\tau \in \mathrm{Cov}(B)$ be given. Let $\mu \in \tau^*$. Choose a cover π as above. We can assume that π refines both σ and μ. Since the class $\mathcal{F}$ is also $N_p^{\mathcal{B}}$-surjective, there is a $\varrho \in \pi^+$ such that for every proper map $f\colon X \to Y$ from $\mathcal{F}$ and every M_p^{ϱ}-function $F\colon X \to B$ there is an M_p^{π}-function $K\colon Y \to B$ with $F \overset{\pi}{\sim} K \circ f$.

Consider a member X of $\mathcal{F}'$ and M_p^{ϱ}-functions F and G from X into B and assume that $F \overset{\sigma}{\sim} G$. Let $f\colon X \to Y$ be a map from the class $\mathcal{F}$. Choose two M_p^{π}-functions K and L from Y into B such that $F \overset{\pi}{\sim} K \circ f$ and $G \overset{\pi}{\sim} L \circ f$. The last two relations imply the relation $K \circ f \overset{\lambda}{\sim} L \circ f$. It follows that $K \overset{\alpha}{\sim} L$ and therefore that $K \overset{\mu}{\sim} L$. Thus, we obtain the following chain of relations : $F \overset{\pi}{\sim} K \circ f \overset{\tau}{\sim} L \circ f \overset{\pi}{\sim} G$. From here we conclude that $F \overset{\tau}{\sim} G$. $\Diamond$

7. Covered and extended classes

In this section we shall explore dependence of all proper shape invariants which were defined on classes of spaces involved under the assumption that these classes are connected by either surjections or injections. The connection can be through one of the following two notions.

Let $\mathcal{F}$ be a class of proper maps and let $\mathcal{B}$ and $\mathcal{C}$ be classes of spaces. We shall say that the class $\mathcal{C}$ is *$\mathcal{F}$-covered* by $\mathcal{B}$ provided for every $C \in \mathcal{C}$ there is a $B \in \mathcal{B}$ and an $h\colon B \to C$ from $\mathcal{F}$. Similarly, the class $\mathcal{C}$ is *$\mathcal{F}$-extended* by $\mathcal{B}$ provided for every $C \in \mathcal{C}$ there is a $B \in \mathcal{B}$ and a $k\colon C \to B$ from $\mathcal{F}$.

For a class of spaces $\mathcal{B}$ we shall use $\mathcal{B}_i$, $\mathcal{B}_s$, and $\mathcal{B}_b$ to denote the classes of all $M_p^{\mathcal{B}}$-injections, $M_p^{\mathcal{B}}$-surjections, and $M_p^{\mathcal{B}}$-bijections. Also, $\mathcal{B}^i$, $\mathcal{B}^s$, and $\mathcal{B}^b$ denote the classes of all $N_p^{\mathcal{B}}$-injections, $N_p^{\mathcal{B}}$-surjections, and $N_p^{\mathcal{B}}$-bijections. Moreover, if $\mathcal{F}$ and $\mathcal{G}$ are classes of maps we let $\mathcal{F}\mathcal{G}$ denote the intersection $\mathcal{F} \bigcap \mathcal{G}$.

We begin with the result on $M_p^{\mathcal{B},\mathcal{C}}$-smooth spaces and continue to cover all our proper shape invariants. The proofs are mostly omitted.

7.1 Theorem. *Let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, and $\mathcal{D}$ be classes of topological spaces. If a space X is $M_p^{\mathcal{A},\mathcal{D}}$-smooth and either*
(cc) *$\mathcal{B}$ is $\{X\}^i$-covered by $\mathcal{A}$ and $\mathcal{D}$ is $\{X\}^i$-covered by $\mathcal{C}$,*
(ce) *$\mathcal{B}$ is $\{X\}^i$-covered by $\mathcal{A}$ and $\mathcal{D}$ is $\mathcal{A}^s$-extended by $\mathcal{C}$,*
(ec) *$\mathcal{B}$ is $\mathcal{D}_s\{X\}^s$-extended by $\mathcal{A}$ and $\mathcal{D}$ is $\{X\}^i$-covered by $\mathcal{C}$, or*
(ee) *$\mathcal{B}$ is $\mathcal{C}_s\{X\}^s$-extended by $\mathcal{A}$ and $\mathcal{D}$ is $\mathcal{A}^s$-extended by $\mathcal{C}$,*
then X is also $M_p^{\mathcal{B},\mathcal{C}}$-smooth.

7.2 Theorem. *Let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$, and $\mathcal{X}$ be classes of spaces. If $\mathcal{X}$ is $N_p^{\mathcal{A},\mathcal{D}}$-smooth and either*
(cc) *$\mathcal{B}$ is $\mathcal{X}^i$-covered by $\mathcal{A}$ and $\mathcal{D}$ is $\mathcal{X}^i$-covered by $\mathcal{C}$,*
(ce) *$\mathcal{B}$ is $\mathcal{X}^i$-covered by $\mathcal{A}$ and $\mathcal{D}$ is $\mathcal{A}_s$-extended by $\mathcal{C}$,*
(ec) *$\mathcal{B}$ is $\mathcal{D}_s\mathcal{X}^b$-extended by $\mathcal{A}$ and $\mathcal{D}$ is $\mathcal{X}^i$-covered by $\mathcal{C}$, or*
(ee) *$\mathcal{B}$ is $\mathcal{D}_s\mathcal{X}^s$-extended by $\mathcal{A}$ and $\mathcal{D}$ is $\mathcal{B}^s$-extended by $\mathcal{C}$, then $\mathcal{X}$ is also $N_p^{\mathcal{B},\mathcal{C}}$-smooth.*

7.3 Theorem. *Let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$, and $\mathcal{X}$ be classes of topological spaces such that $\mathcal{B}$ is $\mathcal{X}_b$-covered by $\mathcal{A}$ and $\mathcal{X}$ is both $N_p^{\mathcal{A},\mathcal{D}}$-smooth and $M_p^{\mathcal{D},\mathcal{C}}$-tame. Then $\mathcal{X}$ is also $N_p^{\mathcal{B},\mathcal{C}}$-smooth.*

Proof. Let a member B of $\mathcal{B}$ and a cover σ of B be given. Let $\mu \in \sigma^*$. Since the class $\mathcal{B}$ is $\mathcal{X}_b$-covered by the class $\mathcal{A}$, there is an $A \in \mathcal{A}$ and an $M_p^{\mathcal{X}}$-bijection $h\colon A \to B$. Let $\delta = h^{-1}(\mu)$. We utilize now the assumption that the class $\mathcal{X}$ is $N_p^{\mathcal{A},\mathcal{D}}$-smooth to select an $\varepsilon \in \mathrm{Cov}(A)$ such that for M_p^{ε}-functions P and Q from a member X of $\mathcal{X}$ into A the relation $P \overset{\varepsilon}{\sim}_{\mathcal{D}} Q$ implies the relation $P \overset{\delta}{\sim} Q$. Since h is an $M_p^{\mathcal{X}}$-injection, there is a $\lambda \in \mathrm{Cov}(A)$ and a $\nu \in \mu^+$ such that for M_p^{λ}-functions P and Q from a member of $\mathcal{X}$ into A the relation $h \circ P \overset{\nu}{\sim} h \circ Q$ implies the relation $P \overset{\varepsilon}{\sim} Q$. Let $\kappa \in \nu^*$. At last, choose the required cover $\tau \in \kappa^+$ of B using the fact that h is an $M_p^{\mathcal{X}}$-surjection such that for every M_p^{τ}-function F from a member X of $\mathcal{X}$ into B there is an M_p^{λ}-function $P\colon X \to A$ with $F \overset{\kappa}{\sim} h \circ P$.

Consider an $X \in \mathcal{X}$ and M_p^{τ}-functions $F, G\colon X \to B$ and assume that $F \overset{\tau}{\sim}_{\mathcal{C}} G$. Pick M_p^{λ}-functions $P, Q\colon X \to A$ and M_p^{κ}-homotopies V and W joining F and $h \circ P$ and G and $h \circ Q$, respectively.

Our goal now is to show that $P \overset{\varepsilon}{\sim}_{\mathcal{D}} Q$. In order to do this, we must find a cover ξ of X so that $P \circ N \overset{\varepsilon}{\sim} Q \circ N$ for every M_p^{ξ}-function N from a member of $\mathcal{D}$ into X. First, observe that the assumption about F implies the existence of a $\theta \in \operatorname{Cov}(X)$ such that the relation $F \circ M \overset{\tau}{\sim} G \circ M$ holds for every M_p^{θ}-function M from a member of $\mathcal{C}$ into X. Let $\zeta \in \theta^{+}$ be from the intersection of sets $D(V, \kappa)$, $D(W, \kappa)$, $S(F, \tau)$, and $S(G, \tau)$. Since X is $M_p^{\mathcal{D}, \mathcal{C}}$-tame, there is a $\xi \in \operatorname{Cov}(X)$ with the property that for every M_p^{ξ}-function N from a member D of $\mathcal{D}$ into X there is a $C \in \mathcal{C}$ and an M_p^{ζ}-function $M \colon C \to X$ such that for every $\gamma \in \operatorname{Cov}(C)$ we can find an M_p^{γ}-function $K \colon D \to C$ with $N \overset{\zeta}{\sim} M \circ K$.

Let $D \in \mathcal{D}$ and let $N \colon D \to X$ be an M_p^{ξ}-function. Pick a C and an N as above. By assumption, there is an M_p^{τ}-homotopy Z joining $F \circ M$ and $G \circ M$. Let $\gamma \in D(Z, \tau)$. Choose a K as above. Our choices imply that $h \circ (P \circ N) \overset{\kappa}{\sim} F \circ N \overset{\tau}{\sim} F \circ M \circ K \overset{\tau}{\sim} G \circ M \circ K \overset{\tau}{\sim}$ $\overset{\tau}{\sim} G \circ N \overset{\kappa}{\sim} h \circ (Q \circ N)$. It follows that $h \circ (P \circ N) \overset{\nu}{\sim} h \circ (Q \circ N)$ so that $P \circ N \overset{\varepsilon}{\sim} Q \circ N$ and our claim has been verified.

Now, we conclude that $P \overset{\delta}{\sim} Q$ and therefore $h \circ P \overset{\mu}{\sim} h \circ Q$. Thus, we obtain now $F \overset{\kappa}{\sim} h \circ P \overset{\mu}{\sim} h \circ Q \overset{\kappa}{\sim} G$. Hence, $F \overset{\sigma}{\sim} G$. ◊

7.4 Theorem. *Let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$, and $\mathcal{X}$ be classes of spaces. If $\mathcal{X}$ is $N_p^{\mathcal{A}, \mathcal{D}}$-smooth and either*

(cc) *$\mathcal{B}$ is $\mathcal{C}_s\mathcal{X}_i$-covered by $\mathcal{A}$ and $\mathcal{C}$ is $\mathcal{A}^i$-covered by $\mathcal{D}$,*

(ce) *$\mathcal{B}$ is $\mathcal{C}_s\mathcal{X}_i$-covered by $\mathcal{A}$ and $\mathcal{C}$ is $\mathcal{A}^s$-extended by $\mathcal{D}$,*

(ec) *$\mathcal{B}$ is $\mathcal{D}_i$-extended by $\mathcal{A}$ and $\mathcal{C}$ is $\mathcal{B}^i$-covered by $\mathcal{D}$, or*

(ee) *$\mathcal{B}$ is $\mathcal{X}_i$-extended by $\mathcal{A}$ and $\mathcal{C}$ is $\mathcal{B}^s\mathcal{X}_s$-extended by $\mathcal{D}$,*

then $\mathcal{X}$ is also $P_p^{\mathcal{B}, \mathcal{C}}$-smooth.

7.5 Theorem. *Let $\mathcal{B}$ and $\mathcal{C}$ be classes of spaces. If a space X is $M_p^{\mathcal{B}}$-calm and the class $\mathcal{C}$ is either $\{X\}^i$-covered or $\{X\}^b$-extended by $\mathcal{B}$, then X is also $M_p^{\mathcal{C}}$-calm.*

7.6 Theorem. *Let $\mathcal{X}$, $\mathcal{B}$, and $\mathcal{C}$ be classes of spaces. If $\mathcal{X}$ is $N_p^{\mathcal{B}}$-calm and the class $\mathcal{C}$ is either $\mathcal{X}_b$-covered or $\mathcal{X}_i$-extended by $\mathcal{B}$, then $\mathcal{X}$ is also $N_p^{\mathcal{C}}$-calm.*

Proof. ($\mathcal{C}$ is $\mathcal{X}_i$-extended by $\mathcal{B}$). Let a member C of $\mathcal{C}$ be given. Since the class $\mathcal{C}$ is $\mathcal{X}_i$-extended by the class $\mathcal{B}$, there is a $B \in \mathcal{B}$ and an $M_p^{\mathcal{X}}$-injection $k \colon C \to B$. Since $\mathcal{X}$ is $N_p^{\mathcal{B}}$-calm, there is a cover $\sigma \in \operatorname{Cov}(B)$ such that for every $\tau \in \operatorname{Cov}(B)$ there is a $\varrho \in \operatorname{Cov}(B)$ with the property that for every $X \in \mathcal{X}$ and all M_p^{ϱ}-functions K and L from X into B the relation $K \overset{\sigma}{\sim} L$ implies the relation $K \overset{\tau}{\sim} L$. Let $\gamma = k^{-1}(\sigma)$. Then γ is the required cover of C.

Let δ be a cover of C. Since k is an $M_p^{\mathcal{X}}$-injection, there is a $\theta \in$ $\in \mathrm{Cov}(C)$ and a $\tau \in \mathrm{Cov}(B)$ such that for M_p^{θ}-functions F and G from a member of $\mathcal{X}$ into C the relation $k \circ F \overset{\tau}{\sim} k \circ G$ implies the relation $F \overset{\delta}{\sim} G$. Pick a ϱ as above and let $\varepsilon \in \mathrm{Cov}(C)$ be a common refinement of θ and $k^{-1}(\varrho)$.

Consider M_p^{ε}-functions F and G from a member X of $\mathcal{X}$ into C and assume that $F \overset{\gamma}{\sim} G$. Let K and L be the compositions $k \circ F$ and $k \circ G$. Then K and L are M_p^{ϱ}-functions with $K \overset{\sigma}{\sim} L$. It follows that $K \overset{\tau}{\sim} L$ and therefore that $F \overset{\delta}{\sim} G$. $\Diamond$

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MEROMORPHIC STARLIKE UNIVALENT FUNCTIONS WITH ALTERNATING COEFFICIENTS

M. K. **Aouf**

Department of Mathematics, Faculty of Science, University Mansoura, Egypt

H. E. **Darwish**

Department of Mathematics, Faculty of Science, University Mansoura, Egypt

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Abstract: Coefficient estimates and distortion theorems are obtained for meromorphic starlike univalent functions with alternating coefficients. Further class preserving integral operators are obtained.

1. Introduction

Let Σ denote the class of functions of the form

(1.1) $$f(z) = \frac{1}{z} + \sum_{m=1}^{\infty} a_m z^m$$

which are regular in the punctured disc $U^* = \{z : 0 < |z| < 1\}$. Define

$$D^0 f(z) = f(z),$$

$$D^1 f(z) = \frac{1}{z} + 3a_1 z + 4a_2 z^2 + \quad = \frac{(z^2 f(z))'}{z},$$

$$D^2 f(z) = D(D^1 f(z)).$$

and for $n = 1, 2, 3, \dots$

$$D^n f(z) = D(D^{n-1} f(z)) = \frac{1}{z} + \sum_{m=1}^{\infty} (m+2)^n a_m z^m = \frac{(zD^{n-1} f(z))'}{z}.$$

In [4] Uralegaddi and Somanatha obtained a new criteria for meromorphic starlike univalent functions via the basic inclusion relationship $B_{n+1}(\alpha) \subset B_n(\alpha)$, $0 \leqq \alpha < 1$, $n \in \mathbb{N}_0 = \{0, 1, \dots\}$, where $B_n(\alpha)$ is the class consisting of functions in $\sum$ satisfying

$$\text{(1.2)} \quad \operatorname{Re}\left\{ \frac{D^{n+1} f(z)}{D^n f(z)} - 2 \right\} < -\alpha, \quad |z| < 1, \quad 0 \leqq \alpha < 1, \quad n \in \mathbb{N}_0.$$

The condition (1.2) is equivalent to

$$\text{(1.3)} \qquad \frac{D^{n+1} f(z)}{D^n f(z)} = \frac{1 + (3 - 2\alpha) w(z)}{1 + w(z)},$$

$w(z) \in H = \{w \text{ regular}, w(0) = 0 \text{ and } |w(z)| < 1, z \in U = \{z \colon |z| < 1\}\}$, or, equivalently,

$$\text{(1.4)} \qquad \left| \frac{\frac{D^{n+1} f(z)}{D^n f(z)} - 1}{\frac{D^{n+1} f(z)}{D^n f(z)} + 2\alpha - 3} \right| < 1.$$

We note that $B_0(\alpha) = \Sigma^*(\alpha)$, is the class of meromorphically starlike functions of order α $(0 \leq \alpha < 1)$ and $B_0(0) = \Sigma^*$, is the class of meromorphically starlike functions.

Let σ_A be the subclass of Σ which consists of functions of the form

$$\text{(1.5)} \quad f(z) = \frac{1}{z} + a_1 z - a_2 z^2 + a_3 z^3 \dots = \frac{1}{2} + \sum_{m=1}^{\infty} (-1)^{m-1} a_m z^m, \quad a_m \geq 0$$

and let $\sigma^*_{A,n}(\alpha) = B_n(\alpha) \cap \sigma_A$.

In this paper coefficient inequalities, distortion theorems for the class $\sigma^*_{A,n}(\alpha)$ are determined. Techniques used are similar to these of Silverman [2] and Uralegaddi and Ganigi [3]. Finally, the class preserving integral operators of the form

(1.6) $$F(z) = \frac{c}{z^{c+1}} \int_0^z t^c f(t)dt \qquad (c > 0)$$

is considered.

2. Coefficient inequalities

Theorem 1. *Let* $f(z) = \frac{1}{z} + \sum_{m=1}^{\infty} a_m z^m$. *If*

(2.1) $$\sum_{m=1}^{\infty} (m+2)^n (m+\alpha)|a_m| \leq (1-\alpha),$$

then $f(z) \in B_n(\alpha)$.

Proof. Suppose (2.1) holds for all admissible values of α and n. It suffices to show that

$$\left| \frac{\frac{D^{n+1}f(z)}{D^n f(z)} - 1}{\frac{D^{n+1}f(z)}{D^n f(z)} + 2\alpha - 3} \right| < 1 \quad \text{for } |z| < 1.$$

We have

$$\left| \frac{\frac{D^{n+1}f(z)}{D^n f(z)} - 1}{\frac{D^{n+1}f(z)}{D^n f(z)} + 2\alpha - 3} \right| = \left| \frac{\sum_{m=1}^{\infty} (m+2)^n (m+1) a_m z^{m+1}}{2(1-\alpha) - \sum_{m=1}^{\infty} (m+2)^n (m-1+2\alpha) a_m z^{m+1}} \right| \leq$$

$$\leq \frac{\sum_{m=1}^{\infty} (m+2)^n (m+1)|a_m|}{2(1-\alpha) - \sum_{m=1}^{\infty} (m+2)^n (m-1+2\alpha)|a_m|}.$$

The last expression is bounded above by 1, provided

$$\sum_{m=1}^{\infty} (m+2)^n (m+1)|a_m| \leq 2(1-\alpha) - \sum_{m=1}^{\infty} (m+2)^n (m-1+2\alpha)|a_m|$$

which is equivalent to (2.1), and this is true by hypothesis. ◊

For functions in $\sigma^*_{A,n}(\alpha)$ the converse of the above theorem is also true.

Theorem 2. *A function $f(z)$ in σ_A is in $\sigma^*_{A,n}(\alpha)$ if and only if*

$$\sum_{m=1}^{\infty}(m+2)^n(m+\alpha)a_m \leq (1-\alpha). \tag{2.2}$$

Proof. In view of Th. 1 it suffices to show the only if part. Suppose

$$\operatorname{Re}\left\{\frac{D^{n+1}f(z)}{D^n f(z)}-2\right\}= \tag{2.3}$$
$$=\operatorname{Re}\left\{\frac{-\frac{1}{z}+\sum\limits_{m=1}^{\infty}(-1)^{m-1}(m+1)^n m a_m z^m}{\frac{1}{z}+\sum\limits_{m=1}^{\infty}(-1)^{m-1}(m+2)^n a_m z^m}\right\}<-\alpha.$$

Choose values of z on the real axis so that $\left(\frac{D^{n+1}f(z)}{D^n f(z)}-2\right)$ is real. Upon clearing the denominator in (2.3) and letting $z\to -1$ through real values, we obtain

$$1-\sum_{m=1}^{\infty}(m+2)^n m a_m \geq \alpha\Big(1+\sum_{m=1}^{\infty}(m+2)^n a_m\Big)$$

which is equivalent to (2.2). ◊

Corollary 1. *Let the function $f(z)$ defined by (1.5) be in the class $\sigma^*_{A,n}(\alpha)$. Then*

$$a_m \leq \frac{(1-\alpha)}{(m+2)^n(m+\alpha)} \qquad (m\geq 1).$$

Equality holds for the functions of the form

$$f_m(z)=\frac{1}{z}+(-1)^{m-1}\frac{(1-\alpha)}{(m+2)^n(m+\alpha)}z^m.$$

3. Distortion theorems

Theorem 3. *Let the function $f(z)$ defined by (1.5) be in the class $\sigma^*_{A,n}(\alpha)$. Then for $0<|z|=r<1$,*

$$\frac{1}{r}-\frac{1-\alpha}{3^n(1+\alpha)}r \leq |f(z)| \leq \frac{1}{r}+\frac{1-\alpha}{3^n(1+\alpha)}r \tag{3.1}$$

with equality for the function

$$f(z) = \frac{1}{z} + \frac{1-\alpha}{3^n(1+\alpha)} z \quad at\ z = r, ir. \tag{3.2}$$

Proof. Suppose $f(z)$ is in $\sigma^*_{A,n}(\alpha)$. In view of Th. 2, we have

$$3^n(1+\alpha) \sum_{m=1}^{\infty} a_m \le \sum_{m=1}^{\infty} (m+2)^n (m+\alpha) a_m \le (1-\alpha)$$

which evidently yields

$$\sum_{m=1}^{\infty} a_m \le \frac{1-\alpha}{3^n(1+\alpha)}.$$

Consequently, we obtain

$$|f(z)| \le \frac{1}{r} + \sum_{m=1}^{\infty} a_m r^m \le \frac{1}{r} + r \sum_{m=1}^{\infty} a_m \le \frac{1}{r} + \frac{1-\alpha}{3^n(1+\alpha)} r.$$

Also

$$|f(z)| \ge \frac{1}{r} - \sum_{m=1}^{\infty} a_m r^m \ge \frac{1}{r} - r \sum_{m=1}^{\infty} a_m \ge \frac{1}{r} - \frac{1-\alpha}{3^n(1+\alpha)} r.$$

Hence the results (3.1) follow. ◊

Theorem 4. *Let the function $f(z)$ defined by (1.5) be in the class $\sigma^*_{A,n}(\alpha)$. Then for $0 < |z| = r < 1$,*

$$\frac{1}{r^2} - \frac{1-\alpha}{3^n(1+\alpha)} \le |f'(z)| \le \frac{1}{r^2} + \frac{1-\alpha}{3^n(1+\alpha)}. \tag{3.3}$$

The result is sharp, the extremal function being of the form (3.2).

Proof. From Th. 2, we have

$$3^n(1+\alpha) \sum_{m=1}^{\infty} m a_m \le \sum_{m=1}^{\infty} (m+2)^n (m+\alpha) a_m \le (1-\alpha)$$

which evidently yields

$$\sum_{m=1}^{\infty} m a_m \le \frac{1-\alpha}{3^n(1+\alpha)}.$$

Consequently, we obtain

$$|f'(z)| \leq \frac{1}{r^2} + \sum_{m=1}^{\infty} m a_m r^{m-1} \leq \frac{1}{r^2} + \sum_{m=1}^{\infty} m a_m \leq \frac{1}{r^2} + \frac{1-\alpha}{3^n(1+\alpha)}.$$

Also

$$|f'(z)| \geq \frac{1}{r^2} - \sum_{m=1}^{\infty} m a_m r^{m-1} \geq \frac{1}{r^2} - \sum_{m=1}^{\infty} m a_m \geq \frac{1}{r^2} - \frac{1-\alpha}{3^n(1+\alpha)}.$$

This completes the proof. ◊

Putting $n = 0$ in Th. 4, we get

Corollary 2. *Let the function* $f(z)$ *defined by* (1.5) *be in the class* $\sigma^*_{A,0}(\alpha) = \sigma^*_A(\alpha)$. *Then for* $0 < |z| = r < 1$,

$$\frac{1}{r^2} - \frac{1-\alpha}{1+\alpha} \leq |f'(z)| \leq \frac{1}{r^2} + \frac{1-\alpha}{1+\alpha}.$$

The result is sharp.

We observe that our result in Cor. 2 improves the result of Uralegaddi and Ganigi [3, Th. 3 (Equation 4)].

4. Class preserving integral operators

In this section we consider the class preserving integral operators of the form (1.6).

Theorem 5. *Let the function* $f(z)$ *be defined by* (1.5) *be in the class* $\sigma^*_{A,n}(\alpha)$. *Then*

$$F(z) = cz^{-c-1} \int_0^z t^c f(t)dt = \frac{1}{z} + \sum_{m=1}^{\infty} (-1)^{m-1} \frac{c}{c+m+1} a_m z^m, \quad c > 0$$

belongs to the class $\sigma^*_{A,n}(\beta(\alpha, n, c))$, *where*

$$\beta(\alpha, n, c) = \frac{(1+\alpha)(c+2) - c(1-\alpha)}{(1+\alpha)(c+2) + c(1-\alpha)}.$$

The result is sharp for

$$f(z) = \frac{1}{z} + \frac{1-\alpha}{3^n(1+\alpha)}z.$$

Proof. Suppose $f(z) \in \sigma^*_{A,n}(\alpha)$, then

$$\sum_{m=1}^{\infty} (m+2)^n(m+\alpha)a_m \leq (1-\alpha).$$

In view of Th. 2 we shall find the largest value of β for which

$$\sum_{m=1}^{\infty} \frac{(m+2)^n(m+\beta)}{(1-\beta)} \cdot \frac{c}{c+m+1}a_m \leq 1.$$

It suffices to find the range of values of β for which

$$\frac{c(m+2)^n(m+\beta)}{(1-\beta)(c+m+1)} \leq \frac{(m+2)^n(m+\alpha)}{(1-\alpha)} \quad \text{for each } m.$$

Solving the above inequality for β we obtain

$$\beta \leq \frac{(m+\alpha)(c+m+1) - mc(1-\alpha)}{(m+\alpha)(c+m+1) + c(1-\alpha)}.$$

For each α and c fixed let

$$F(m) = \frac{(m+\alpha)(c+m+1) - mc(1-\alpha)}{(m+\alpha)(c+m+1) + c(1-\alpha)}.$$

Then

$$F(m+1) - F(m) = \frac{A}{B} > 0 \quad \text{for each } m,$$

where

$$A = c(m+1)(m+2)(1-\alpha)$$

and

$$B = [(m+1+\alpha)(c+m+2)+c(1-\alpha)][(m+\alpha)(c+m+1)+c(1-\alpha)].$$

Hence $F(m)$ is an increasing function of m. Since

$$F(1) = \frac{(1+\alpha)(c+2) - c(1-\alpha)}{(1+\alpha)(c+2) + c(1-\alpha)}$$

the result follows. ◊

Remark. Putting $n = 0$ in the above theorems, we have the results obtained by Uralegaddi and Ganigi [3].

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ON THE FELLER STRONG LAW OF LARGE NUMBERS FOR FIELDS OF B-VALUED RANDOM VARIABLES

Zbigniew A. **Lagodowski**

Department of Mathematics, Technical University, 20–618 Lublin, Nadbystrzycka 38, Poland

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Abstract: The purpose of this note is to provide Feller type strong law of large numbers for sums of i.i.d. B-valued random variables with multidimensional indices.

1. Introduction

Let $\{X_{\bar{n}}, \bar{n} \in \mathbb{N}^r\}$ be a field of independent and identically distributed (i.i.d.) random variables taking values in a separable Banach space $(B, \| \ \|)$. $\mathbb{N}^r$ denotes the positive integer r-dimensional lattice points, r is positive integer. Assume that points of $\mathbb{N}^r$ are denoted by $\bar{n} = (m_1, ..., m_r)$, $\bar{n} = (n_1, ..., n_r)$ etc. and ordered by coordinatewise partial ordering. For $\bar{n} = (n_1, ..., n_r) \in \mathbb{N}^r$, we define $S_{\bar{n}} = \sum\limits_{\bar{k} \leq \bar{n}} X_{\bar{n}}$ and $|\bar{n}| = \prod\limits_{i=1}^{r} n_i$. Throughout this paper, $\bar{n} \to \infty$ means $|\bar{n}| \to \infty$. Further, let $\{a_{\bar{n}}, \bar{n} \in \mathbb{N}^r\}$ be an increasing directed (directed upwards) family of positive numbers, i.e $a_{\bar{m}} \leq a_{\bar{n}}$ whenever $\bar{m} \leq \bar{n}$ and $a_{\bar{n}} \to \infty$ as $\bar{n} \to \infty$.

In order to bring into focus the main aim of this paper we start with a description of Fazekas' result [2]. Let B be separable Banach space, $1 \leq p < 2$ and $\{X_{\bar{n}}, \bar{n} \in \mathbb{N}^r\}$ be i.i.d. B-valued random variables. Assume that $E\|X_1\|^p(\log^+ \|X_1\|)^{d-1} < \infty$. If $S_{\bar{n}}/|\bar{n}|^{1/p} \xrightarrow{P} 0$ as $\bar{n} \to \to \infty$ then $S_{\bar{n}}/|\bar{n}|^{1/p} \to 0$ a.s. as $\bar{n} \to \infty$. Our main aim is to establish the strong law of large numbers with an arbitrary normalizing family and no moment restriction assuming on a random field $\{X_{\bar{n}}, \bar{n} \in \mathbb{N}^r\}$, which is also a generalization of some results obtained for a field of real random variables (cf. Gut [3], Klesov [4]). We exploit the concept of Mikosh and Norvaisa [8], and assume similar properties for normalizing family of positive numbers $\{a_{\bar{n}}, \bar{n} \in \mathbb{N}^r\}$.

2. Auxiliary lemmas

In this section we collect some auxiliary results needed later on. Let $\{a_{\bar{n}}, \bar{n} \in \mathbb{N}^r\}$ be a field of positive numbers such that $\lim_{\bar{n}\to\infty} a_{\bar{n}} = \infty$ and let there exist a sequence D_k, $k \geq 1$ of finite subsets of $\mathbb{N}^r$ such that $D_k \uparrow \mathbb{N}^r$, and satisfies the following conditions:

(A) Set $I_k = D_k - D_{k-1}$, $k \geq 1$. If $\bar{n} \in I_k$, then $(\bar{n}) \subseteq D_k$;

(B) There are constants $d > 1$, $C_1, C_2 > 0$ such that for every $k, \bar{n} \in I_k$, the relation $C_1 d^k \leq a_{\bar{n}} \leq C_2 d^k$ holds;

(C) For every k there exist disjoint rectangles E_{kl} and an appropriate index set R_k such that $I_k = \bigcup_{l \in R_k} E_{kl}$;

(D) $\nu_0 \overline{\lim_{k\to\infty}} \max_{\bar{n}\in I_k} d^{-k} \sum_{i=1}^{k} d^i \big|\{t \in R_i : E_{it} \cap (\bar{n}) \neq \emptyset\}\big| < \infty.$

Conditions (A), (B), (C) and (D) come from Mikosh and Norvaisa [8] and field of numbers satisfying them is said to have the *weak star property*. For examples of the weak star property see Mikosh [7], Mikosh, Norvaisa [8] and the *star property* see Li, Wang, Rao [5–6].

Lemma 2.1 (Mikosh, Norvaisa [8]). *Let $\{X_{\bar{n}}, \bar{n} \in \mathbb{N}^r\}$ be a field of independent, symmetric B-valued r.v.'s. Assume that* (A)–(D) *hold. Then the condition*

$$\sum_k \sum_{l \in R_k} P\big(\|S_{E_{kl}}\| > \varepsilon d^k\big) < \infty, \quad \forall \varepsilon > 0 \tag{2.1}$$

is equivalent to the strong law of large numbers

$$S_{\bar{n}}/a_{\bar{n}} \to 0, \quad a.s.\ as\ \bar{n} \to \infty \tag{2.2}$$

Lemma 2.2. *Let $\{X_{\bar{n}}, \bar{n} \in \mathbb{N}^r\}$ be a field of independent, symmetric B-valued r.v.'s. Assume that* (A)–(C) *hold and*

(2.3) $$\|X_{\bar{k}}\| \le a_{\bar{k}} \quad a.s. \quad (\bar{k} \in \mathbb{N}^r),$$

(2.4) $$S_{\bar{n}}/a_{\bar{n}} \to 0 \quad \textit{in probability}.$$

Then for all $p > 0$, $E\|S_{E_{kl}}/d^k\|^p \to 0$ *as* $k \to \infty$ *uniformly in* $l \in R_k$.

Proof. Lemma V-1-1 of Neveu [9] implies that, the family $\{x_{\bar{n}}, \bar{n} \in \mathbb{N}^r\}$ converges to x if the net $\{x_{\bar{n}}, \bar{n} \in K\}$ converges to x for any infinite linearly ordered subset $(K, \le)$ of $\mathbb{N}^r$, thus by Lemma 3.1 of de Acosta [1] $E\|S_{\bar{n}}/a_{\bar{n}}\|^p \to 0$ as $\bar{n} \to \infty$. Then it is easy to see that for an arbitrary $l \in R_k$, $\lim\limits_{k\to\infty} E\|S_{E_{kl}}\|^p/d^{kp} \to 0$. Since for every k, R_k are finite, therefore $\lim\limits_{k\to\infty} \max\limits_{l\in R_k} E\|S_{E_{kl}}\|^p/d^{kp} \to 0$. ◊

3. Results

Let $M_j = \mathrm{card}\{\bar{n} \in \mathbb{N}^r : a_{\bar{n}} \le j\}$ and $m_j = M_j - M_{j-1}$ for every integer $j \ge 1$.

Theorem 3.1. *Let* $\{X_{\bar{n}}, \bar{n} \in \mathbb{N}^r\}$ *be a field of i.i.d. Banach space valued random variables and* $\{a_{\bar{n}}, \bar{n} \in \mathbb{N}^r\}$ *be an increasing directed field of positive numbers. Suppose that there exist* $j_0 \ge 1$ *and positive numbers* C_3, C_4 *such that*

(3.1) $$\forall_{j>j_0} M_j \le C_3 M_{j-1}, \quad \sum_{i\ge j} i^{-3} M_i \le C_4 j^{-2} M_j.$$

Then

(3.2) $$\sum_{\bar{n}} P\big(|X_{\bar{n}}| \ge a_{\bar{n}}\big) < \infty,$$

(3.3) $$S_{\bar{n}}/a_{\bar{n}} \to 0 \quad \textit{in probability}$$

are equivalent to

(3.4) $$S_{\bar{n}}/a_{\bar{n}} \to 0 \quad a.s.$$

Proof. It is enough to prove (3.2) and (3.3)⇒(3.4). We assume, without loss of generality, that $\{X_{\bar{n}}, \bar{n} \in \mathbb{N}^r\}$ are symmetric. Let us put

$$Y_{\bar{j}} = X_{\bar{j}}\big(\|X_j\| > a_{\bar{j}}\big) \quad \text{and} \quad T_{\bar{n}} = \sum_{\bar{j}\le\bar{n}} Y_{\bar{j}}.$$

By the virtue of (3.2) and from the Borel–Cantelli lemma follows

(3.5) $$(S_{\bar n} - T_{\bar n})/a_{\bar n} \to 0 \quad \text{a.s. as } \bar n \to \infty.$$

Therefore it is enough to prove $T_{\bar n}/a_{\bar n} \to 0$ a.s. as $\bar n \to \infty$. Let us put

$$V_{kl} = \|T_{E_{kl}}\| - E\|T_{E_{kl}}\|.$$

Thus by Th. 2.1 of de Acosta [1] we have

$$\sum_{k=1}^{\infty} P\big(\max_{l\in R_k} |V_{kl}|/d^k > \varepsilon\big) \le \sum_{k=l}^{\infty} \sum_{l\in R_k} P\big(|V_{kl}|/d^k > \varepsilon\big) \le$$

$$\le \frac{1}{\varepsilon^2}\sum_{k=1}^{\infty}\sum_{l\in R_k} E|V_{kl}|^2/d^{2k} \le \frac{4}{\varepsilon^2}\sum_{k=1}^{\infty}\sum_{l\in R_k}\sum_{\bar j\in E_{kl}} E\|Y_{\bar j}\|^2/d^{2k} \le$$

$$\le \frac{4C_2^2}{\varepsilon^2}\sum_{k=1}^{\infty}\sum_{l\in R_k}\sum_{\bar j\in E_{kl}} E\|Y_{\bar j}\|^2/a_{\bar j}^2 \le \frac{4C_2^2}{\varepsilon^2}\sum_{\bar n\in\mathbb{N}^r} E\|Y_{\bar n}\|^2/a_{\bar n}^2.$$

On the other hand

$$\sum_{\bar k} E\|Y_{\bar k}\|^2/a_{\bar k}^2 = \sum_{\bar k} E\{\|X\|^2 I(\|X\| < a_{\bar k})\}/a_{\bar k}^2 \le$$

$$\le C + C\sum_{i\ge 1} i^{-2} m_i E\{\|X\|^2 I(\|X\| < i)\} \le$$

$$\le C + C\sum_{j\ge 1} E\{\|X\|^2 I(j-1 \le \|X\| < 1)\}\sum_{i\ge j} i^{-2} m_i$$

for some constants C.

Now let us observe that by assumptions and Abel transform we get

$$\sum_{i\ge j} i^{-2} m_i \le C\sum_{i\ge j} i^{-3} M_i \le C j^{-2} M_j.$$

Hence

(3.6) $$\sum_{\bar k} E\|Y_{\bar k}\|^2/a_{\bar k}^2 \le C + C\sum_{j\ge 1} j^{-2} M_j E\{\|X\|^2 I(j-1\le\|X\|<j)\} \le$$
$$\le C + C\sum_{i\ge 1} m_i P(\|X\|\ge i) \le C + C\sum_{\bar n} P(\|X\|\ge a_{\bar n}) < \infty.$$

Therefore by the Borel–Cantelli Lemma

(3.7) $$\max_{l\in R_k} |V_{kl}|/d^k \to 0 \quad \text{a.s.}$$

It follows at once from (3.3) and (3.5) that $T_{\bar n}/a_{\bar n} \xrightarrow{P} 0$ and by Lemma 2.2

$$(3.8) \qquad E\|T_{E_{kl}}/d^k\| \to 0 \quad \text{uniformly in } l \in R_k.$$

Now, let us observe that (3.7) and (3.8) imply

$$T_{E_{kl}}/d^k \to 0 \quad \text{a.s. uniformly in } l \in R_k$$

and applications of Lemma 2.1 complete the proof. ◊

Remark. Let us observe that condition (3.2) is essential. However, it is necessary for (3.4), but (3.3) does not imply (3.2). We will give appropriate example. Let $\{X_k, k \geq 1\}$ be a sequence of i.i.d. random variables with probability density function

$$f(x) = \begin{cases} k(1+\ln|x|)/(x^2\ln^2|x|), & \text{for } |x| \geq 2 \\ \frac{1}{4}(1+\ln 2)/(1+\ln 2), & \text{for } |x| \leq 2 \end{cases}$$

where constant $k = \ln^2 2/(1+2\ln 2)$.

It is easy to see that $\sum_{i=1}^{n} X_i/n \to 0$ in probability but condition (3.2) is not satisfied.

Corollary 3.1. *Let* $a_{\bar{n}} = |\bar{n}|^{1/p}$, $1 \leq p < 2$ *then* $M_j = \mathrm{O}(j^p(\log j)^{r-1})$ (cf. Smythe [10]) *and* (3.1) *is satisfied. Convergence of series* (3.2) *is equivalent to* $E\|X\|^p(\log_+\|X\|^{r-1}) < \infty$.

Thus by Th. 3.1 we can obtain immediately result of Fazekas [2].

The following corollary is not only generalization of Marcinkiewicz SLLN but give us a better and deeper understanding of strong laws for random variables with multidimensional indices.

Corollary 3.2. *Let* $a_{\bar{n}} = n_1^{1/p_1} \cdot \ldots \cdot n_r^{1/p_r}$, $1 \leq p_i < 2$, $1 \leq i \leq r$, $t = \max(p_1, \ldots, p_r)$, $q = \mathrm{card}\{i : p_i = t, 1 \leq i \leq r\}$. *Thus* $M_j = \mathrm{O}(j^t(\log_+ j)^{q-1})$ *and* (3.2) *imply* $E\|X\|^t(\log_+\|X\|^{q-1}) < \infty$. *Then the following are equivalent:*

$$\text{(i)} \qquad S_{\bar{n}}/n_1^{1/p_1} \cdot \ldots \cdot n_r^{1/p_r} \to 0 \quad a.s. \ as \ \bar{n} \to \infty;$$

$$\text{(ii)} \qquad S_{\bar{n}}/n_1^{1/p_1} \cdot \ldots \cdot n_r^{1/p_r} \xrightarrow{P} 0 \quad as \ \bar{n} \to \infty.$$

Furthermore, let us observe that for $r=1$ all increasing to infinity, positive sequences $\{a_{\bar{n}}, \bar{n} \in \mathbb{N}^r\}$ have the weak star property and assumption (3.1) implies the well-known Feller condition $\sum_{k \geq n} a_k^{-2} = \mathrm{O}(n/a_n^2)$.

Th. 3.1 and corollaries are established for separable Banach space $(B, \| \ \|)$. In what follows we will assume geometric conditions on the space $(B, \| \ \|)$.

Theorem 3.2. *Let Banach space* $(B, \| \ \|)$ *be of type 2,* $\{X_{\bar{n}}, \bar{n} \in \mathbb{N}^r\}$, $\{a_{\bar{n}}, \bar{n} \in \mathbb{N}^r\}$ *are such as in* Th. 1 *and* $EX_{\bar{n}} = 0$, $\bar{n} \in \mathbb{N}^r$. *If moreover* (3.1) *and* (3.2) *are satisfied then*

$$S_{\bar{n}}/a_{\bar{n}} \to 0 \quad a.s. \ as \ \bar{n} \to \infty.$$

Proof. A family $\{x_{\bar{n}}, \bar{n} \in \mathbb{N}^r\}$ is said to be Cauchy if for every $\varepsilon > 0$ we have $d(x_{\bar{n}}, x_{\bar{m}}) < \varepsilon$ whenever $\bar{n}, \bar{m} \geq \bar{k}_\varepsilon$ for suitably chosen $\bar{k}_\varepsilon$ in $\mathbb{N}^r$. Since B is of type 2 one can prove using estimation as in (3.6) that, $\{Y_{\bar{n}}/a_{\bar{n}}, \bar{n} \in \mathbb{N}^r\}$ is a Cauchy family. Hence $\{Y_{\bar{n}}/a_{\bar{n}}, \bar{n} \in \mathbb{N}^r\}$ convergence in L^2, then converges almost surely. Further, by multidimensional version of Kronecker Lemma and arguments as in proof of Th. 3.1, we deduce the assertion of Th. 3.2. ◊

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ON THE CALCULATION OF EVOLUTIONARILY STABLE STRATEGIES

Helmut **Länger**

Technische Universität, Institut für Algebra und Diskrete Mathematik, Wiedner Hauptstraße 8-10, A-1040 Wien, Austria

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Abstract: A simple procedure is developed in order to calculate all evolutionarily stable strategies not involving at most three pure strategies. By means of this procedure especially all evolutionarily stable strategies of an at most four-dimensional payoff matrix can be determined.

Applying game-theoretical methods to problems in population dynamics, Maynard Smith and Price ([15]) introduced the notion of an evolutionarily stable strategy (ESS). Such a strategy is in some sense robust against new strategies invading the population. For literature concerning theoretical investigations on ESS's cf. e.g. [1]–[14] and [16]. The aim of this paper is to give a simple necessary condition for ESS's and to show how one can determine for a given payoff matrix all ESS's with "large" support.

In the following let n denote a positive integer and let $I = \{i_1, \dots, i_s\} \subseteq N := \{1, \dots, n\}$ with $i_1 < \dots < i_s (0 \le s \le n)$. If not stated otherwise, all indices run from 1 to n. Let $A = (a_{ij})$ and $B = (b_{ij})$ be real matrices with $b_{ii} = 1$ and $b_{ij} = b_{ji}$ for all i, j. Further let $a, b, c \in \mathbb{R}$ and put

$$C := \begin{pmatrix} 1 & a \\ a & 1 \end{pmatrix} \quad \text{and } D := \begin{pmatrix} 1 & a & b \\ a & 1 & c \\ b & c & 1 \end{pmatrix}.$$

Let $\mathbb{R}^n$ denote the set of all n-dimensional real row vectors. For a vector x and an index i let x_i denote the component of x corresponding to i. By $x^T \leq y^T (x, y \in \mathbb{R}^n)$ we mean $x_i \leq y_i$ for all i (here and in the following ^{T} denotes transposition). Put $S := \{x \in \mathbb{R}^n | x_i \geq \geq 0$ for all i and $\sum x_i = 1\}$. $p \in S$ is called a *Nash-equilibrium* of A if $pAp^T \geq xAp^T$ for all $x \in S$. Let $N(A)$ denote the set of all Nash equilibria of A. $p \in N(A)$ is called an *ESS* of A if $pAx^T > xAx^T$ for all $x \in S \setminus \{p\}$ with $xAp^T = pAp^T$. Let $E(A)$ denote the set of all ESS's of A. For $x \in \mathbb{R}^n$ define the support supp x of x by supp $x := \{i | x_i \neq 0\}$. Put $S(I) := \{x \in S | \text{supp } x = I\}$. For every i let e_i denote the unique element of $S(\{i\})$. A is called *strictly I-copositive* if $xAx^T > 0$ for all $x \in \mathbb{R}^n \setminus \{(0, \ldots, 0)\}$ with $x_i \geq 0$ for all $i \in I$. A is called *strictly copositive* if it is strictly N-copositive. Observe that in case $|I| \leq 1$ strict I-copositiveness coincides with positive definiteness.

A fundamental problem in evolutionary biology is the determination of $E(A)$ for a given payoff matrix A. The following lemma says that for the sake of determining $N(A)$ or $E(A)$ we can restrict ourselves to matrices having 0 in their main diagonal:

Lemma 1 (cf. [3]). *$N(A) = N((a_{ij} - a_{jj}))$ and $E(A) = E((a_{ij} - a_{jj}))$.*

Proof. $\sum_{i,j}(x_i - y_i)(a_{ij} - a_{jj})z_j = (x - y)^T Az$ for all $x, y, z \in S$. ◊

A further simplification of the problem of determining $E(A)$ is provided by the following lemma:

Lemma 2 (cf. [2] and [13]). *The supports of two different ESS's of A are incomparable.*

Proof. Assume there exist two distinct ESS's p, q of A with supp $p \subseteq \subseteq$ supp q. From $qAq^T = \sum q_i(e_iAq^T) \leq \sum q_i(qAq^T) = qAq^T$ it follows that $e_iAq^T = qAq^T$ for all $i \in$ supp q. Since supp $p \subseteq$ supp q, we would have $pAq^T = \sum p_i(e_iAq^T) = \sum p_i(qAq^T) = qAq^T$ and hence, because of $q \in E(A)$, also $qAp^T > pAp^T$. But this contradicts $p \in E(A)$. ◊

Next we want to characterize the Nash equilibria of A:

Theorem 3 (cf. [13]). *Let $p \in S(I)$, assume $k \in I$ and put $p_0 := := -e_kAp^T$. Then t. f. a. e.:*
(i) $p \in N(A)$.
(ii) (a) *and* (b) *hold:*
(a) $(p_0, p_{i_1}, \ldots, p_{i_s})$ *is a solution of*

$$\begin{pmatrix} 0 & 1 & \dots & 1 \\ 1 & a_{i_1 i_1} & \dots & a_{i_1 i_s} \\ \vdots & \vdots & & \vdots \\ 1 & a_{i_s i_1} & \dots & a_{i_s i_s} \end{pmatrix} \begin{pmatrix} x_0 \\ x_{i_1} \\ \vdots \\ x_{i_s} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix};$$

(b) $(p_0, \dots, p_n)$ *is a solution of*

$$\begin{pmatrix} 1 & a_{11} & \dots & a_{1n} \\ \vdots & \vdots & & \vdots \\ 1 & a_{n1} & \dots & a_{nn} \end{pmatrix} \begin{pmatrix} x_0 \\ \vdots \\ x_n \end{pmatrix} \le \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Proof. (i) $\Rightarrow$ (ii):
$pAp^T = \sum p_i(e_iAp^T) \le \sum p_i(pAp^T) = pAp^T$ shows that $e_iAp^T = = pAp^T = e_kAp^T = -p_0$ for all $i \in I$ and that $e_iAp^T \le pAp^T = = e_kAp^T = -p_0$ for all i.
(ii) $\Rightarrow$ (i):
$xAp^T = \sum x_i(e_iAp^T) \le \sum x_i(-p_0) = \sum p_i(-p_0) = \sum p_i(e_iAp^T) = = pAp^T$ for all $x \in S$.

Now we are going to show a way of calculating all ESS's of A:
Theorem 4 (cf. [1] and [4]). *Let $p \in S(I)$, assume $k \in I$ and put $p_0 := -e_kAp^T$. Then t. f. a. e.:*
(i) $p \in E(A)$.
(ii) (a) - (c) *hold:*
(a) $(p_0, p_{i_1}, \dots, p_{i_s})$ *is the unique solution of* (1) *over* $\mathbb{R}$;
(b) $(p_0, \dots, p_n)$ *satisfies* (2);
(c) *if* $|J| > 1$ *then* $(a_{ik} + a_{kj} - a_{ij} - a_{kk})_{i,j \in J\setminus\{k\}}$ *is strictly* $(J\setminus I)$*-copositive, where* $J := \{i | (e_k - e_i)Ap^T = 0\}$.

Proof. Assume $p \in E(A)$. Then, by Th. 3, $(p_0, p_{i_1}, \dots, p_{i_s})$ is a solution of (1) over $\mathbb{R}$. Suppose, there exists another solution of (1) over $\mathbb{R}$. Then there would exist such a solution $(q_0, q_{i_1}, \dots, q_{i_s})$ with $q_i > 0$ for all $i \in I$. But then $q \in \mathbb{R}^n$ defined by $q_i := 0$ for all $i \notin I$ would be an element of $S \setminus \{p\}$ with $qAp^T = -p_0 = pAp^T$ and $pAq^T = = -q_0 = qAq^T$ contradicting $p \in E(A)$. Hence (1) is uniquely solvable over $\mathbb{R}$. The rest of the proof follows from Th. 3 and [12].
Remark. Let F denote the matrix in (1) and assume $k \in I$. Then $|F|$ can be expanded in the following way: Subtract the row corresponding to k from the rows corresponding to the elements of $I \setminus \{k\}$, expand the resulting determinant along the first column, subtract the column

corresponding to k from the columns corresponding to the elements of $I \setminus \{k\}$ and expand the resulting determinant along the first row. This shows $|F| = (-1)^{|I|}|(a_{ik} + a_{kj} - a_{ij} - a_{kk})_{i,j\in I\setminus\{k\}}|$, the latter determinant being positive in case there exists some $p \in E(A)$ with support I, according to [12]. Hence $(-1)^{|I|}|F| > 0$ in this case and therefore, using Cramer's rule, (a) and (b) can be translated in this case into equations and inequalities involving certain determinants.

The next problem is to decide whether a given quadratic matrix is I-copositive. First we remark that for the sake of investigating strict I-copositiveness of a quadratic matrix we can restrict ourselves to symmetric matrices having 1 in their main diagonal:

Lemma 5 (cf. [11]). *T. f. a. e.:*

(i) *A is strictly I-copositive.*

(ii) (a) *and* (b) *hold:*

(a) $a_{ii} > 0$ *for all i;*

(b) $\left(\frac{a_{ij}+a_{ji}}{2\sqrt{a_{ii}a_{jj}}}\right)$ *is strictly I-copositive.*

Proof. We have $a_{ii} = e_iAe_i^T$ for all i, and in case $a_{ii} > 0$ for all i we have $\sum\limits_{i,j} \frac{a_{ij}+a_{ji}}{2\sqrt{a_{ii}a_{jj}}}(x_i\sqrt{a_{ii}})(x_j\sqrt{a_{jj}}) = xAx^T$ for all $x \in \mathbb{R}^n$.

The following lemma shows how one can reduce strict I-copositiveness of an n-dimensional matrix in case $n > \max(1,|I|)$ to strict I-copositiveness of a matrix of dimension $n-1$:

Lemma 6. *Assume $n > 1$ and $k \notin I$. Then t. f. a. e.:*

(i) *B is strictly I-copositive.*

(ii) $(b_{ij} - b_{ik}b_{jk})_{i,j\neq k}$ *is strictly I-copositive.*

Proof. $xBx^T = (x_k + \sum\limits_{i\neq k} b_{ik}x_i)^2 + \sum\limits_{i,j\neq k}(b_{ij} - b_{ik}b_{jk})x_ix_j$ for all $x \in \mathbb{R}^n$.

By Lemma 6, strict I-copositiveness of an n-dimensional matrix can be reduced to strict I-copositiveness of a matrix of dimension $\max(1,|I|)$. Hence, in order to settle the case $|I| \leq 3$ completely, one has to characterize strict copositivity of matrices of dimension two and three. This is done by the following theorem:

Theorem 7 (cf. [11]).

(i) *C is strictly copositive iff $a > -1$.*

(ii) *D is strictly copositive iff $a, b, c > -1$ and ($a + b + c > -1$ or $|D| > 0$ (or both)).*

Proof. (i) follows from $xCx^T = (x_1 - x_2)^2 + 2(a+1)x_1x_2$ for all $x \in \mathbb{R}^2$ and (ii) was proved in [11].

Concluding remark. By the described method, for a given n-dimensional payoff matrix all ESS's with a support of cardinality $\geq n-3$ can be determined. Especially, all ESS's of an at most four-dimensional payoff matrix can be calculated in this way.

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SOME PROBLEMS IN NUMBER THEORY, COMBINATORICS AND COMBINATORIAL GEOMETRY

Paul **Erdős**

Mathematical Institute of the Hungarian Academy of Sciences, 1364 Budapest, P. O. Box 127, Hungary

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Abstract: Some open problems are posed on topics as given in the title. The solution of some of the problems will be awarded in U.S.$ by the author.

1. Number theory

1.1. In a forthcoming paper of A. Sárközy, V. T. Sós and myself we investigate the following problem: Denote by $F_k(n)$ the size of the largest set of integers $1 \leq a_1 < a_2 < \cdots < a_l \leq n$, $l = F_k(n)$ for which the product of no k a's can be a square. We obtain fairly accurate upper and lower bounds for $F_k(n)$.

We also discussed the following problem which is not mentioned in our paper: Let $1 \leq a_1 < a_2 \dots$ be an infinite sequence of integers, assume that the product of an odd number of a's is never a square. I. Ruzsa proved that the density of such a sequence is at most $1/2$ (Th. 4.1 in the cited paper). Clearly it can be $1/2$. To see this let the a's have an odd number of distinct prime factors.

A related problem states as follows: Let $a_1 < a_2 < \cdots < a_l \leq n$, assume that the product of an odd number of a's is never a square. Denote $\max l = g(n)$. *Determine or estimate $g(n)$ as accurately as possible.* It is easy to see that for a fixed but small $c > 0$, $g(n) > n\left(\frac{1}{2}+c\right)$, and I. Ruzsa showed that $g(n) < n(1-c)$ (Th. 4.2 in the cited paper).

ERDŐS, P., SÁRKÖZY, A. and SÓS, V. T.: On product representation of integers I. The paper will soon appear in the *European Journal of Combinatorics*
RUZSA, I.: General multiplicative functions, *Acta Arithm.* **32** (1977), 313–347.

1.2. This is a problem of P. Cameron and myself. Denote by $f(n)$ the number of sequences of integers $1 \leq a_1 < a_2 < \cdots < a_t \leq n$ (t is not fixed) for which no a_i is the distinct sum of other a's. *Is it true that*

$$f(n) = 2^{\frac{n}{2}(1+o(1))} \ ? \tag{1}$$

The integers $\frac{n}{2} \leq t \leq n$ show that $f(n) > 2^{\frac{n}{2}}$, but we expect that $f(n)$ is not very much larger than $2^{\frac{n}{2}}$.

CAMBERON, P. and ERDŐS, P.: On the number of sets of integers with various properties, Number Theory, Banff (A.B. 1988) de Gruyter, Berlin, 1990, 61–79.

1.3. Let $A = \{a_1 < a_2 < \dots\}$ be an infinite sequence of integers. $f(n)$ denotes the number of solutions of $n = a_i + a_j$. $A + A$ will denote the set of integers which can be written in the form $a_i + a_j$ i.e. the set of integers for which $f(n) > 0$.

A is called a basis of order r if every integer is the sum of r or fewer a's. It is called an asymptotic basis of order r if all but a finite number of integers are the sum of r or fewer a's. An old conjecture of P. Turán and myself states that *if A is an asymptotic basis of order* 2 *then*

$$\overline{\lim} f(n) = \infty \tag{1}$$

and perhaps there is an $\varepsilon > 0$ for which for infinitely many n

$$f(n) > \varepsilon \log n. \tag{2}$$

I offer for a proof or disproof of (1) 500 dollars.

Perhaps (1) *holds already if we only assume that the upper density of the integers for which $f(n) > 0$ is* 1.

It follows from an old result of mine that (2) if true is best possible (apart from the value of ε).

Unfortunately these old problems seem unattackable at the moment. Perhaps the following related problem is not hopeless. Denote by

$g_A(x)$ the number of integers $n < x$ for which $f(n) > 0$ $\Big($i.e. $g_A(x) = = \sum_{\substack{n<x \\ n=a_i+a_j}} 1, \ \ a_i, a_j \in A\Big)$. *Is it true that for every $\varepsilon > 0$ there is a sequence A for which for every x*

$$g(x) > (1-\varepsilon)x$$

but for every n, $f(n) < c(\varepsilon)$. In other words $f(n)$ is bounded but most of the numbers can be written in the form $a_i + a_j$. I would be satisfied if one would show *that there is a sequence A for which the upper density of the integers n with $f(n) > 0$ is $> 1-\varepsilon$ but $f(n) < c(\varepsilon)$.* Both A. Sárközy and I believe that *if $f(n) < c$ then the upper density of the integers n with $f(n) > 0$ is $< 1-\varepsilon$ where $\varepsilon = \varepsilon(c)$.*

For the literature see the excellent book of HALBERSTAM, H. and ROTH, K. F.: Sequences, Springer Vrlg, Berlin.

1.4. M. Nathanson, J. Spencer and I proved a few years ago that there is a basis of order three for which $f(n) \le 2$ with at most finitely many exceptions. We used the probability method. *Perhaps there is a basis of order three for which $f(n) = 1$ for all but finitely many exceptions* (i.e. there is a sequence A for which every integer $n = a_i + a_j + a_k$ but the integers $a_M + a_N$ are all distinct with a possible finite number of exceptions). It is not very likely that the probability method will help here.

1.5. St. Burr and I posed a few years ago the following problem. Let A be a sequence of integers for which the density of the integers with $f(n) > 0$ is positive. *Can one always decompose the sequence A as the union of two disjoint subsequences $A = A_1 \cup A_2$ for which the density of $A_1 + A_1$ and $A_2 + A_2$ is also positive?* As far as I remember we could not settle this question. While writing this paper I several times thought that I can prove that *there is a basis A of order two for which for any decomposition $A = A_1 \cup A_2$ the sequences $A_1 + A_1$ and $A_2 + A_2$ can not both have bounded gaps.* But unfortunately I could never quite finish the proof. Thus the problem is still open.

For problems 1.3, 1.4 and 1.5 the interested reader should consult besides the book of H. Halberstam and K. F. Roth also several recent papers of M. Nathanson, J. Spencer, P. Tetali and myself, many of which are joint papers.

1.6. A sequence $A = \{a_1 < a_2 < \cdots < a_l \le n\}$ is called a Sidon sequence if the sums $a_i + a_j$ are all distinct. Put $\max l = g(n)$. P. Turán and I proved

$$g(n) < n^{\frac{1}{2}} + cn^{\frac{1}{4}}. \tag{1}$$

Perhaps

$$g(n) = n^{\frac{1}{2}} + O(1). \tag{2}$$

(2) is perhaps too optimistic, but I am fairly sure that *for every* $\varepsilon > 0$

$$g(n) = n^{\frac{1}{2}} + o(n^{\varepsilon}). \tag{3}$$

I offer 500 dollars for a proof or disproof of (3).

I conjecture that *for every* t *and* $n > n_0(t)$

$$g(n+t) \le g(n) + 1 \tag{4}$$

and *perhaps for* $t < \varepsilon n^{\frac{1}{2}}$

$$g(n+t) \le g(n) + 1. \tag{5}$$

(4) and (5) if true would imply that the growth of $g(n)$ is fairly regular.

The older literature on Sidon sequences can be found in the book of H. Halberstam and K. F. Roth "Sequences".

2. Combinatorics

2.1. Let $G(n)$ be a graph of n vertices. Assume that there is a k for which every subgraph of m vertices $(1 \le m \le n)$ of our $G(n)$ has an independent set of size $\frac{m}{2} - k$ (k is fixed, n is arbitrary). Is it then true that *the vertex set of* G *can be decomposed into three disjoint sets* $S_1 \cup S_2 \cup S_3$ where S_1 and S_2 *are independent and* $S_3 < f(k)$, *i.e.* $G(n)$ *is the union of a bipartite graph and a bounded set*? The question is open even for $k = 1$.

A. Hajnal and I proved that there is a graph G of infinite chromatic number every subgraph of m vertices of which contains an independent set of size $\frac{m}{2}(1-\varepsilon)$. *Does this remain true if* $\frac{m}{2}(1-\varepsilon)$ *is replaced by* $\frac{m}{2} - f(m)$ *where* $f(m) \to \infty$?

A. Hajnal, E. Szemerédi and I have the following very annoying unsolved problem: Let $h(n)$ tend to infinity arbitrarily slowly. Is it true that *there is a* G *of infinite chromatic number every subgraph of* n *vertices of which can be made bipartite by the omission of* $\leq h(n)$ *edges*? I offer 250 dollars for a proof or disproof.

ERDŐS, P., HAJNAL, A. and SZEMERÉDI, E.: On almost bipartite large chromatic graphs, *Annals of Discrete Math.* **12** (1982), 117–123.

2.2. Denote by $G(n; f(n))$ a graph of n vertices and $f(n)$ edges. It is easy to see that every $G(2n+1; 3n+1)$ contains an even cycle and $3n+1$ is best possible.

Let now $\mathcal{S}$ be a subsequence of the even numbers. Let $f(n;\mathcal{S})$ be the smallest integer for which every $G(n; f(n;\mathcal{S}))$ contains a cycle whose length is in $\mathcal{S}$. In particular *can we find a sequence* $\mathcal{S}$ *of density* 0 *for which* $f(n;\mathcal{S}) < cn$? A. Gyárfás and I conjectured that if the sequence $\mathcal{S}$ consists of the powers of 2 then $f(n;\mathcal{S})/n \to \infty$. We have no guess *what happens if* $\mathcal{S}$ *consists of the numbers* $2u^2$. It follows from an old result of B. Bollobás that if $\mathcal{S}$ is an arithmetic progression of even numbers then $f(n;\mathcal{S}) < cn$.

BOLLOBÁS, B.: Cycles modulo *k*, *Bull. London Math. Soc.* **9** (1977), 97–98.

2.3. Problem 2.2 originated as follows. A. Hajnal and I conjectured that *if* G *has infinite chromatic number and if* $n_1 < n_2 < \dots$ *is the sequence of the sizes of distinct odd cycles of* G *then* $\sum \frac{1}{n_i} = \infty$ *and perhaps* $\sum_{n_i < x} 1 > cx$ *holds for infinitely many* x. We never could get anywhere with this problem. P. Mihok and I later conjectured that *if* G *has infinite chromatic number then for infinitely many* u G *has a cycle of length* 2^u. This problem also remained unattackable.

Three years ago A. Gyárfás and I conjectured that if G is a graph every vertex of which has degree ≥ 3 then G has a cycle of length 2^u. We finally thought that this was not true but could not find a counterexample, we concluded that *probably for every* k *there is a graph every vertex of which has degree* $\geq k$ *but there is no cycle of length* 2^u. The problem is still open and perhaps is not very difficult.

2.4. Here is a problem of R. Faudree and myself. Consider all the graphs of n vertices. Denote by

$$(1) \qquad 3 \le a_1 < a_2 < \cdots < a_t \le n$$

the lengths of the cycles occurring in any of these graphs. Denote by $f(n)$ the number of possible sequences (1). Clearly $f(n) \le 2^{n-2}$. We easily showed $f(n) > 2^{\frac{n}{2}}$. *Probably*

$$f(n)^{\frac{1}{n}} \to c, \qquad \sqrt{2} \le c < 2.$$

It is easy to see that for $n \ge 5$, $f(n) < 2^{n-2}$ but we could not prove *that* $f(n)/2^n \to 0$ *and that* $f(n)/2^{\frac{n}{2}} \to \infty$.

2.5. Let $f(n)$ be the smallest integer for which every graph of n vertices every vertex of which has degree $\ge f(n)$ contains a C_4 (i.e. a cycle of length 4). *Is it true that for* $n > n_0$

$$(1) \qquad f(n+1) \ge f(n)?$$

If this is too optimistic is it at least true that *there is an absolute constant* c *for which for every* $m > n$

$$(2) \qquad f(m) > f(n) - c?$$

The proof of (2) is perhaps easy, but so far the problem is open.

2.6. Let G be a four-chromatic graph, $m_1 < m_2 < \ldots$ be the lengths of the cycles of G, can $\min(m_{i+1} - m_i)$ *be arbitrarily large? Can this happen if the girth of* G *is large?*

2.7. R. Faudree, R. H. Schelp and I have the following question: Let $G(n)$ have girth $> 2s$ and every vertex has degree $\ge k$. Is it then true that *the number of cycles of distinct lengths of our* $G(n)$ *is* $> ck^s$? We proved this conjecture only for $s = 2$ and ran into unexpected difficulties already for $s = 3$.

2.8. The n-dimensional cube $C^{(n)}$ has 2^n vertices and $n2^{n-1}$ edges. A very old conjecture of mine states that *every subgraph of* $(1+\varepsilon)n2^{n-2}$

edges of $C^{(n)}$ *contains a* C_4 (i.e. a $C^{(2)}$). This conjecture is still open and I offer 100 dollars for a proof or disproof. Denote by $f(n)$ the smallest integer for which every subgraph of $C^{(n)}$ of $f(n)$ edges contains a C_4. I conjectured that

$$f(n) < n2^{n-2} + C2^n$$

for sufficiently large C, but here I overconjectured since H. Harborth and his students proved

$$f(n) > n2^{n-2} + n^{\alpha}2^n$$

for some fixed positive α. *The exact determination of* $f(n)$ *is perhaps not hopeless and would even be of some interest for small values of* n. *Is there an Erdős–Stone type theorem for the subgraphs of* $C^{(n)}$?

ERDŐS, P. and STONE, A.: On the structure of linear graphs, *Bull. Amer. Math. Soc.* **52** (1940), 1087–1091.
CHUNG, F. K. R.: Subgraphs of a hypercube containing no small even cycles, *J. Graph Theory* **16** (1992), 273–286.
CONDER, M.: Hexagon free subgraphs of hypercubes, *J. Graph Theory* **17** (1993), 477–479.

3. Combinatorial geometry

3.1. Let $f_k(n)$ be the largest integer for which there are n points $x_1, \dots, x_n$ in k-dimensional Euclidean space for which for every x_i there are at least $f_k(n)$ points x_j equidistant from x_i. First let us discuss $k = 2$ i.e. our points are in the plane. I conjectured $f_2(n) < n^{\varepsilon}$ *for every* $\varepsilon > 0$ *if* $n > n_0(\varepsilon)$, *and perhaps* $f_2(n) < n^{c/\log\log n}$. The lattice points show that $f_2(n) > n^{c_1/\log\log n}$.

In 1946 I conjectured that *among any* n *points in the plane the same distance can occur at most* $n^{1+c/\log\log n}$ *times* and the above conjectures would give a very considerable strengthening. I offer 500 dollars for a proof of $f_2(n) = o(n^{\varepsilon})$, but only 50 dollars for a counterexample. $f_2(n) < cn^{\frac{1}{2}}$ is trivial and J. Pach points out that a result of J. Pach and M. Sharir gives $f_2(n) < cn^{2/5}$. Any further improvement would be very welcome. A recent letter of P. C. Fishburn shows that perhaps the set $x_1, \dots, x_n$ which gives the largest value of $f_2(n)$ may not be given by the lattice points. Fishburn proved that 6 is the smallest integer for which $f_2(6) = 3$ and 8 the smallest integer for which $f_2(8) = 4$.

E. Makai, J. Pach and I proved

$$(1) \qquad c_1 n^{\frac{1}{3}} < f_3(n) < c_2 n^{\frac{3}{4}}$$

and

$$(2) \qquad \frac{n}{2} + 2 \leq f_4(n) < \frac{n}{2}(1 + o(1)).$$

In (1) the upper bound holds in the stronger form that if among our n points $x_1, \dots, x_n$ there are $k = c\sqrt{n}$ points, say $y_1, \dots, y_k$, no three of these on a line, then there exists an i, $1 \leq i \leq k$, such that y_i does not have more than $c_2 n^{3/4}$ equidistant points x_j from it. *Probably*

$$f_4(n) = \frac{n}{2} + O(1).$$

In a finite time we may publish a quadruple paper about these results.

3.2. Denote by $\mathrm{d}(x, y)$ the distance between the points x and y and by $\mathrm{D}(x_1, \dots, x_n)$ denote the diameter i.e. the maximum of $\mathrm{d}(x_i, x_j)$. Let $x_1, x_2, \dots, x_n$ be n points in the plane for which $\mathrm{d}(x_i, x_j) \geq 1$ and the diameter is minimal. It has been known since A. Thue (1910) that asymptotically the minimum is given by the triangular lattice. Let $x_1, \dots, x_n$ be a set which implements the minimum of the diameter. Denote by $h(n)$ the number of incongruent sets which implement the minimum of the diameter. I would guess that $h(n) \to \infty$ *as* $n \to \infty$ but as far as I know it has not even been proved that $h(n) \geq 2$ *for* $n > n_0$. It is generally believed that *for* $n > n_0$ *no subset of the triangular lattice implements the minimum of the diameter but that any set* $x_1, \dots, x_n$ *which implements the minimum of the diameter has a large intersection with the triangular lattice*, but as far as I know nothing has been proved.

I conjectured that *if* $x_1, \dots, x_n$ *implements the minimum of the diameter and if* $n > n_0$ *then our set contains an equilateral triangle of sides* 1. I could not prove it but felt that it should not be hard. To my great surprise both B. H. Sendov and M. Simonovits doubted the truth of this conjecture. I offer 100 dollars for a counterexample for infinitely many values of n but only 50 dollars for a proof.

3.3. Inscribe n non-overlapping squares into a unit square. Denote by $a_1, a_2, \dots, a_n$ the sides of these squares. Let

$$f(n) = \max \sum_{i=1}^{n} a_i.$$

It is easy to see that

$$f(k^2) = k$$

and *I conjectured that*

$$f(k^2+1) = k. \tag{1}$$

I conjectured (1) more than 60 years ago. Perhaps the proof (or disproof) of (1) will not be difficult.

3.4. Let $x_1, x_2, \dots, x_n$ be n points in the plane in general position i.e. no three on a line and no four on a circle. Let $h(n)$ be the largest integer for which these points determine at least $h(n)$ distinct distances. I conjectured *that* $h(n)/n \to \infty$ but could not even prove *that* $h(n) > n-1$, in fact I could not exclude *that* $h(n) < cn$ *for some* $c < 1$.

3.5. It is easy to see by the well-known construction of H. Lenz that one can give $3n$ points in six-dimensional space which determine $n^3 + 6n^2$ equilateral triangles of size 1 for $4 \mid n$. It suffices to take three suitable orthogonal circles and take n points on each of them which form $\frac{n}{4}$ inscribed squares. I conjectured that *in six-dimensional space one cannot have* $3n$ *points which determine* $(1+\varepsilon)n^3$ *equilateral triangles of size* 1. If one just asks *for equilateral triangles of any size* one can of course get somewhat more equilateral triangles, but *their maximal number is probably less than* $(1+\varepsilon)n^3$. Perhaps I overlook a trivial point. Many related questions can be asked but I leave their formulation to the interested reader.

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SMALL n-DOMINATING SETS

Zsolt **Tuza**

Computer and Automation Institute, Hungarian Academy of Sciences, H–1111 Budapest, Kende u. 13–17, Hungary

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Abstract: We prove the inequality $\gamma_n(G) + (n-1)\gamma_n^t(G) \le p$, where G is any connected graph on $p \ge 2n-1$ vertices, and $\gamma_n(G)$ and $\gamma_n^t(G)$ denote the minimum cardinalities of vertex sets D and D_t such that each vertex x is at distance less than n from some $y \in D_t$, $y \ne x$, and each $x \notin D$ is at distance less than n from some $y \in D$. Our method yields a very short proof of a recent theorem due to Henning et al. [Math. Pannon. **5**/1 (1994), 67–77].

In this note we provide a very short proof of a recent result due to Henning et al. [1] on generalized domination parameters of graphs. Our method, at the same time, also yields a somewhat stronger assertion. (For further related results, see [2].)

Let G be a *connected* graph with a p-element vertex set $V(G)$ $(p > 1)$ and with edge set $E(G)$. The *distance* $\mathrm{d}(x, y)$ of two vertices $x, y \in V(G)$ is the smallest number of edges in a path joining x to y. Let $n > 1$ be an integer. Adopting the terminology of [1], a $D \subseteq V(G)$ is a $P_{\le n}$*-dominating set* (*total* $P_{\le n}$*-dominating set*, respectively) if each vertex $x \in V(G) - D$ (each $x \in V(G)$) is at distance less than n from some $y \in D$, $y \ne x$. (Sometimes such a D is simply called a (total) $(n-1)$-dominating set in the literature.) The minimum cardinality of a $P_{\le n}$-dominating set and of a total $P_{\le n}$-dominating set is denoted by $\gamma_n(G)$ and $\gamma_n^t(G)$, respectively.

The main result of [1] states

$$\gamma_n(G) + \gamma_n^t(G) \le 2p/n \tag{1}$$

provided that G is connected and its order, p, is at least $2n$. Here we prove the following stronger assertion.

Theorem 1. *If G is a connected graph of order $p \ge 2n-1$, $n \ge 2$, then*

$$\gamma_n(G) + (n-1)\gamma_n^t(G) \le p\,. \tag{2}$$

Proof. Since every (total) $P_{\le n}$-dominating set of any spanning tree T of G is a (total) $P_{\le n}$-dominating set in G as well, it suffices to prove the assertion for trees. Hence, let T be a tree of order $p \ge 2n-1 \ge$ ≥ 3. We denote $\mathrm{d}(T) = \max_{x,y\in V(T)} \mathrm{d}(x,y)$ (the *diameter* of T) and $r(T) = \min_{x\in V(T)} \max_{y\in V(T)} \mathrm{d}(x,y)$ (the *radius*).

Suppose first that $\mathrm{d}(T) \le 2n-2$. Then $r(T) \le n-1$, and a 'central' vertex within distance $n-1$ from every vertex of T forms a $P_{\le n}$-dominating set. Therefore, $\gamma_n(T) = 1$ and $\gamma_n^t(T) = 2$, i.e., $\gamma_n(T) + (n-1)\gamma_n^t(T) = 2n-1 \le p$. Hence, the assertion is valid for 'small' diameter (which is always the case for $p = 2n-1$), allowing us to apply induction on p. This can be done if T has an edge e such that both components T_1, T_2 of $T-e$ contain at least $2n-1$ vertices (and, in particular, whenever $\mathrm{d}(T) \ge 4n-3$). Indeed, in this case $\gamma_n(T) \le \gamma_n(T_1) + \gamma_n(T_2)$ and $\gamma_n^t(T) \le \gamma_n^t(T_1) + \gamma_n^t(T_2)$, thus $\gamma_n(T) +$ $+(n-1)\gamma_n^t(T) \le (\gamma_n(T_1)+(n-1)\gamma_n^t(T_1)) + (\gamma_n(T_2)+(n-1)\gamma_n^t(T_2)) \le$ $\le |V(T_1)| + |V(T_2)| = p$ follows by induction.

Suppose $2n-1 \le d(T) \le 4n-4$, and that $T-e$ contains a 'small' component for each edge $e \in E(T)$. Choose an edge $e = uv$ such that the smaller component (the one of order $\le 2n-2$), say the component containing u, is as large as possible. Then *all* components $T_1, T_2, \ldots, T_m$ of $T-v$ have orders at most $2n-2$. Define the height h_i of T_i as the length of a longest path $P_i \subseteq \langle T_i \cup v\rangle$ starting at v, where $\langle T_i \cup v\rangle$ denotes the subgraph induced by $V(T_i) \cup \{v\}$. By the assumption $\mathrm{d}(T) \ge 2n-1$, some T_i have $h_i \ge n$; say, $h_i \ge n$ for $1 \le$ $\le i \le k$ and $h_i < n$ for $k < i \le m$ (where $k = m$ is possible). For $i \le k$ we denote by v_i the vertex of P_i at distance $h_i - n + 1$ from v. Since $|V(T_i)| \le 2n-2$, v_i $P_{\le n}$-dominates T_i for each i, moreover v $P_{\le n}$-dominates $\{v_1, \ldots, v_k\} \cup \{T_j \mid k < j \le m\}$. Hence, $\gamma_n(T) \le$ $\le \gamma_n^t(T) \le k+1$, implying (2) for $p \ge (k+1)n$.

To obtain sharper bounds on $\gamma_n(T)$ and $\gamma_n^t(T)$, we consider the subtree $T' = \langle T'' \cup \{v_1, \dots, v_k\}\rangle$, where T'' is the connected component of $T - \{v_1, \dots, v_k\}$ containing v. If $d(T'') < n - 3$, we have $\gamma_n(T) = \gamma_n^t(T) = k$ and $p > |V(T_1)| + \dots + |V(T_k)| \geq kn$, implying $\gamma_n(T) + (n-1)\gamma_n^t(T) < p$. On the other hand, if $d(T'') \geq n - 2$, then $p \geq (k+1)n - 1$ holds, with equality if and only if T'' has order $n - 1$ and the subtree rooted at v_i is a path of length $n - 1$ in each T_i. In this case, however, $\{v_1, \dots, v_k\}$ is already a dominating set, therefore $\gamma_n(T) = k$, $\gamma_n^t(T) \leq k + 1$, $\gamma_n(T) + (n-1)\gamma_n^t(T) \leq (k+1)n - 1 = p$. ◊

Remarks. Since $\gamma_n(G) \leq \gamma_n^t(G)$ whenever G has no isolated vertices, the inequality (2) immediately implies (1). Certainly, every example showing the tightness of (1) (see [1] where an infinite family of graphs G with $\gamma_n(G) = \gamma_n^t(G) = p/n$ is exhibited) yields that (2) is tight, too. However, (2) is best possible in a much stronger sense as well; namely, its left-hand side cannot be replaced by $(1-\varepsilon)\gamma_n(G) + (n - 1 + \varepsilon)\gamma_n^t(G)$, for any $\varepsilon > 0$. To see this, take $k - 1$ (≥ 1) vertex-disjoint paths $T_1, \dots, T_{k-1}$ of length $n-1$ and one path of length $2n-2$. Joining a new vertex v with one endpoint of each T_i, we obtain a tree T of order $p = (k+1)n - 1$, with $\gamma_n(T) = k$ and $\gamma_n^t(T) = k + 1$, hence $\gamma_n(T) + (n-1)\gamma_n^t(T) = p$ and $\gamma_n(T) < \gamma_n^t(T)$. Further 'isolated' examples are the paths on $3n - 1$, $4n - 2$, $4n - 1$ vertices (the corresponding parameters are $\gamma_n = 2, 2, 3$ and $\gamma_n^t = 3, 4, 4$), and all connected graphs G of order $p = 2n - 1$ ($\gamma_n(G) = 1$ and $\gamma_n^t(G) = 2$). It may be true, however, that if p is 'sufficiently large' with respect to n, then $\gamma_n(G) + (n-1)\gamma_n^t(G) < p$ holds unless $\gamma_n(G) = \gamma_n^t(G) = p/n$, or G is a k-branched tree constructed above plus possibly a few additional edges among its 'short' branches.

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A HAJÓS-TYPE RESULT ON FACTORING FINITE ABELIAN GROUPS BY SUBSETS

Keresztély **Corrádi**

Department of Computer Sciences, Eötvös University Budapest, H–1088 Budapest, Múzeum krt. 6–8, Hungary

Sándor **Szabó**

Department of Mathematics, University of Bahrain, P. O. Box 32038 Isa Town, State of Bahrain

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Abstract: Hajós' theorem asserts that if a finite abelian group is a direct product of cyclic subsets, then in fact at least one of the factors must be a subgroup of the group. A cyclic subset is the "front end" of a cyclic subgroup. The main result of the paper is an analogous result. Namely, that the same conclusion holds for finite abelian groups of odd order with certain more general type of factors. The proofs mainly rely on characters of finite abelian groups.

1. Introduction

Throughout the paper the word group is used to mean finite abelian group. The groups are written multiplicatively with identity element e. We need the concept of factoring subsets into subsets. Let G be a finite abelian group. If $B, A_1, \ldots, A_n$ are subsets of G such that each b in B is uniquely expressible in the form

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$$b = a_1 \cdots a_n, \qquad a_1 \in A_1, \dots , a_n \in A_n,$$

and each product $a_1 \cdots a_n$ belongs to B, that is, if the product $A_1 \cdots A_n$ is direct and is equal to B, then we say that B is factored by subsets $A_1, \dots , A_n$. The equation $B = A_1 \cdots A_n$ is also said to be a *factorization* of B. If $e \in B \cap A_1 \cap \cdots \cap A_n$, then the factorization $B = A_1 \cdots A_n$ and the subsets $B, A_1, \dots , A_n$ are said to be *normed.* Clearly the product $A_1 \cdots A_n$ is a factoring of B if and only if $A_1 \cdots A_n = B$ and $|A_1| \cdots |A_n| = |B|$. Direct product of subsets is a straightforward generalization of direct product of subgroups which is a commonly used construction. However factoring a finite abelian group into certain type of subsets also admits important applications.

The subset A of G is called *cyclic* if it is of form $\{e, a, a^2, \dots , a^{r-1}\}$ where a is an element of $G \setminus \{e\}$ and r is a positive integer. We denote this subset of G shortly by $[a, r]$. Loosely speaking the cyclic subset $[a, r]$ consists of the "first consecutive" r elements of $\langle a \rangle$ the cyclic subgroup generated by the element a. We would like to point out that it is assumed that $|a|$ the order of a is at least r.

To settle a famous geometric conjecture of H. Minkowski G. Hajós [2] proved that *if a finite abelian group is a direct product of cyclic subsets, then at least one of the factors must be a subgroup of the group.* In order to generalize Hajós' theorem we can try to extend the family of subsets that occur in a factorization of a given finite abelian group. Of course this extended family should contain the cyclic subsets. Beside cyclic subsets we will consider subsets of form $[a, r] \cup g[a, s]$, where the union is disjoint. We would like to show that if a finite abelian group is factored into the above type of subsets, then at least one of the factors must be a subgroup. We are able to verify this fact in the special case when the order of the finite abelian group is odd. This is the main result of this note. On the other hand the result does not extend to abelian groups of even order as the following example shows. Let G be the direct product of two cyclic groups of order four, say $G = \langle x \rangle \times \langle y \rangle$, where $|x| = |y| = 4$. Choose the subsets A and B to be $A = [x, 2] \cup \cup y^2[x, 2]$, $B = [y, 2] \cup x^2y[y, 2]$. Then as it is easy to verify $G = AB$ is a factorization of G and none of the factors A and B is a subgroup of G.

2. Result

If A and A' are subsets of G such that for every subset B of G, if $G = AB$ is a factorization of G, then $G = A'B$ is also a factorization of G, then we shall say that A is *replaceable* by A'. Rédei [3] made use of group characters to study replaceable factors. If A is a subset and χ is a character of G, then $\chi(A)$ denotes the sum

$$\sum_{a \in A} \chi(a).$$

Rédei showed that the factor A can be replaced by A' if $|A| = |A'|$ and if from $\chi(A) = 0$ it follows $\chi(A') = 0$ for each character χ of G. The set of characters χ of G for which $\chi(A) = 0$ we call the *annihilator* of the subset A and it is denoted by $\mathrm{Ann}(A)$. Using this concept Rédei's test reads that if $|A| = |A'|$ and $\mathrm{Ann}(A) \subset \mathrm{Ann}(A')$, then the subset A can be replaced by the subset A'.

Lemma 1. *Let G be a finite abelian group of odd order and let A be a subset of G such that $A = [a, r] \cup g[a, s]$, where the union is disjoint and $r + s$ is odd. Then* $\mathrm{Ann}(A) \subset \mathrm{Ann}([a, r + s])$.

Proof. Let $B = [a, r + s]$. First note that $\mathrm{Ann}(B)$ consists of each character χ of G for which $\chi(a) \neq 1$ and $\chi(a^{r+s}) = 1$. Indeed, if $\chi(a) = 1$, then $\chi(B) = r + s$ and if $\chi(a) \neq 1$, then

$$\chi(B) = \frac{1 - \chi(a^{r+s})}{1 - \chi(a)}$$

which proves the claim. Thus it is enough to verify that from $\chi(A) = 0$ it follows that (i) $\chi(a) \neq 1$ and (ii) $\chi(a^{r+s}) = 1$.

To prove (i) assume the contrary that χ is a character of G for which $\chi(A) = 0$ and $\chi(a) = 1$. Now $0 = \chi(A) = r + \chi(g)s$ or equivalently $\chi(g) = -(r/s)$. Taking the absolute values of both sides we have $s = r$. Hence $r + s$ is even which is not the case.

To prove (ii) consider a character χ of G with $\chi(A) = 0$. Now $0 = \chi(A)\chi(a) = \chi(Aa)$. From $\chi(A) = \chi(Aa)$ after cancelling we get $\chi(e) + \chi(g) = \chi(a^r) + \chi(ga^s)$. Drawing complex numbers on the plane the reader can verify easily that as the roots of unity occurring are of odd order $\chi(e)$, $\chi(g)$ is a rearrangement of $\chi(a^r)$, $\chi(ga^s)$. Hence $\chi(e)\chi(g) = \chi(a^r)\chi(ga^s)$, which is equivalent to $1 = \chi(a^{r+s})$. ◊

If $G = AC$ is a factorization of the finite abelian group G, where $A = [a, r] \cup g[a, s]$, then by Lemma 1 A can be replaced by $B = [a, r + s]$ to get factorization $G = BC$. Now B must contain $r + s$ elements and so $|a| \geq r+s$. Thus when A is a factor of a factorization then $|a| \geq r+s$ holds. We would like to point out that this is not the case in general. Let $G = \langle x\rangle \times \langle y\rangle$, $|x| = 5$, $|y| = 3$ and $A = [x,3] \cup y[x,4]$. Now $|A| = 7$ and $|x| < 7$.

A subset A of G is called *periodic* if there is an element $g \in G\setminus\{e\}$ such that $Ag = A$. The element g is called a *period* of A.

Lemma 2. *Let G be a finite abelian group of odd order and let A be a subset of G such that $A = [a,r] \cup g[a,s]$, where the union is disjoint and $r+s$ is odd. If A is periodic and $|a| \geq r+s$, then $A = \langle a\rangle$.*

Proof. As $|a| \geq r+s$ so it is enough to prove that (i) $a^{r+s} = e$ and (ii) $g = a^r$.

If $\chi(a^{r+s}) = 1$ for each character χ of G, then $a^{r+s} = 1$. So to prove (i) we consider $\overline{C} = \{\chi : \chi(a^{r+s}) = 1\}$ and we show that $\overline{C}$ in fact coincides with the character group $\overline{G}$ of G. Note that $\overline{C}$ is a subgroup of $\overline{G}$ and $\mathrm{Ann}(A) \subset \overline{C}$. Let x be a period of A. By Th. 1 of [4], $\chi(A) = 0$ whenever $\chi(x) \neq 1$. Counting the number of characters χ of G for which $\chi(x) \neq 1$ we get a lower bound for $|\mathrm{Ann}(A)|$.

$$|\mathrm{Ann}(A)| \geq |\overline{G}| - |G : \langle x\rangle| = |G| - |G : \langle x\rangle| =$$
$$= |G|(1-(1/|x|)) \geq |G|(1-(1/p)) > |G|(1/2) = (1/2)|\overline{G}|.$$

Here p is the smallest prime divisor of $|G|$. As $|\mathrm{Ann}(A)| > (1/2)|\overline{G}|$, $\mathrm{Ann}(A)$ generates $\overline{G}$ and consequently $\overline{C} = \overline{G}$.

To prove $g = a^r$ assume the contrary that $g \neq a^r$. Let χ be a character of G for which $\chi(A) = 0$. Applying χ to $g \neq a^r$ we face to two possibilities, (a) $\chi(g) = \chi(a^r)$ and (b) $\chi(g) \neq \chi(a^r)$. We establish an upper bound for $|\mathrm{Ann}(A)|$. If $\chi(g) = \chi(a^r)$, then $\chi(ga^{-r}) = 1$ and the number of these characters is $|G : \langle ga^{-r}\rangle| - 1 \leq |G|/p - 1$ since $\chi(A) = |A| \neq 0$ for the principal character χ of G. Turn to the case when $\chi(g) \neq \chi(a^r)$ and let $B = [a,r] \cup a^r[a,s] = [a, r+s]$. By Lemma 1, from $\chi(A) = 0$ it follows that $\chi(B) = 0$ and so

$$0 = \chi(A) - \chi(B) =$$
$$\chi([a,r]) + \chi(g)\chi([a,s]) - \chi([a,r]) - \chi(a^r)\chi([a,s]) =$$
$$= \chi([a,s])\big(\chi(g) - \chi(a^r)\big).$$

Hence $\chi([a,s]) = 0$ and consequently $\chi([a,r]) = 0$. Therefore $\chi(a) \neq 1$, $\chi(a^s) = 1$, $\chi(a^r) = 1$. If t is the greatest common divisor of s and

r, then $\chi(a^t) = 1$. The number of these characters is $|G : \langle a^t \rangle| - 1 \leq$ $\leq |G|/p - 1$. We now combine the lower and upper bounds for $|\mathrm{Ann}(A)|$ together.

$$|G|(1 - (1/p)) \leq |\mathrm{Ann}(A)| \leq |G|/p - 1 + |G|/p - 1 < |G|(2/p).$$

Cancelling $|G|$ we get $1 - (1/p) < (2/p)$ or equivalently $p < 3$ which is not the case. ◊

Theorem 1. *Let G be a finite abelian group of odd order and $A_1, \ldots, A_n$ be subsets of G such that $A_i = [a_i, r_i] \cup g_i[a_i, s_i]$. If $G = A_1 \cdots A_n$ is a factorization of G, then A_i is a subgroup of G for some i, $1 \leq i \leq n$.*

Proof. For $n = 1$ the theorem holds and so we proceed by induction on n. Replace the factor A_i by $B_i = [a_i, r_i + s_i]$ for each i, $1 \leq i \leq n$ in the factorization $G = A_1 \cdots A_n$ to get the factorization $G = B_1 \cdots B_n$. By Lemma 1 this can be done. From the factorization $G = B_1 \cdots B_n$ by Hajós' theorem it follows that at least one of the factors B_i is a subgroup of G. We may assume that $B_1 = H$ is a subgroup of G since this is only a matter of indexing the factors. In the factorization $G = A_1 A_2 \cdots A_n$ replace A_1 by $B_1 = H$ to get the factorization $G = HA_2 \cdots A_n$. From this we get the factorization $G/H = (A_2 H)/H \cdots (A_n H)/H$ of the factor group G/H. By the inductive assumption some of the factors $(A_i H)/H$, say $(A_2 H)/H$, is a subgroup of G/H, that is, HA_2 is a subgroup of G. Continuing in this way we have that

$$H,\ HA_2,\ HA_2A_3,\ \ldots,\ HA_2 \cdots A_n$$

are subgroups of G. Let $K = HA_2 \cdots A_{n-1}$. If $g_1 \in K$, then $A_1 \subset K$ and so $K = A_1 A_2 \cdots A_{n-1}$ is a factorization of K. By the inductive assumption one of the factors is a subgroup of K and so of G.

In the remaining part of the proof we assume that $g_1 \notin K$. Let $b \in A_n$. From the factorizations $G = A_1 A_2 \cdots A_n$ and $G = HA_2 \cdots A_n$ by multiplying with b^{-1} we have that $G = A_1 A_2 \cdots A_{n-1}(b^{-1}A_n)$ and $G = HA_2 \cdots A_{n-1}(b^{-1}A_n) = K(b^{-1}A_n)$ are also factorizations of G. As $G = K(b^{-1}A_n)$ is a factorization of G, $b^{-1}A_n$ is a complete set of representatives of G modulo K. There is an element t_b in $b^{-1}A_n$ such that $t_b^{-1}K$ contains g_1. Now $g_1 t_b \in K$. Let $C_b = [a_1, r_1] \cup$ $\cup [a_1, s_1] g_1 t_b$. We claim that $K = C_b A_2 \cdots A_{n-1}$ is a factorization of K. Indeed, products coming from $C_b A_2 \cdots A_{n-1}$ occur among the product coming from $A_1 A_2 \cdots A_{n-1}(b^{-1}A_n)$. But these latter ones are distinct as $G = A_1A_2 \cdots A_{n-1}(b^{-1}A_n)$ is a factorization of G. From $K = C_bA_2 \cdots A_{n-1}$ by the inductive assumption we have that one of the factors is a subgroup of K. If this is not C_b, then we are done. Thus we suppose that C_b is a subgroup of K. Now C_b is periodic and so by Lemma 2, $C_b = \langle a_1 \rangle$. Consequently $g_1t_b = a_1^{r_1}$ or equivalently $t_b = g_1^{-1}a_1^{r_1} \in b^{-1}A_n$. If $t_b = e$ for some $b \in A_n$, then $g_1 = a_1^{r_1}$ and $A_1 = \langle a_1 \rangle$. If $t_b \neq e$ for each $b \in A_n$, then

$$e \neq t_b = g_1^{-1}a_1^{r_1} \in \bigcap_{b \in A_n} b^{-1}A_n$$

and so by Lemma 4 of [1], A_n is periodic. Now by Lemma 2, $A_n = \langle a_n \rangle$. ◊

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LOCALISATION OF A COMMUTATIVITY CONDITION FOR *S*-UNITAL RINGS

Veselin **Perić**

Department of Mathematics, University of Montenegro, YU–71000 Podgorica/Montenegro, Yugoslavia

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Abstract: We consider here *s*-unital rings R satisfying the following condition: For each subset F of R having at most four elements, there exist non-negative integers $m = m(F)$, $n = n(F)$, $r = r(F)$, $s = s(F)$, $t = t(F)$ and $t' = t'(F)$ such that $m > 0$ or $n > 0$, $n + t + t' \neq r + 1$ or $m + s > 1$ if $m = n > 0$, and $x^{t'}[x^n, y]x^t = \pm x^r[x, y^m]y^s$ for all $x, y \in F$. Under appropriate additional conditions we prove some commutativity theorems for R. Especially, in the "global case", where $F = R$, instead of $F \subseteq R$, $|F| \leq 4$, i.e. where m, n, r, s, t and t' are fixed, we improve some earlier results obtained by several authors, among them H. A. S. Abujabal and the present author.

1. Introduction

We investigate here the commutativity of a ring R satisfying the following property

(P_4) For each subset F of R having at most four elements, $|F| \leq$ ≤ 4, there exist non-negative integers $m = m(F)$, $n = n(F)$, $r = r(F)$, $s = s(F)$, $t = t(F)$ and $t' = t'(F)$ such that

$$(1) \qquad x^{t'}[x^n, y]x^t = \pm x^r[x, y^m]y^s \quad \text{for all} \quad x, y \in F.$$

Our general assumption on the above integers will be:

$$(2) \qquad m(F) > 0 \text{ or } n(F) > 0 \text{ for each } F \subseteq R,\ |F| \leq 4,$$

and moreover

$$(3) \qquad \text{if } m(F) = n(F), \text{ then } n(F) + t(F) + t'(F) \neq r(F) + 1 \text{ or } m(F) + s(F) > 1 \text{ for each } F \subseteq R,\ |F| \leq 4.$$

The assumption (2) is natural, since for $m(F) = 0$ and $n(F) = 0$ the condition (1) is trivially satisfied. Concerning the condition (3) we remark that if $m(F) = n(F)$, $n(F)+t(F)+t'(F) = r(F)+1$ and $m(F)+ + s(F) = 1$, i.e. $s(F) = 0$, then for every ring R with the additional condition

$$(4) \qquad [x,[x,y]] = 0 \text{ for all } x,y \in R$$

the condition (1_+) (resp. (1_-)) is surely fulfilled (if R is of characteristic 2). Namely, for such a ring, in view of (4), (1) is equivalent (see Lemma 2) with

$$(5) \qquad nx^{n+t+t'-1}[x,y] = \pm mx^r[x,y]y^{m+s-1} \text{ for all } x,y \in F.$$

Since a non-commutative ring can satisfy (4), the condition (3) is thus also reasonable. Thereby, we have denoted by (i_+), resp. by (i_-) the condition obtained from a condition (i) which contains the sign $\pm$ by setting $+$, resp. $-$ instead of $\pm$.

A similar condition denoted by (Q_4) with

$$(1') \qquad x^{t'}[x^n,y]x^t = \pm y^{s'}[x,y^m]y^s \text{ for all } x,y \in F$$

instead of (1), we consider in another paper [13].

If in (P_4),$r(F) = 0$ for each $F \subseteq R$, $|F| \leq 4$, resp. in (Q_4), $s'(F) = 0$ for all $F \subseteq R$, $|F| \leq 4$, then (P_4), resp. (Q_4) is a special case of (Q_4), resp. of (P_4). Otherwise, these two conditions are not easily comparable.

We remark that instead of (1) we could consider

$$(1'') \qquad x^{t'}[x^n,y]x^t = \pm y^s[x,y^m]x^r \text{ for all } x,y \in F.$$

But going from R to the opposite ring R', ($1''$) becomes

$$(1''') \qquad x^t[x^n,y]x^{t'} = \pm x^r[x,y^m]y^s \text{ for all } x,y \in F',$$

where $F' = F$ is considered as a subset of R'. Thus, we see that it suffices to consider only one of the conditions (1) and ($1''$): the results concerning one can be obtained as corollaries from the results concerning the other of the conditions.

For $F=R$ instead of $F\subseteq R$, $|F|\leq 4$, the condition (P_4) reduces to

(P) There exist non-negative integers m,n,r,s,t and t' such that

$$x^{t'}[x^n,y]x^t = \pm x^r[x,y^m]y^s \text{ for all } x,y\in R, \tag{6}$$

and

(7) $m>0$ or $n>0$, and if $m=n$, then $n+t+t'\neq r+1$ or $m+s>1$.

We tell (P) the globalisation of (P_4), and (P_4) a localisation of (P), and in this connection we will talk about the local and the global case of a statement or a condition.

Occasionally we will make other additional conditions on R, or on integers $m(F)$, $n(F)$, $r(F)$, $s(F)$, $t(F)$ and $t'(F)$. For the local case we will additionally assume (4) or

(I–A) For each $x\in R$, $x\in A$ or there exists $f(X)\in X^2\mathbb{Z}[X]$ such that $x-f(x)\in A$, for a suitable non-void subset A of R.

In both cases we will need one of the conditions:

$$m[x,y]=m[x,y]f(y)\Rightarrow [x,y]=[x,y]kf(y), \tag{8}$$

$$n[x,y]=n[x,y]f'(y)\Rightarrow [x,y]=[x,y]k'f'(y), \tag{8'}$$

$$(n\mp 1)[x,y]=[x,y]f''(y)\Rightarrow [x,y]=[x,y]k''f''(y), \tag{8''}$$

$$2[x,y]=[x,y]f'''(y)\Rightarrow [x,y]=[x,y]k'''f'''(y) \tag{8'''}$$

for all $x,y\in F$ in the local case, and for all $x,y\in R$ in the global case. Thereby, k,k',k'' and k''' are appropriate integers (or polynomials in $X\mathbb{Z}[X]$ taken at $X=y$), and

$$f(X)=-(X+1)^{m+s-1}+X^{m+s-1}+1, \tag{9}$$

$$f'(X)=-(X+1)^{n+t+t'-1}+X^{n+t+t'-1}+1, \tag{9'}$$

$$f''(X)=-n(X+1)^{n+t+t'-1}+nX^{n+t+t'-1}\pm(X+1)^r\mp X^r+(n\mp 1), \tag{9''}$$

and

$$f'''(X)=-(X+1)^{t+t'}+X^{t+t'}-(X+1)^r+X^r+2 \tag{9'''}$$

are polynomials in $X\mathbb{Z}[X]$ for suitable values of m,n,r,s,t and t'.

Obviously, (8), resp. (8') implies

$$m[x,y]=0\Rightarrow [x,y]=0 \text{ for all } x,y\in F,\ [x,y]y^2=0, \tag{8)*}$$

resp.

$$n[x,y]=0\Rightarrow [x,y]=0 \text{ for all } x,y\in R,\ [x,y]y^2=0. \tag{8')*}$$

Concerning $(8'')$, resp. $(8''')$ we have only

$$(n \mp 1)[x,y] = [x,y]f''(y) \Rightarrow (n \mp 1)^2[x,y] = 0,$$
$$[x,y] = [x,y]k''f''(y) \Rightarrow [x,y] = 0,$$

resp.

$$2[x,y] = [x,y]f''(y) \Rightarrow 2^2[x,y] = 0,$$
$$[x,y] = [x,y]k''f''(y) \Rightarrow [x,y] = 0$$

for all $x, y \in F$, $[x,y]y^2 = 0$. Therefore, the conditions

$(8'')^*$ $\quad (n \mp 1)[x,y] = 0 \Rightarrow [x,y] = 0$ for all $x, y \in F$, $[x,y]y^2 = 0$

resp.

$(8''')^*$ $\quad 2^2[x,y] = 0 \Rightarrow [x,y] = 0$ for all $x, y \in F$, $[x,y]y^2 = 0$,

which we will also consider here, does not follow from $(8'')$, resp. $(8''')$.

We remark that in the local case, the condition (8), $(8)^*$, resp. $(8')$, $(8')^*$, resp. $(8'')^*$, resp. $(8''')^*$ follow from

$Q(q)$ $\quad q[x,y] = 0 \Rightarrow [x,y] = 0$ for all $x, y \in R$
with $q = m$, resp. $q = n$, resp. $q = n \mp 1$, resp. $q = 2$.

But the conditions (8), $(8')$ are surely satisfied if $[x,y]f(y)$, resp. $[x,y]f'(y)$ is torsion-free or m, resp n is relatively prime to the additive order of $[x,y]f(y)$, resp of $[x,y]f'(y)$ Similarly, the conditions $(8'')$, $(8''')$ are satisfied provided $[x,y]f''(y)$, resp $[x,y]f'''(y)$ is a torsion element, and $n \mp 1$, resp 2 is relatively prime to the additive order of the element $[x,y]f''(y)$, resp. $[x,y]f'''(y)$ Finally, $(8)^*$, $(8')^*$, $(8'')^*$ and $(8''')^*$ are satisfied if and only if $[x,y]$ is torsion-free or the additive order of $[x,y]$ is relatively prime to $m, n, n \mp 1$ and 2, respectively.

In the global case some commutativity results were obtained by H.A.S. Abujabal and the present author in [2], and more generally in [4] For a very special form of $(1''_+)$ with $s = t = 1$ (in our notations), but for $m > 1$ and $r > 0$ depending on x and y were combined by Ashraf and Quadri with (I–A) for some commutative subset A of $N(R)$. They proved that each ring R satisfying these two conditions must be commutative ([4], Th. 1). Later H.A.S. Abujabal, M. Ashraf and M. Obaid enlarged this result to the case where $t' > 0$ also depending on x and y ([4], Th. 1). They also proved the corresponding result for (1_+) instead of $(1''_+)$ (assuming that R is s-unital ([1], Th. 2). Recently, H. Komatsu, T. Nishinaka and H. Tominaga [11] proved somewhat more ($f(y)$ with $f(X) \in X^2\mathbb{Z}[X]$ and $f(1) = \pm 1$ instead of y^m, and $A = N$ instead of $A \subseteq N$ and A commutative) ([11], Th. 3.2)) and without condition (I–A) for $t' \geq 0$ and $r \geq 0$ fixed) ([11], Th. 3.3; see also [12], Th. 1).

Our aim is to prove here two commutativity theorems for the local case, and three theorems for the global case. The second local theorem is partially connected with the results of Ashraf and Quadri, resp. of Abujabal, Ashraf and Obaid mentioned above. The second global theorem we proved here improves all results of [2] and [3].

2. Results

In the local case we combine first the condition (P_4) with (4) and prove

Theorem 1. *Let R be an s-unital (a left, or right s-unital) ring satisfying (P_4) and (4). Then R is commutative provided for each $F \subseteq R$, $|F| \leq 4$, any one of the following conditions is fulfilled:*

1) *$m(F) > 0$, $n(F) > 0$; $m(F) > 1$ or $s(F) > 0$; F satisfies (8) if $r(F) > 0$ and (8) or (8′) if $r(F) = 0$ and $(m(F), n(F)) \neq 1$;*
2) *$n(F) = 0$, $m(F) > 0$; F satisfies (8) (and $r(F) = 0$ or $s(F) = 0$);*
3) *$m(F) = 0$, $n(F) > 0$; F satisfies (8′) (and $t(F) = 0$ or $t'(F) = 0$);*
4) *$m(F) = 1$, $n(F) > 1$, $s(F) = 0$; and F satisfies (8″);*
5) *$m(F) = n(F) = 1$, $s(F) = 0$.*

Theorem 2. *Let R be an s-unital (a left, resp. right s-unital) ring satisfying (P_4) and (I–N). Then R is commutative provided for each $F \subseteq R$, $|F| \leq 4$, one of the following conditions is fulfilled:*

i) *$m(F) > 0$, $n(F) > 0$; $m(F) > 1$ or $s(F) > 1$; ($t(F) = 0$ or $t(F) > > 0$ and $s(F) = 0$, resp. $t'(F) = 0$ or $t'(F) > 0$ and $r(F) = 0$); F satisfies $(8)^*$ for $r(F) > 0$, except for $n(F) = 1$, $t(F) = 0$ or $t'(F) = 0$, and $(8)^*$ or $(8')^*$ for $r(F) = 0$ and $(m(F), n(F)) \neq 1$;*

ii) *$n(F) = 0$, $m(F) > 0$; ($s(F) = 0$, resp. $r(F) = 0$); and F satisfies $(8)^*$;*

iii) *$m(F) = 0$, $n(F) > 0$; ($t(F) = 0$, resp. $t'(F) = 0$); and F satisfies $(8')^*$;*

iv) *$m(F) = 1$, $n(F) > 1$, $s(F) \le 1$; ($t(F) = 0$ or $t(F) > 0$ and $s(F) = 0$, resp. $t'(F) = 0$ or $t'(F) > 0$ and $r(F) = 0$); and F satisfies $(8'')^*$ for $s(F) = 0$;*

v) *$m(F) = n(F) = 1$, $s(F) \le 1$; ($t(F) = 0$ or $t(F) > 0$ and $s(F) = 0$, resp. $t'(F) = 0$ or $t'(F) > 0$ and $r(F) = 0$); and F satisfies (1_-) and $(8''')^*$.*

Th. 1 we use in several occasions. Th. 2 is connected with the above mentioned results ([1], Th. 1) and ([4], Th. 1). The localisation in our theorem is not properly complete, but the equation (1) is more general, and moreover, in (I–A) we assume $A = N$, and not that A is a commutative subset N.

In the global case, for a semi-prime ring R we can drop the condition (I–N) in Th. 2 and weak all of the other conditions in this theorem. Precisely, the following theorem holds true:

Theorem 3. *Let R be a semi-prime ring satisfying* (P). *Then R is commutative provided one of the following conditions is satisfied:*

a) *$m > 0$, $n > 0$; $m > 1$ or $s > 0$;*

b) *$n = 0$, $m > 0$; and $r = 0$ or $s = 0$ for m even;*

c) *$m = 0$, $n > 0$; and $t = 0$ or $t' = 0$ for n even;*

d) *$m = 1$, $n > 0$, $s = 0$; $t > 0$ or $t = 0$ and n even, or n and $t' - r$ odd.*

Similarly, in the global case, it is possible to drop the condition (4) in Th. 1, for any ring R, under an appropriate sharpening of other conditions in this theorem:

Theorem 4. *Let R be a left, resp. right s-unital (an s-unital) ring satisfying condition* (P). *Then R is commutative provided one of the following conditions is fulfilled:*

A) *$m > 0$, $n > 0$; $m > 1$ or $s > 1$; $t = 0$ or $s = 0$, resp. $t' = 0$ or $r = 0$ (only if R is left, resp. right s-unital); R satisfies (8) for $r > 0$, except for $n = 1$ and $t = 0$ or $t' = 0$, and (8) or (8′) for $r = 0$ and $(m, n) \ne 1$;*

B) *$n = 0$, $m > 0$; $r = 0$ or $s = 0$ (for m even); and R satisfies (8);*

C) *$m = 0$, $n > 0$; $t' = 0$ or $t = 0$ (for n even); and R satisfies (8′);*

D) *$m = 1$, $n > 0$, $s \le 1$; $t = 0$ or $s = 0$, resp. $t' = 0$ or $r = 0$ (only if R is left, resp. right s-unital); and R satisfies (8‴) for $s = 0$.*

The conditions (8), resp. (8) or (8′) in Th. 4 A), which are trivially satisfied if $m = 1$, can be eliminated if we assume that R satisfies (P) for $m = m_j$, $n = n_j$, $r = r_j$, $s = s_j$, $t = t_j$ and $t' = t'_j$, where j

runs over a finite index set J such that the greatest common divisor $(m_j: j \in J)$ is equal to 1:

Theorem 5. *Let R be a left, resp. right s-unital (an s-unital) ring satisfying* (P) *for $m = m_j$, $n = n_j$, $r = r_j$, $s = s_j$, $t = t_j$ and $t' = t'_j$, where j runs over a finite index set J, such that $(m_j: j \in J) = 1$. If moreover, for each $j \in J$,*

$$m_j > 0,\ n_j > 0;\ m_j > 1 \ \ or\ \ s_j > 0;$$
$$t_j = 0 \ \ or\ \ s_j = 0,\ t'_j = 0 \ \ or\ \ r_j = 0$$
(only if R is left, resp. right s-unital),

then R is commutative.

Since, as just mentioned, $Q(m(F)) \Rightarrow (8) \Rightarrow (8)^*$, and $Q(n(F)) \Rightarrow \Rightarrow (8') \Rightarrow (8')^*$, we have especially the following results:

Corollary 1. *Let R be an s-unital (a left, resp. right s-unital) ring satisfying* $(\mathrm{P})_4$ *and* (4). *Then R is commutative if for each $F \subseteq R$, $|F| \leq 4$, one of the following conditions is fulfilled:*

1′) *$m(F) > 0$, $n(F) > 0$; $m(F) > 1$ or $s(F) > 0$; R satisfies $Q(m(F))$ if $r(F) > 0$, and $Q(m(F))$ or $Q(n(F))$ if $r(F) = 0$ and $(m(F), n(F)) \neq 1$;*

2′) *$n(F) = 0$, $m(F) > 0$; R satisfies $Q(m(F))$ (and $r(F) = 0$ or $s(F) = 0$);*

3′) *$m(F) = 0$, $n(F) > 0$; R satisfies $Q(n(F))$ (and $t(F) = 0$ or $t'(F) = 0$);*

4′) *$m(F) = n(F) = 1$, and $s(F) = 0$.*

Corollary 2. *Let R be an s-unital (a left, resp. right s-unital) ring satisfying* (P_4) *and* (I–N). *Then R is commutative if for each $F \subseteq R$, $|F| \leq 4$, one of the following conditions is fulfilled:*

i′) *$m(F) > 0$, $n(F) > 0$; $m(F) > 1$ or $s(F) > 1$; ($t(F) = 0$ or $T(F) > 0$ and $s(F) = 0$, resp. $t'(F) = 0$ or $t'(F) > 0$ and $r(F) = = 0$); F satisfies $Q(m(F))$ for $r(F) > 0$, except for $n(F) = 1$, $t(F) = 0$ or $t'(F) = 0$, and $Q(m(F))$ or $Q(n(F))$ if $r(F) = 0$ and $(m(F), n(F)) \neq 1$;*

ii′) *$n(F) = 0$, $m(F) > 0$; ($s(F) = 0$, resp. $r(F) = 0$); and F satisfies $Q(m(F))$;*

iii′) *$m(F) = 0$, $n(F) > 0$; ($t(F) = 0$, resp. $t'(F) = 0$); and F satisfies $Q(n(F))$;*

iv′) *$m(F) = 1$, $n(F) > 0$, $s(F) = 1$; ($t(F) = 0$, resp. $t'(F) = 0$, or $t'(F) > 0$ and $r(F) = 0$).*

Corollary 3. *Let R be a left, resp. right s-unital (an s-unital) ring satisfying condition* (P). *Then R is commutative provided one of the following conditions is fulfilled:*

A$'$) $m > 0$, $n > 0$; $m > 1$ *or* $s > 1$; $t = 0$ *or* $s = 0$, *resp.* $t' = 0$ *or* $r = 0$ *(only if R is left, resp. right s-unital); and R satisfies $Q(m)$ for* $r > 0$, *except for* $n = 1$ *and* $t = 0$ *or* $t' = 0$, *and* $Q(m)$ *or* $Q(n)$ *for* $r = 0$ *and* $(m,n) \neq 1$;

B$'$) $n = 0$, $m > 0$; $s = 0$ *or* $r = 0$ *(for m even); and R satisfies* $Q(m)$;

C$'$) $m = 0$, $n > 0$; $t = 0$ *or* $t' = 0$ *(for n even); and R satisfies* $Q(n)$;

D$'$) $m = 1$, $n > 0$, $s = 1$; $t = 0$, *resp.* $t' = 0$ *or* $r = 0$ *(only if R is left, resp. right s-unital).*

Cor. 1 follows immediately from Th. 1, Cor. 2 from Th. 2, and Cor. 3 from Th. 4. Cor. 2 is related to the mentioned results of [1] and [4], and Cor. 3 contains almost all main results of [2] and also of [3].

3. Preparations for the proofs

In all of our theorems, except for Th. 3, we suppose that R is an s-unital (a left or/resp. right s-unital) ring. It is well known that in an s-unital (a left, resp. right s-unital) ring R, for arbitrary elements x, y there exists an element e such that $ex = xe = x$ and $ey = ye = y$ ($ex = x$ and $ey = y$, resp. $xe = x$ and $ye = y$). We call such an element e a local (left, resp. right) unity for x and y. Also, it is known ([14], Lemma) that if R is a left, resp. right s-unital ring, and for every two elements x and y in R there exists a positive integer $k = k(x, y)$ and an element $e = e(x, y)$ such that $x^k e = x^k$ and $y^k e = y^k$, resp. $ex^k = x^k$ and $ey^k = y^k$, then R is s-unital.

We state first some known results we will use in this paper. The following two lemmas are well known and easy to prove.

Lemma 1. *Let R be a ring with unity element* 1, *and* $x, y \in R$. *If* $x^k y = (x+1)^{k'} y = 0$ *or* $yx^k = y(x+1)^{k'} = 0$ *for some non-negative integers k and k', then* $y = 0$.

Lemma 2. *Let x and y be given elements of an arbitrary ring R. If* $[x, [x, y]] = 0$, *then* $[x^k, y] = kx^{k-1}[x, y]$ *for all integers* $k \geq 1$.

The next lemma, for $f(X) \in X\mathbb{Z}[X]$ a fixed polynomial, is a very special case of a result due to Streb ([14], Hauptsatz 3; see also [2], Th. S), and is also generally valid by a result due to Bell ([6], Th. 1). In the present form it was proved in a simple manner by this author

([13], Lemma 3).

Lemma 3. *Let R be a ring with the following property:*

(10) *for arbitrary elements x, y in R there exists a polynomial $f(X) \in X\mathbb{Z}[X]$ such that $[x,y] = [x,y]f(y)$.*

Then R is commutative.

The first of the next two results is due to Kezlan and Bell, and the second to Herstein.

Theorem KB ([5], Th. 1; [10], Theorem). *Let f be a polynomial in n non-commuting indeterminates $X_1, \ldots, X_n$ with (relatively prime) coefficients. Then the following are equivalent:*

1) *for any ring R satisfying the polynomial identity*

$$f(x_1, \ldots, x_n) = 0 \text{ for all } x_1, \ldots, x_n \in R$$

the commutator ideal $C = C(R)$ of R is a nil ideal;

2) *for every prime p, the matrix ring $R = M_2(GF(p))$ fails to satisfy the above identity;*
3) *every semi-prime ring R satisfying the above identity is commutative.*

Theorem H ([8], Theorem). *Let R be a ring. If for each $x \in R$ there exists a polynomial $f(X) \in X\mathbb{Z}[X]$ such that $x - f(x) \in Z(R)$, then R is commutative.*

4. Proofs

The next three lemmas concern a ring R which satisfies the condition (P_4). The first of them shows that the ring R in all our theorems, except for Th. 3, is in fact an s-unital ring. This will enable us to prove these theorems for a ring with unity element 1 (see [9], Prop. 1).

Lemma 4. *Let R be a left, resp. right s-unital ring satisfying* (P_4). *Then R is s-unital, provided for each $F \subset R$, $|F| \leq 4$, any one of the following conditions is fulfilled:*

1'') *$m(F) > 0$, $n(F) > 0$; $t(F) = 0$ and $m(F) > 1$ or $s(F) > 0$, or $t(F) > 0$ and $s(F) > 0$, resp. $t'(F) = 0$ and $m(F) > 1$ or $r(F) > 0$ or $s(F) > 0$, or $t'(F) > 0$ and $r(F) = 0$;*

2'') *$n(F) = 0$, $m(F) > 1$; and $s(F) = 0$, resp. $r(F) = 0$;*

3'') *$m(F) = 0$, $n(F) > 1$; and $t(F) = 0$, resp. $t'(F) = 0$.*

Proof. Let x and y be arbitrary elements of R, and e a left, resp. right local unity for x and y. Set $F = \{x, y, x+1, y+1\}$.

Case 1″): Let e be a left local unity for x and y. If $t(F) = 0$, then

$$e^{t'}[e^n, x] = \pm e^r[e, x^m]x^s, \text{ hence } x = xe^n \pm x^{m+s} \mp x^m e x^s,$$

and thus, $x \in xR$ if $m(F) > 1$ or $s(F) > 0$. If $t(F) > 0$ and $s(F) = 0$, then

$$z^{t'}[z^n, e]z^t = \pm z^r[z, e^m], \text{ i.e. } z^{r+1}e^m = z^{r+1} \text{ for } z \in \{x, y\}.$$

Let now e be a right local unity for x and y. If $t'(F) = 0$, then

$$[e^n, x]e^t = \pm e^r[e, x^m]x^s, \text{ hence } x = e^n x \pm e^r x^{m+s} \mp e^{r+1}x^{m+s},$$

and thus, $x \in Rx$ if $m(F) > 1$ or $r(F) > 0$ or $s(F) > 0$. If $t'(F) > 0$, and $r(F) = 0$, then

$$x^{t'}[x^n, e]x^t = \pm[x, e^m]e^s, \text{ hence } x = e^m x.$$

Case 2″): Let e be left, resp. right local unity for x, y. Then

$$\text{for } s(F) = 0, \; z^r[z, e^m] = 0, \text{ i.e. } z^{r+1} = z^{r+1}e^m,$$

resp.

$$\text{for } r(F) = 0, \; [e, z^m]z^s = 0, \text{ i.e. } z^{m+s} = ez^{m+s},$$

for $z \in \{x, y\}$.

Case 3″): For $t(F) = 0$ and a left local unity e resp. for $t'(F) = 0$ and a right local unity e, we have

$$e^{t'}[e^n, x] = 0, \text{ resp. } [e^n, x]e^t = 0,$$

hence,

$$x = xe^n, \text{ resp. } x = e^n x. \quad \Diamond$$

The next lemma we need only for the global case, but we state it in the general local case:

Lemma 5. *No matrix ring $M_2(GF(p))$, p prime, can satisfy* (P_4), *such that for each $F \subset R = M_2(GF(p))$, $|F| \leq 4$, one of the following conditions is fulfilled:*

a) *$m(F) > 0$, $n(F) > 0$, and $m(F) > 1$ or $s(F) > 0$;*
b) *$n(F) = 0$, $m(F) > 0$, and $r(F) = 0$ or $s(F) = 0$ for $m(F)$ even;*
c) *$m(F) = 0$, $n(F) = 0$, and $t'(F) = 0$ or $t(F) = 0$ for $n(F)$ even;*
d) *$m(F) = 1, n(F) > 0$, $s(F) = 0$; and $t(F) > 0$ or $t(F) = 0$ and $n(F)$ even, or $n(F)$ and $t'(F) - r(F)$ odd.*

Proof. Let e_{ij} be the matrix in R with entree 1 on the position i, j,and with 0 elsewhere. Set $F = \{e_{11}, e_{22}, e_{12}, e_{12} + e_{21}\}$.

Case a): $x = e_{12} + e_{21}$ and $y = e_{12}$ for $n(F)$ odd, resp. $y = e_{11}$ for $n(F)$ even fail to satisfy (1).

Case b): $x = e_{12} + e_{21}$ and $y = e_{11}$ for $m(F)$ odd, and $y = e_{12}$, $x = e_{22}$, resp. $x = e_{11}$ for $m(F)$ even and $t'(F) = 0$, resp. $t(F) = 0$, fail to satisfy (1).

Case c): Similar to *Case* b).

Case d): $x = e_{11}$ and $y = e_{12}$ for $t(F) > 0$, and $x = e_{12} + e_{21}$ and $y = e_{11}$ for $t(F) = 0$ and $n(F)$ even or $n(F)$ and $t'(F) - r(F)$ odd, fail to satisfy (1).

The following Lemma will be used in the proof of Th. 2 and that of Th. 4, and thus we need its general local form:

Lemma 6 (cf. [2], Lemma 5). *Let R be a ring with unity 1 which satisfies (P_4). Then every nilpotent element of R is central, i.e. $N(R) \subseteq Z(R)$ provided for each $F \subseteq R$, $|F| \leq 4$, one of the following conditions is fulfilled:*

i″) *$m(F) > 0$, $n(F) > 0$; $m(F) > 1$ or $s(F) > 1$; and F satisfies $(8)^*$ for $r(F) > 0$, except for $n(F) = 1$, $t'(F) = 0$ or $t(F) = 0$, and $(8)^*$ or $(8')^*$ if $r(F) = 0$ and $(m(F), n(F)) \neq 1$;*

ii″) *$n(F) = 0$, $m(F) > 0$ and F satisfies $(8)^*$;*

iii″) *$m(F) = 0$, $n(F) > 0$ and F satisfies $(8')^*$;*

iv″) *$m(F) = 1$, $n(F) > 1$, $s(F) \leq 1$, and F satisfies $(8'')^*$ for $s(F) = 0$;*

v″) *$m(F) = n(F) = 1$, $s(F) \leq 1$, and for $s(F) = 0$, F satisfies (1_-) and $(8''')^*$.*

Proof. Let $x \in R$ and $a \in N(R)$ be arbitrary, but fixed elements. We have to prove that $[x, a] = 0$. Since $a \in N(R)$, there exists a minimal positive integer p such that

$$[x, a^k] = 0 \text{ for all integers } k \geq p.$$

If $p = 1$, we have nothing to prove. Suppose that $p > 1$ and set $b = a^{p-1}$. Then

$$b^k[x, b] = [x, b^k] = [x, b]b^k \text{ for all integers } k \geq 1 \tag{11}$$

and

$$b[x, b] = -[x, b]b. \tag{11'}$$

We will prove that $[x, b] = 0$, which in view of the minimality of p contradicts to the assumption that $p > 1$. For this purpose set $F = \{x, b, x+1, b+1\}$.

Case i''): Setting b for y in (1), in view of (11) we get

$$x^{t'}[x^n, b]x^t = 0. \tag{12}$$

Now, setting $b+1$ in (1) and using (12), we obtain

$$x^r[x(b+1)^m](b+1)^s = 0.$$

Since $b+1$ is invertible, the last equation, in view of (11) and Lemma 1 yields

$$m[x, b] = 0. \tag{13}$$

If $n = 1$, then setting $x+1$ for x in (12), we get

$$(x+1)^{t'}[x, b](x+1)^t = 0,$$

and if $t = 0$ or $t' = 0$, then by Lemma 1, follows $[x, b] = 0$. Also, for $r(F) > 0$, (8)* and (13) imply $[x, b] = 0$. Let now $r(F) = 0$. Then for $n = 1$, $t > 0$ and $t' > 0$, and also for $n > 1$, (1) with b, resp. x instead of x, resp. y, gives

$$[b, x^m]x^s = 0$$

and thus (1), for $b+1$, resp. x instead of x, resp. y, becomes

$$(b+1)^{t'}[(b+1)^n, x](b+1)^t = 0.$$

In view of (11) and the invertibility of $b+1$, this yields

$$n[x, b] = 0. \tag{13'}$$

If $(m, n) = 1$, then (13) and (13′) imply $[x, b] = 0$. If $(m, n) \neq 1$, then we have (8)* or (8′)*,and then (13) or (13′) implies $[x, b] = 0$ again.

Case ii''): If $n(F) = 0$, $m(F) > 0$, then (1) for $y = b+1$ becomes

$$x^r[x, (b+1)^m](b+1)^s = 0,$$

hence by (11) and the invertibility of $b+1$, after applying Lemma 1, we get $m[x, b] = 0$. This in view of (8)* implies $[x, b] = 0$.

Similarly, we can get $[x, b] = 0$ in *Case* iii'').

Case iv''): If $s(F) = 1$, then (1) for $y = b$ and the same equation for $y = b+1$ easily give $x^n[x, b] = 0$, and this using Lemma 1 implies $[x, b] = 0$.

Let now $s(F) = 0$. Setting $b+1$ for x, and x for y in (1), we get in view of (11) and (11′),

$$(n \mp 1)[b, x] = (-nt' + nt \pm r)b[b, x],$$

hence

$$(n \mp 1)^2[x, b] = 0,$$

and thus, by $(8'')^*$, $[x,b]=0$.

Case v''): For $s=1$, we can get $[x,b]=0$ as in *Case* iv'). Let now $s=0$. Then for $x=b+1$, and $y=x$, (1_-) becomes

$$(b+1)^{t'}[b,x](b+1)^t=-(b+1)^r[b,x],$$

hence, in view of (11) and (11') we have

$$2[x,b]=(-t'+t-r)b[x,b], \text{ i.e. } 2^2[x,b]=0,$$

which by $(8''')^*$ gives $[x,b]=0$. ◊

Now we can go to the proofs of our theorems.

By Lemma 4, all rings in our theorems, except for Th. 3, are s-unital, and according to ([9], Prop. 1) we can and will assume that, in all these theorems, R is a ring with unity 1.

Proof of Th. 1. In view of (4) and Lemma 2, the identity (1) is equivalent with (5), and also with

$$x^{t+t'}[x^n,y]=[x^n,y]x^{t+t'}=\pm x^r[x,y^m]y^s=\pm y^s[x,y^m]x^r. \tag{14}$$

From (14) and Lemma 4 we easily see that R is s-unital. Now we can assume that R is a ring with unity 1. We fix x and y in R, and set $F=\{x,y,x+1,y+1\}$.

Case 1): Setting in (5) $y+1$ for y, and combining equality (5) with the obtained one, we get

$$mx^r[x,y](1-f(y))=0,$$

hence, using Lemma 1,

$$m[x,y]=m[x,y]f(y), \tag{15}$$

where $f(X)\in X\mathbb{Z}[X]$ is given by (9). Similarly, from (5), if $r(F)=0$, we get

$$n[x,y]=n[x,y]f'(y), \tag{15'}$$

where $f'(X)\in X\mathbb{Z}[X]$ is given by (9').

Using (15) and Lemma 1, from (5) we easily obtain

$$n[x,y]=n[x,y]f(y), \tag{16}$$

and similarly, from (5) and (15'), if $r(F)=0$ we can get

$$m[x,y]=m[x,y]f'(y). \tag{16'}$$

If $r(F)>0$, then by assumption, F satisfies (8), and thus (15) implies

$$[x,y]=[x,y]kf(y). \tag{17}$$

Let now $r(F)$=0. If $(m,n)=1$, then from (15) and (16), resp. (15′) and (16′) follows (17), resp.

$$[x,y]=[x,y]k'f'(y). \tag{17′}$$

If $(m,n)\neq 1$, then, by assumption, F satisfies (8) or (8′), and thus (15) implies (17) or (15′) implies (17′).

Case 2): In this case instead of (5) we have

$$mx^r[x,y]y^{m+s-1}=0,\ \text{resp. } n[x,y]x^{n+t+t'-1}=0,$$

which by applying Lemma 1 give

$$m[x,y]=0,\quad \text{resp.}\quad n[x,y]=0,$$

i.e. in view of $(8)^*$, resp. $(8')^*$, $[x,y]=0$.

Case 3): Since $m=1$, and $s=0$, in this case (5) becomes

$$n[x,y]x^{n+s+t-1}=\pm[x,y]x^r \text{ for all } x,y\in F.$$

From this equation we easily get

$$(n\mp 1)[x,y]=[x,y]f''(y),$$

where $f''(X)\in X\mathbb{Z}[X]$ is given by (9″). But the last equation, in view of (8″), implies

$$[x,y]=[x,y]k''f''(y). \tag{17″}$$

Case 4): Now (5_-) becomes

$$[x,y]x^{t+t'}=-[x,y]x^r \quad \text{for all} \quad x,y\in F.$$

This implies

$$2[x,y]=[x,y]f'''(y),$$

i.e. by (8‴),

$$[x,y]=[x,y]k'''f'''(y), \tag{17‴}$$

where $f'''(X)\in X\mathbb{Z}[X]$ is given by (9‴).

Thus, the ring R in Th. 1 satisfies the condition of Lemma 3, and so R is commutative. ◊

Remark. Since obviously, for every prime p, the matrix ring $R=$ $=M_2(GF(p))$ fails to satisfy (4), then by Th. KB, for every ring R, (4) implies

(4′) $[x, y] \in N(R)$ for all $x, y \in R$.

For an s-unital (a left, resp. right s-unital) ring R in Th. 1, we can replace (4) by the weaker condition (4′), assuming the conditions i″)–v″) of Lemma 6 (and the conditions 1″)–3″) of Lemma 4) were satisfied. Namely (by Lemma 4, R is unital, and) we can assume that R has a unity 1. Moreover, according to Lemma 6, $N(R) \subseteq Z(R)$, hence by (4), $[x, y] \in Z(R)$ for all $x, y \in R$, and especially we have (4).

Proof of Th. 2. By Lemma 4, the ring R in this theorem is s-unital, and we can assume that R is with unity 1. But, then, according to Lemma 6, $N(R) \subseteq Z(R)$. Since, moreover, R satisfies (I–N), R in fact satisfies the conditions in Th. H, and hence R is commutative. ◊

Proof of Th. 3. By Lemma 5, no matrix ring $M_2(GF(p))$, p prime, can satisfy (P) in situation of Th. 3. Since R is semi-prime and R satisfies (P), R must be commutative in view of Th. KB. ◊

Proof of Th. 4. According to Lemma 4, the ring R in this theorem is s-unital, and thus we can assume that R is a ring with unity 1. But, then R satisfies all conditions of Lemma 6, and thus, $N(R) \subseteq Z(R)$. Moreover, in the situation of Th. 4, in view of Lemma 5, no matrix ring $M_2(GF(p))$, p prime, can satisfy (P), hence in view of Th. KB, $C(R) \subseteq$ $\subseteq N(R)$. Therefore, $C(R) \subseteq Z(R)$, and thus R satisfies all conditions of Th. 1, and so R is commutative. ◊

Proof of Th. 5. The ring R in this theorem obviously satisfies condition 1′) of Lemma 4. Therefore, R is s-unital, and so we can assume that R is a ring with unity 1. Moreover, since $m_j > 1$ or $s_j > 0$, similarly as we have got $m[x, b] = 0$ in the proof of Lemma 6, we can get $m_j[x, b] = 0$ for all $j \in J$. But, this with $(m_j : j \in J) = 1$ implies $[x, b] = 0$. Hence $N(R) \subseteq Z(R)$. Also, for each $j \in J$, R satisfies condition 1″) of Lemma 5, and thus, in view of Th. KB, $C(R) \subseteq N(R)$. Hence, $C(R) \subseteq Z(R)$, and R surely satisfies (4). Now, as in the proof of Th. 1, we can prove that

(15j) $m_j[x, y] = m_j[x, y] f_j(y)$ for all $x, y \in J$,

where $f_j(X) \in X\mathbb{Z}[X]$. This, in view of $(m_j : j \in J) = 1$, yields

$$[x, y] = [x, y] \sum_{j \in J} m_j m_j' f_j(y) \text{ for all } x, y \in R,$$

and some integers m_j' $(j \in J)$. Therefore, R is commutative according to Lemma 3. ◊

For the appropriate examples in the local, resp. global case, and also for other related results, we refer to [1] and [4], resp. [2] and [3].

We give here an example showing that a ring R can satisfy a local condition of the form (P_4), which seems to does not satisfy a global condition of the form (P).

Example 1. Let $M = M(p)$ be the ring of all infinite lower triangular matrices over a Galois field $GF(p)$, p prime, and k be a positive integer. We denote by $M_k = M_k(p)$ the subring of M generated by the identity matrix I and all matrices $(a_{ij}) \in M$ such that $a_{ii} = 0$ for $i = 1, 2, \ldots$ and $a_{ij} = 0$ for $i \geq k$, or $j \geq k$. Let R be the union of all the M_k. Then R is a ring with unity I.

The regular matrices in M_k form a finite group of order $(p-1)p^{\frac{k(k-1)}{2}} =: m_k$ and thus, $a^{m_k} = I$ for all regular matrices $a \in \in M_k$. A non-regular matrix $b \in M_k$ is nilpotent, and $b^k = 0$ for all such matrices b. For sufficiently large p, we have $m_k \geq k$, and hence

$$x^{m_k} \in Z(R) \text{ for all } x \in M_k. \tag{18}$$

Now, for each finite subset F of R, there exists an integer k such that $F \subseteq M_k$, and thus for

$$m = m(F) = m_k, n = n(F) = \varrho(F)m_k \tag{19}$$

and arbitrary non-negative integers $r = r(F)$, $s = s(F)$, $t = t(F)$ and $t' = t'(F)$ we have

$$x^{t'}[x^n, y]x^t = \pm x^r[x, y_m]y^s \text{ for all } x, y \in F.$$

Thereby, $\varrho = \varrho(F)$ is a non-negative integer. Especially, the ring R satisfies a condition of the form (P_4). Since R is obviously non-commutative, R fails to satisfy some of the additional conditions in our Th. 1 and Th. 2. Actually, the condition (8)* is not fulfilled, and thus also the condition (8) cannot be satisfied. Namely, for $b = e_{12} \in M_k$, $k > 1$, we have $b^2 = 0$. On the other hand, for $x = e_{2k} \in M_k$, $k > 2$, we have

$$m_k[x, b] = 0, \quad \text{but} \quad [x, b] = e_{1k} \neq 0.$$

We remark that in view of (18), $C(R) \subseteq N(R)$ by a well known result due to Herstein [7].

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