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ERDŐS–MORDELL INEQUALITY FOR SPACE N–GONS

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Abstract: The Erdős–Mordell inequality is extended on the space closed n-gons in E^3. The inequality holds for any point O of the convex hull of the n-gon. The equality is attained only for regular n-gons with the center O.

H. Ch. Lenhard [3] proved the following statement: *Let $\mathcal{A}$ be a closed n-gon with vertices $A_0, A_1, \dots, A_{n-1}$ bounding a star shaped region in the plane and let O be a point in the interior of $\mathcal{A}$ such that all sides of $\mathcal{A}$ are visible from O. Denote by R_i the distance of the points O and A_i and let r_i denote the distance of O to the line A_iA_{i+1} (where $A_n = A_0$). Then*

$$\cos\frac{\pi}{n}\sum_{i=0}^{n-1} R_i \geq \sum_{i=0}^{n-1} r_i \tag{1}$$

holds. Equality holds only if $\mathcal{A}$ is a regular n-gon with center O.

The inequality (1) for $n = 3$ is known as the Erdős–Mordell inequality. L. Fejes-Tóth [2] conjectured (1) for convex plane n-gons. Lenhard's result confirms and generalizes Fejes-Tóth's conjecture, because (1) holds even for non-convex n-gons. In the present work we will give a further generalization of (1), which is given in the following

Theorem 1. *Let $\mathcal{A}$ be a closed n-gon in E^3 with vertices $A_0, A_1, \dots, A_{n-1}$ and let O be a point in the convex hull $K(\mathcal{A})$ of $\mathcal{A}$. Denote by*

R_i the distance of the points O and A_i and denote by r_i the distance of O to the line A_iA_{i+1}. Then inequality (1) holds. Equality in (1) is attained iff $\mathcal{A}$ is a plane regular n-gon and O is its center.

To prove Th. 1 we shall need the following lemma:

Lemma. *Let $\mathcal{A}$ be a closed space n-gon in E^3 with vertices $A_0, A_1, \dots, A_{n-1}$ and let O be a point in the convex hull $K(\mathcal{A})$ of $\mathcal{A}$. Writing $\varphi_i = |\sphericalangle A_iOA_{i+1}|$ we have*

$$\sum_{i=0}^{n-1} \varphi_i \geq 2\pi \,. \tag{2}$$

Proof. Our proof is based on the following statement given by I. Fáry [1]: *Let $\vec{u}, \vec{v}$ be two vectors and φ their angle. Denote by $\varphi(\sigma)$ the angle between the orthogonal projections of $\vec{u}$ and $\vec{v}$ in the direction σ. Then*

$$\varphi = \frac{1}{4\pi} \int_{\Omega} \varphi(\sigma)\, d\Omega \quad \sigma \in \Omega,$$

where Ω is a unit sphere and σ its point determined by the direction σ. By applying this statement to the n-gon $\mathcal{A}$ we get

$$\sum_{i=0}^{n-1} \varphi_i = \frac{1}{4\pi} \int_{\Omega} \sum_{i=0}^{n-1} \varphi_i(\sigma) d\Omega \,.$$

This equality reduces the space case to a planar one. To prove (2) it suffices to show that

$$\sum_{i=0}^{n-1} \varphi_i(\sigma) \geq 2\pi \quad \text{for all} \quad \sigma \in \Omega \,.$$

Let $\mathcal{A}_\sigma$ denote the orthogonal projection of $\mathcal{A}$ in the direction σ. From the definition of the convex hull, it follows that the point O belongs to the convex hull $K(\mathcal{A}_\sigma)$ of $\mathcal{A}_\sigma$. The convex hull $K(\mathcal{A}_\sigma)$ of $\mathcal{A}_\sigma$ is a polygon, whose vertices form a subset of the set of vertices of $\mathcal{A}_\sigma$. Because of convexity of $K(\mathcal{A}_\sigma)$, the sum of the angles between O and the vertices of $K(\mathcal{A}_\sigma)$ equals 2π. Our assertion now readily follows. $\Diamond$

For the proof of Th. 1 we shall need another geometrical result, known as a discrete case of Wirtinger's inequality: *Let $\mathcal{A}$ be a closed space n-gon in E^3 with vertices $A_0, A_1, \dots, A_{n-1}$ and with the centroid at the origin of the coordinate system. Then*

$$\sum_{k=0}^{n-1} |A_kA_{k+1}|^2 \geq 4\sin^2\frac{\pi}{n} \sum_{k=0}^{n-1} |A_k|^2 \,. \tag{3}$$

Equality holds iff $\mathcal{A}$ is a plane affine-regular n-gon. Inequality (3) was stated for plane n-gons by B. H. Neumann [4]. For a proof of the general case see [5], [6].

Proof of Th. 1. We will proceed similarly as H. Ch. Lenhard. It is more convenient to use now the notation $|\sphericalangle A_iOA_{i+1}| = 2\varphi_i$. We will show that even

$$\cos\frac{\pi}{n}\sum_{i=0}^{n-1} R_i \geq \sum_{i=0}^{n-1}\sqrt{R_iR_{i+1}}\cos\varphi_i \geq \sum_{i=0}^{n-1} r_i \tag{4}$$

holds. Namely,

$$|OP_i| = \frac{2R_iR_{i+1}}{R_i + R_{i+1}}\cos\varphi_i\,,$$

where P_i is the intersection of the bisector of the angle R_iOR_{i+1} with the side A_iA_{i+1}. The second inequality in (4) now follows from the inequality between the harmonic and geometric mean, with equality only for $R_i = R_{i+1}$.

To prove the first inequality in (4), construct the central symmetric $2n$-gon $\mathcal{B}$ with vertices $B_0, B_1, \ldots, B_{2n-1}$, with $B_{2n} = B_0$, with the centroid at the point O as the origin of the coordinate system, so that

$$|B_i| = \sqrt{R_i}\,,\quad |\sphericalangle B_iOB_{i+1}| = \varphi_i\,,\quad B_{n+i} = -B_i\,,\quad i = 0, 1, \ldots, n-1\,.$$

Inequality (2) ensures, that this construction always gives at least one $2n$-gon $\mathcal{B}$. By applying the inequality (3) to the $2n$-gon $\mathcal{B}$ we get

$$\sum_{k=0}^{2n-1} |B_kB_{k+1}|^2 \geq 4\sin^2\frac{\pi}{2n}\sum_{k=0}^{2n-1} |B_k|^2\,, \tag{5}$$

which is equivalent to

$$\cos\frac{\pi}{n}\sum_{k=0}^{2n-1} |B_k|^2 \geq \sum_{k=0}^{2n-1} |B_k|\cdot|B_{k+1}|\cos\varphi_k\,.$$

Dividing both sides by 2, we get the left inequality in (4). Equality in (5) is attained if and only if the $2n$-gon is a plane affine-regular one which, together with the condition $R_i = R_{i+1}$, $i = 0, 1, \ldots, n-1$, gives that equality in (1) is attained iff the n-gon $\mathcal{A}$ is regular and O is its center. ◊

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TORSION-FREE MODULES AND SYZYGIES

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Abstract: We show how a number of well-known results follow from a characterization of torsion-free modules.

In this note we bring to light a result which seems to be lying beneath the surface of a number of well known theorems and, once stated, from which these theorems may be readily derived. To wit, let R be a Noetherian ring and M a finitely generated R-module admitting a finite free resolution,

$$\mathbf{F}\colon 0 \longrightarrow F_n \xrightarrow{\phi_n} \cdots \xrightarrow{\phi_2} F_1 \xrightarrow{\phi_1} F_0 \longrightarrow M \longrightarrow 0.$$

We observe that M is torsion-free if and only if the ideal of minors associated to the ith map in the resolution has depth greater than or equal to $i+1$. The similarity with this statement and the one appearing in the celebrated exactness theorem of Buchsbaum and Eisenbud [3] is not coincidental. The result is essentially equivalent to their theorem. However, it seems that bringing it to the fore allows one to see precisely how the conditions of their theorem yield exactness. The result also serves as the inductive step in an analogous characterization for M to be a kth syzygy. Using this result, we can derive a theorem of Auslander–Bridger appearing in [1] and extend a result of Bruns concerning the structure of kth syzygies to non-Cohen–Macaulay local rings. Finally, though the proposition below doesn't seem to explicitly appear in any of the standard references on the subject, undoubtedly it is not new. Our primary purpose here is to demonstrate the central place it occupies.

Let $\mathbf{F}$ be given as above i.e., each F_i is a free R-module of finite rank and ϕ_i is a rank$(F_{i-1}) \times$ rank(F_i) matrix with entries in R. The rank of ϕ_i is the size of the largest non-vanishing minor of ϕ_i and we will write $I(\phi_i)$ for the ideal generated by minors of size rank(ϕ_i). If rank$(\phi_i) = 0$, take $I(\phi_i) = R$. With this we may state the result as follows.

Proposition. *Let M and $\mathbf{F}$ be as above. Then M is a torsion-free R-module if and only if* depth$(I(\phi_i)) \geq i+1$ *for* $i = 1, \ldots, n$.

Proof. We begin the proof with a couple of remarks. First, recall that for M as above, M is torsion-free if and only if every prime ideal associated to M is an associated prime of R. Furthermore, as the hypotheses and conclusions of the proposition are preserved under localization, we are free to localize at a prime ideal at any point in the argument. Finally, recall that if R is local and the projective dimension of M (denoted p.d.$_R(M)$) is finite, then the Auslander–Buchsbaum formula states that depth(M) + p.d.$_R(M)$ = depth(R) (see [2]).

Now, suppose that M is torsion-free. Let i be the largest integer for which depth$(I(\phi_i)) \leq i$. We seek a contradiction. Select a prime ideal P containing $I(\phi_i)$ such that depth$(R_P) \leq i$. It follows that $I(\phi_j) \nsubseteq P$, for $j > i$. Therefore, upon localizing at P, the sequence $\mathbf{F}$ splits at F_i. Localizing at P and changing notation, we have $I(\phi_{i+1}) = R$, $I(\phi_i) \neq R$ and depth$(R) \leq i$. Thus F_i = image$(\phi_{i+1}) \oplus F_i'$ and we may truncate $\mathbf{F}$ to obtain an exact sequence

$$\mathbf{F}' : 0 \longrightarrow F_i' \xrightarrow{\phi_i} \cdots \xrightarrow{\phi_2} F_1 \xrightarrow{\phi_1} F_0 \longrightarrow M \longrightarrow 0.$$

Thus p.d.$_R(M) \leq i$. If p.d.$_R(M) < i$, image(ϕ_j) is projective (free) for some $j \leq i-1$, so the truncated sequence splits to the left of F_j and it follows that $I(\phi_i) = R$, which isn't so. Thus p.d.$_R(M) = i$. Since depth$(R) \leq i$, the Auslander–Buchsbaum implies i = depth(R) = p.d.$_R(M)$. Consequently depth$(M) = 0$. Therefore P, the maximal ideal of R, is an associated prime of M, and hence R, since M is torsion-free. Thus i = depth$(R) = 0$ and this is the contradiction we sought.

Conversely, suppose the depth condition holds. Let P be an associated prime of M. We must show that P is an associated prime of R. We may assume that R is a local ring and P is its maximal ideal.

Since depth$(M) = 0$, depth$(R) \leq n$ (by the Auslander–Buchsbaum formula). Moreover, as depth$(I(\phi_n)) \geq n+1$, we must have $I(\phi_n) = R$. Thus the sequence $\mathbf{F}$ splits at F_{n-1}, so we may truncate as before to obtain

$$\mathbf{F}' : 0 \longrightarrow F'_{n-1} \xrightarrow{\phi_{n-1}} \cdots \xrightarrow{\phi_2} F_1 \xrightarrow{\phi_1} F_0 \longrightarrow M \longrightarrow 0.$$

By induction on n (the case $n = 0$ is trivial), M is torsion-free, so P is associated to R, as desired. ◊

Corollary A (Buchsbaum–Eisenbud). *Let R be a Noetherian domain and*

$$\mathbf{F} : 0 \longrightarrow F_n \xrightarrow{\phi_n} \cdots \xrightarrow{\phi_2} F_1 \xrightarrow{\phi_1} F_0 \longrightarrow 0$$

a complex of finitely generated free R-modules. Then $\mathbf{F}$ is acyclic (i.e., $\ker(\phi_i) = \text{image}(\phi_{i+1})$ *for $i > 0$) if and only if:* (i) $\text{rank}(\phi_i) + \text{rank}(\phi_{i+1}) = \text{rank}(F_i)$ *and* (ii) depth$(I(\phi_i)) \geq i$, *for $i = 1, \ldots, n$.*

Proof. Let K denote the quotient field of R and suppose the conditions hold. We proceed by induction on n. If $n = 1$, the complex is exact by McCoy's theorem. Assume $n > 1$. Condition (i) implies that $\mathbf{F} \otimes K$ is an acyclic complex of vector spaces. Hence the ith homology module is a torsion module, for $i > 0$. In particular, $H_1(\mathbf{F})$ is torsion. On the other hand, by induction $H_i(\mathbf{F}) = 0$ for $i = 2, \ldots, n$. Thus $\mathbf{F}$ resolves the cokernel of ϕ_2. By the Prop., condition (ii) implies that the cokernel of ϕ_2 is torsion-free. Hence its submodule $H_1(\mathbf{F})$ is torsion-free. Thus $H_1(\mathbf{F})$ is both torsion and torsion-free, and therefore zero. That is, $\mathbf{F}$ is acyclic. Conversely, if $\mathbf{F}$ is acyclic, then $\mathbf{F} \otimes K$ is an acyclic complex of vectors spaces, so (i) holds. Clearly depth$(I(\phi_1)) \geq 1$. Moreover, the cokernel of ϕ_2 is torsion-free, so (ii) holds by the proposition. ◊

Remark. Of course the Buchsbaum–Eisenbud theorem holds for any Noetherian ring, but we have presented the domain case to exhibit more clearly how conditions (i) and (ii) determine exactness. However, essentially the same proof works in general (with the aid of some additional linear algebra). For example, one can show that the conditions (i) and depth$(I(\phi_i)) \geq 1$ hold if and only if the complex $\mathbf{F} \otimes K$ is split exact, where K now denotes the total quotient ring of R. Hence if (i) and (ii) hold, $H_i(\mathbf{F})$ is torsion on the one hand and torsion-free on the other (by the Prop., as in the proof above) and therefore zero.

Corollary B. *Let R be a Noetherian ring and M a finitely generated R-module. Then M is the kth syzygy in a finite free resolution of an R-module N if and only if M admits a finite free resolution $\mathbf{F}$ (as before) satisfying* depth$(I(\phi_i)) \geq i + k$ *for $i = 1, \ldots, n$.*

Proof. Suppose that M is the kth syzygy in a finite free resolution of the R-module N. Then $M = \text{image}(\psi_k)$, where ψ_k is the kth map in the resolution of N. We may take for $\mathbf{F}$, the resolution of M, that portion of the resolution for N whose first map is ψ_{k+1}. That $\text{depth}(I(\phi_i)) \geq$ $\geq i + k$ for $i = 1, \dots, n$ now follows from the Buchsbaum–Eisenbud theorem.

Conversely, suppose M admits a finite free resolution $\mathbf{F}$ satisfying the required depth condition. We proceed by induction on k. When $k =$ $= 1$, M is torsion-free (by the Prop.) and it is well known that M can be embedded in a free module (when R is not a domain, this requires that M have finite projective dimension, which we are assuming). Therefore M is the first syzygy in a resolution of the cokernel of this embedding. Now suppose that $k > 1$. Let $f_1, \dots, f_n$ generate $\text{Hom}(M, R)$ and take $u\colon M \to R^n$ to be the so-called *universal pushforward* (see [6]). In other words, for each $m \in M$, $u(m)$ is the column vector whose jth entry is $f_j(m)$. Let $C = \text{cokernel}(u)$. Using * to denote R duals, we have exact sequences

$$0 \longrightarrow M \longrightarrow R^n \longrightarrow C \longrightarrow 0$$

$$0 \longrightarrow C^* \longrightarrow (R^n)^* \longrightarrow M^* \longrightarrow 0$$

where exactness in the first sequence follows because M is torsion-free, and exactness in the second sequence follows from the definition of universal pushforward. Let $Q \subseteq R$ be a prime ideal with $\text{depth}(R_Q) \leq$ $\leq k$. Then $I(\phi_1) \nsubseteq Q$, so M_Q is a free R_Q module. Therefore M_Q^* is free, so the second sequence splits over R_Q. Therefore the dual of the second sequence (i.e., the "double dual") splits over R_Q. Since M_Q is free this shows that $C_Q = C_Q^{**}$ and that the first sequence splits over R_Q. It follows that if we let ϕ_0 denote the composition $F_0 \to M \to R^n$, then $\text{depth}(I(\phi_0)) \geq k + 1$ and C admits a resolution satisfying the given depth condition for $k - 1$. By induction C is a $(k-1)$st syzygy of the required form, so M is a kth syzygy, as desired. $\Diamond$

Corollary C (Auslander–Bridger). *Let R be a local ring satisfying Serre's condition S_k and M a finitely generated R-module having finite projective dimension. Then M is a kth syzygy if and only if M satisfies S_k $(k \geq 1)$.*

Proof. Recall that a finitely generated R-module N satisfies S_k if for all prime ideals P in the support of N, $\text{depth}_{R_P}(N_P) \geq \min(k, \dim(R_P))$. Now, let M satisfy S_k and $\mathbf{F}$ be a projective resolution. We want to see that $\mathbf{F}$ satisfies the depth condition of Cor. B. As in the proof of the

Prop., we let i be the largest integer for which depth $(I(\phi_i)) \leq i+k-1$ and we select a prime ideal P containing $I(\phi_i)$ with depth $(R_P) \leq i+$ $+k-1$. If we localize at P, then p.d.$(M_P) = i > 0$. Thus depth $(M_P) \leq$ $\leq k-1$, by the Auslander–Buchsbaum formula. Since M satisfies S_k, this implies depth $(M_P) =$ depth (R_P). Thus M_P is free, so $i = 0$, contradiction.

Conversely, suppose that M is a kth syzygy and $\mathbf{F}$ is a resolution of M. We may assume that $\mathbf{F}$ satisfies the depth condition of Cor. B. Let $P \subseteq R$ be prime ideal. If $\dim(R_P) \leq k$, then $I(\phi_i) \not\subseteq P$, so M_P is R_P-free. Thus depth $(M_P) =$ depth $(R_P) = \dim(R_P)$, since R satisfies S_k. If $\dim(R_P) \geq k+1$, depth $(R_P) = k+i$, for some $i \geq 0$, as R satisfies S_k. Thus $I(\phi_{i+1}) \not\subseteq P$. Hence, p.d.$(M_P) \leq i$, so depth $(M_P) \geq k$. Thus M satisfies S_k. $\Diamond$

Remark. In [4] Bruns proves the following result which shows how to construct kth syzygies of rank k from kth syzygies having rank greater than k. Let (R, m) be a Cohen–Macaulay local ring and M a finitely generated R-module having finite projective dimension. Suppose that M is a kth syzygy having rank $k+s$, for $s \geq 1$. Then there exists a free submodule $F \subseteq M$ such that $F \cap mM = mF$, rank $(F) = s$, and M/F is a kth syzygy. In the corollary below, we use Cor. B to extend Bruns' theorem to non-Cohen–Macaulay rings. In order to do this, we need to observe that choosing basic elements on subsets of Spec(R) determined by depth conditions can be done analogously to the more standard method of choosing basic elements on subsets of Spec(R) determined by height conditions. We follow the treatment given in [6] (which is based upon [4]).

Basic Element Lemma. *Let (R, m) be a local ring and $M \subseteq R^n$ be a finitely generated R-module with well-defined rank $\geq k+1$. Suppose that M_P is a free summand of $(R^n)_P$ for all prime ideals P satisfying* depth $(R_P) \leq k$. *Then there exists a minimal generator $x \in M$ such that x is basic at P for all P satisfying* depth $(R_P) \leq k$.

Proof. We first recall that a submodule $M' \subseteq M$ is said to be t-*fold basic* at P if at least t minimal generators for M_P can be chosen from the image of M'. The proof now follows along the same lines as that of Cor. 2.6 in [6], once we verify the following statement. Let $\{x_1, \ldots, x_s\}$ be a subset of a set of generators for M such that M', the submodule they generate, is t-fold basic at all primes P satisfying depth $(R_P) \leq j-1$. Then M' is t-fold basic at all but finitely many primes P satisfying depth $(R_P) = j$. To see this, suppose that

$\{x_1, \dots, x_s, \dots, x_m\}$ is a set of generators for M and write A for the $n \times m$ matrix whose columns correspond to the x_i. Let A' denote the corresponding submatrix associated to M'. Then for any prime ideal P such that M_P is a summand of $(R^n)_P$, M' is t-fold basic at P if and only if $I_t(A')$, the ideal of $t \times t$ minors of A', is not contained in P. Now, since M' is t-fold basic at all P satisfying depth$(R_P) \leq$ $\leq j-1$, depth$(I_t(A')) \geq j$. If depth$(I_t(A')) \geq j+1$, the statement follows. Otherwise, letting $a_1, \dots, a_j$ be a maximal regular sequence in $I_t(A')$, it follows that $I_t(A') \subseteq P$ and depth$(R_P) = j$ if and only if $P \in \text{Ass}(R/(a_1, \dots, a_j)R)$. Since $\text{Ass}(R/(a_1, \dots a_j)R)$ is finite, the statement follows in this case as well. $\Diamond$

Corollary D. *Let (R, m) be a local ring and M a finitely generated R-module with finite projective dimension. Suppose that M is a kth syzygy having rank $k+s$, for $s \geq 1$. Then there exists a free submodule $F \subseteq M$ such that $F \cap mM = mF$,* rank$(F) = s$, *and M/F is a kth syzygy.*

Proof. We follow the path laid out in Bruns' original theorem. Let $\mathbf{F}$ as above be a projective resolution of M. We may assume that $\mathbf{F}$ satisfies the depth condition of Cor. B. Since depth$(I(\phi_i)) \geq k+1$, M_P is free for all prime ideals P satisfying depth$(R_P) \leq k$. As in the proof of Cor. B, we may use the universal pushforward of M to further assume that $M \subseteq R^n$ for some n, and M_P is a summand of $(R^n)_P$, whenever M_P is free. We now employ the Basic Element Lemma to find a minimal generator $x \in M$ which is basic at all P satisfying depth$(R_P) \leq k$. In particular, Rx is a free submodule of M and without loss of generality we may assume that x is the "first" generator of M. It follows that a minimal resolution for M/Rx has the form

$$\mathbf{F}': 0 \longrightarrow F_n \xrightarrow{\phi_n} \cdots \xrightarrow{\phi_2} F_1 \xrightarrow{\phi_1'} F_0' \longrightarrow M/Rx \longrightarrow 0$$

where ϕ_1' is the matrix obtained from ϕ_1 by deleting the first row and F_0' is the free R-module on one less generator than F_0. Furthermore, the choice of x implies that $(M/Rx)_P$ is free for all primes P satisfying depth$(R_P) \leq k$, so depth$(I(\phi_1')) \geq k+1$. Hence the resolution for M/Rx satisfies the depth condition of Cor. B. That is, M/Rx is a kth syzygy. The process may be repeated if M/Rx has rank greater than k. $\Diamond$

Remark. Unfortunately, Cor. B does not shed a lot of light on the Evans–Griffith Syzygy Theorem (see [6]), which states that kth syzygies with finite projective dimension have rank $\geq k$ (when the ring R

contains a field). Using Cor. B in a manner analogous to its use in Cor. D, one can easily see that it suffices to find a minimal generator x whose order ideal has depth $\geq k$. For then M/Rx would be a $(k-1)$st syzygy, and induction would yield the result. (For rings containing a field such x exists.) This is exactly the original line of thought followed by Evans and Griffith. The point of Cor. B is that one need not have any standing assumption on the ring (like the Cohen–Macaulay property) as long as one replaces height by depth in a characterization of kth syzygies. (See also [7] or [5], where for rings containing a field, the Evans–Griffith estimates on the ranks of syzygies are improved.)

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SUCCESSIVE APPROXIMATION METHOD FOR INVESTIGATING THREE POINT BOUNDARY VALUE PROBLEM WITH SINGULAR MATRICES

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Abstract: In this paper the existence of the solution of a three-point boundary value problem belonging to a system of nonlinear differential equations

$$\frac{dx}{dt} = f(t,x), \quad x, f \in \mathbb{R}^n, \quad Ax(0) + A_1x(t_1) + Cx(T) = d$$

is investigated by using a new version of the numerical-analytic methods. The approximate solution is determined and an estimation for the error is given.

1. Introduction

In the literature different numerical, analytic and functional-analytic methods are known to investigate both the two- and the n-point boundary value problems depending on the type of the equation and the boundary value condition ([1], [4], [5]).

When the existence of the solution can be supposed some numerical methods aim mainly at the approximate determination of the solution ([2], [6]).

The analytic methods (i.e. those of the continuous closed form) based mostly upon various series expansions are generally used for qualitative investigations (uniqueness, stability) ([3], [10], [11], [12]).

When using functional analytic methods the given boundary value problems are substituted by a suitably chosen equivalent operator equation ([4], [12]). For certain three-point boundary value problems — see papers ([7], [8]) — this operator equation is an integral equation, which is set up by using a suitably chosen Green-function. These integral equations are generally investigated by using contraction and fix-point theorems.

The so-called numerical-analytic methods which have been developed in the last some years [9] give the opportunity of investigating the two most important approach for solving the boundary value problems — the existence and the approximate determination of the solution — simultaneously.

These methods are fairly widely used (see monograph [9]), mostly for handling periodic or two-point nonlinear boundary value problems.

When the boundary value problems are of more general nature (three- or n-point b.v. problems) and, in addition, even degenerate matrices are contained the evaluation and the mathematical foundation of numerical analytic methods based on successive approximations are facing several difficulties. In this connection we mention the determination of the successive approximation satisfying the boundary conditions and the proof of the uniform convergence, the determination of the necessary and sufficient conditions ensuring the existence based upon the features of the approximate solutions.

In this paper both the existence of the solution and the approximate solution of a three-point boundary value problem belonging to a system of nonlinear differential equations are investigated by using a numerical-analytic method. It is worth mentioning that the earlier versions of the numerical-analytic methods are not suitable for solving our problem due to the singularity of the matrices in the boundary value conditions.

Let a nonlinear differential equation be given

$$\dot{x} = f(t, x), \qquad x, f \in \mathbb{R}^n, \qquad t \in [0, T], \tag{1}$$

with a three-point linear boundary value condition

$$Ax(0) + A_1 x(t_1) + Cx(T) = d \tag{2}$$

where x, f, d are points of the n-dimensional Euclidean space $\mathbb{R}^n$ while

A, A_1, C are constant matrices of type $n \times n$ and $t_1 \in (0, T)$. Matrices A, A_1, C are allowed to be singular, but it is supposed that there exist constants k_1 and k_2 ($k_1 \neq k_2$) satisfying

$$\det\Big[k_1 A + k_2 A_1 + \Big[k_1 + \frac{T}{t_1}(k_2 - k_1)\Big] C\Big] \neq 0. \tag{3}$$

It will be shown that a $\{x_m(t, x_0)\}$ sequence of functions depending on the parameter x_0 can be constructed on the set of continuous functions satisfying the boundary value conditions (2) such that for certain value of the parameter x_0 the sequence of functions uniformly converges and its limit is the solution of the nonlinear boundary value problem (1), (2). The existence of the solution for the underlying problem is proved by using the properties of the approximate solution. An estimation for the error of the approximate solution is given.

2. Construction of successive approximations

Let $\mathcal{D}$ denote a closed, connected domain in $\mathbb{R}^n$. Let us suppose that the domain of definition of the right hand side function $f(t, x)$ in Eq. (1) fulfills

$$(t, x) \in [0, T] \times \mathcal{D} \tag{4}$$

and the following conditions hold

(i) $f(t, x)$ is continuous in its domain of definition (4);

(ii) $f(t, x)$ is bounded by the vector M

$$|f(t, x)| \leq M, \quad (t, x) \in [0, T] \times \mathcal{D},$$

and $f(t, x)$ satisfies the Lipschitz-condition in the variable x with matrix K:

$$|f(t, x') - f(t, x'')| \leq K|x' - x''|, \tag{5}$$

where the vector $|f(t, x)|$ is

$$|f(t, x)| = \big(|f_1(t, x)|, \dots, |f_n(t, x)|\big),$$

and both the vector $M = (M_1, M_2, \dots, M_n)$ and the matrix $K = \{K_{ij},\ i, j = 1, \dots, n\}$ contain only non-negative constant elements.

In relations $|f(t, x)| \leq M$ and (5) the inequalities are meant componentwise. Those boundary value problems (1), (2) will be investigated, for which the parameters M, K, A, A_1, C, d, k_1, k_2 and the domain of definition (4) satisfy condition (3) and the following conditions:

1. The set $\mathcal{D}_\beta$, the collection of those points $x_0 \in \mathrm{R}^n$ belonging — together with their β-neighbourhood — to the set $\mathcal{D}$, is non-empty

$$\mathcal{D}_\beta \neq \emptyset, \tag{6}$$

where $\beta(x_0) = \frac{T}{2}M + \beta_1(x_0)$,

$$\beta_1(x_0) = \left[|k_1| + \left|\frac{(k_2-k_1)T}{t_1}\right|\right]\left[|H(d-(A+A_1+C)x_0)| + \frac{T}{2}|HA_1|M\right],$$

where $H = D^{-1}$, $D = k_1A + k_2A_1 + \left[k_1 + \frac{T}{t_1}(k_2-k_1)\right]C$. (The β-neighbourhood of the point x_0 is the following $\{x\colon x \in \mathrm{R}^n, |x - x_0| \leq \leq \beta\}$.)

2. The highest eigenvalue $\lambda(Q)$ for the matrix $Q = \frac{T}{2}(K+G)$ is less than unity

$$\lambda(Q) < 1, \tag{7}$$

where

$$G = \left[|k_1| + \left|\frac{(k_2-k_1)T}{t_1}\right|\right]|HA_1|K.$$

A sequence of functions $\{x_m(t, x_0)\}$ whose elements satisfy the boundary value conditions (2) in every point $x_0 \in \mathcal{D}$ is constructed.

Let us consider the functions determined by the following formula:

$$\begin{aligned} x_m(t,x_0) = x_0 + &\int_0^t \left[f(t, x_{m-1}(t,x_0)) - \frac{1}{T}\int_0^T f(s, x_{m-1}(s,x_0))ds\right]dt + \\ &+\alpha\left[k_1T + \frac{T}{t_1}(k_2-k_1)t\right],\ m = 1,2,\dots\ ;\ x_0(t,x_0) = x_0, \end{aligned} \tag{8}$$

where $x_0 = (x_{01}, ..., x_{0n})$ is a parameter of dimension n, $\alpha = (\alpha_1, ..., \alpha_n)$ is an unknown vector chosen such that the functions (8) satisfy the boundary value conditions (2) for every point $x_0 \in \mathcal{D}_\beta$. Substituting the functions (8) into the boundary value conditions (2) the following system of linear algebraic equations is obtained

$$D\alpha = \frac{1}{T}d(x_0, x_{m-1}), \tag{9}$$

where

$$\begin{aligned} d(x_0, x_{m-1}) &= d - (A + A_1 + C)x_0 - \\ -A_1\int_0^{t_1}&\left[f\big(t, x_{m-1}(t,x_0)\big) - \frac{1}{T}\int_0^T f\big(s, x_{m-1}(s,x_0)\big)ds\right]dt. \end{aligned} \tag{10}$$

From (9) we get $\alpha = \frac{1}{T}Hd(x_0, x_{m-1})$, and from (8) we obtain the sequence of functions we wanted to get

$$x_m(t,x_0) = z_0(x_0, x_{m-1}) +$$

$$+\int_0^t \left[f(t, x_{m-1}(t,x_0)) - \frac{1}{T}\int_0^T f\big(s, x_{m-1}(s,x_0)\big)ds \right] dt + \tag{11}$$

$$+\frac{t}{t_1}(k_2 - k_1)Hd(x_0, x_{m-1}),$$

$$x_0(t,x_0) = x_0, \quad m = 1, 2, \ldots$$

where $z_0(x_0, x_{m-1}) = x_0 + k_1 Hd(x_0, x_{m-1})$, and $d(x_0, x_{m-1})$ can be expressed from (10).

The convergence of the above constructed functions is stated in

Theorem 1. *Let the function* $f(t,x)$ *in* Eq. (1) *be continuous in the domain* (4) *and satisfy the conditions* (5). *Furthermore, if the parameters of the boundary value problem* (1), (2) *satisfy the conditions* (6), (7) *then*

(i) *the functions of sequence* (11) *satisfy the boundary value conditions* (2) *for each* $x_0 \in \mathcal{D}_\beta$;

(ii) $\lim_{m\to\infty} x_m(t,x_0) = x^*(t,x_0)$, *where the limit function is a solution for the integral equation*

$$x(t) = z_0(x_0, x) +$$

$$+\int_0^t \left[f\big(t, x(t)\big) - \frac{1}{T}\int_0^T f\big(s, x(s)\big)ds + \frac{1}{t_1}(k_2 - k_1)Hd(x_0, x) \right] dt, \tag{12}$$

where

$$z_0(x_0, x) = x_0 + k_1 H \Bigg[d - (A + A_1 + C)x_0 -$$

$$-A_1 \int_0^{t_1} \left[f(t, x(t)) - \frac{1}{T}\int_0^T f(s, x(s))ds \right] dt \Bigg];$$

(iii) $x^*(0,x_0) = z_0\big(x_0, x^*(t,x_0)\big)$ *and the limit function* $x^*(t,x_0)$ *satisfies the boundary value conditions* (2) *i.e.* x^* *is a solution for the perturbed boundary value problem*

$$\begin{aligned} \dot{x} &= f(t,x) + \Delta(x_0) \\ Ax(0) + A_1 x(t_1) + Cx(T) &= d, \end{aligned} \tag{13}$$

where

$$\Delta(x_0) = \frac{1}{t_1}(k_2 - k_1)Hd(x_0, x^*(t, x_0)) - \frac{1}{T}\int_0^T f(t, x^*(t, x_0))dt,$$

$$d(x_0, x^*(t, x_0)) = d - (A + A_1 + C)x_0 -$$
$$-A_1 \int_0^{t_1} \left[f(t, x^*(t, x_0)) - \frac{1}{T}\int_0^T f(s, x^*(s, x_0))ds \right] dt;$$

(iv) *the deviation of functions $x^*(t, x_0)$ and $x_m(t, x_0)$ is governed by the inequality*

$$|x^*(t, x_0) - x_m(t, x_0)| \leq Q^m (E - Q)^{-1} \beta(x_0). \tag{14}$$

Proof. It will be shown that in the space $C(0, T)$ of continuous vector functions the sequence of functions given by (11) is a Cauchy-sequence and therefore it is uniformly convergent. First we prove that $x_0 \in \mathcal{D}_\beta$ implies $x_m(t, x_0) \in \mathcal{D}$ for each element of the sequence.

From (11) we get

$$|x_1(t, x_0) - x_0| \leq \left| \int_0^t \left[f(t, x_0(t, x_0)) - \frac{1}{T}\int_0^T f(s, x_0(s, x_0))ds \right] dt \right| +$$
$$+ \left| \frac{T}{t_1}(k_2 - k_1) \right| |Hd(x_0, x_0)| + |k_1||Hd(x_0, x_0)|.$$

Using Lemma 2.1 in [9, p. 31] it is obvious that if $f(t) \in C[0, T]$, then

$$\left| \int_0^t \left[f(t) - \frac{1}{T}\int_0^T f(s)ds \right] dt \right| \leq \left(1 - \frac{t}{T}\right)\int_0^t |f(s)|ds + \frac{t}{T}\int_t^T |f(s)|ds \leq$$
$$\leq \alpha_1(t) \max_{t \in [0,T]} |f(t)|,$$

where $\alpha_1(t) = 2t\left(1 - \frac{t}{T}\right)$, $|\alpha_1(t)| \leq \frac{T}{2}$. Thus

$$|x_1(t, x_0) - x_0| \leq \alpha_1(t)M + \left[\left| \frac{T}{t_1}(k_2 - k_1) \right| + |k_1| \right] |Hd(x_0, x_0)|.$$

Furthermore, from (10)

$$|Hd(x_0, x_0)| \leq$$

$$\leq |H(d - (A + A_1 + C)x_0)| + \left| HA_1 \int_0^{t_1} \left[f(t, x_0) - \frac{1}{T}\int_0^T f(s, x_0)ds \right] dt \right| \leq$$

$$\leq |H(d - (A + A_1 + C)x_0)| + |HA_1| \left| \int_0^{t_1} \left[f(t, x_0) - \frac{1}{T}\int_0^T f(s, x_0)ds \right] dt \right|,$$

$$|Hd(x_0,x_0)| \le \Big[\big|H\big(d-(A+A_1+C)x_0\big)\big| + \frac{T}{2}|HA_1|M\Big]$$

therefore

$$(15) \qquad \big|x_1(t,x_0)-x_0\big| \le \frac{T}{2}M+\beta_1(x_0),$$

and $x_1(t,x_0)\in\mathcal{D}$ when $x_0\in\mathcal{D}_\beta$.

In a similar way, using induction we obtain

$$|x_m(t,x_0)-x_0| \le \frac{T}{2}M+\beta_1(x_0),$$

that is $x_m(t,x_0)\in\mathcal{D}$, when $x_0\in\mathcal{D}_\beta$.

We prove that $\{x_m(t,x_0)\}$ is really a Cauchy-sequence. Let us consider the following difference:

$$x_2(t,x_0)-x_1(t,x_0) = \int_0^t\Big[f\big(t,x_1(t,x_0)\big)-\frac{1}{T}\int_0^T f\big(s,x_1(s,x_0)\big)ds\Big]dt-$$

$$-\int_0^t\Big[f\big(t,x_0(t,x_0)\big)-\frac{1}{T}\int_0^T f\big(s,x_0(s,x_0)\big)ds\Big]dt+$$

$$+\frac{t}{t_1}(k_2-k_1)H\Big[-A_1\int_0^{t_1}\Big[f\big(t,x_1(t,x_0)\big)-\frac{1}{T}\int_0^T f\big(s,x_1(s,x_0)\big)ds\Big]dt+$$

$$+A_1\int_0^{t_1}\Big[f\big(t,x_0(t,x_0)\big)-\frac{1}{T}\int_0^T f\big(s,x_0(s,x_0)\big)ds\Big]dt\Big]+$$

$$+k_1H\big[d(x_0,x_1)-d(x_0,x_0)\big].$$

Rearranging and using Lemma 2.1 of [9, p. 31]

$$|x_2(t,x_0)-x_1(t,x_0)| \le \Big(1-\frac{t}{T}\Big)\int_0^t\big|f\big(t,x_1(t,x_0)\big)-f\big(t,x_0(t,x_0)\big)\big|dt+$$

$$+\frac{t}{T}\int_t^T\big|f\big(t,x_1(t,x_0)\big)-f\big(t,x_0(t,x_0)\big)\big|dt+\Big[|k_1|+\Big|\frac{(k_2-k_1)T}{t_1}\Big|\Big]|HA_1|\cdot$$

$$\cdot\Big[\Big(1-\frac{t_1}{T}\Big)\int_0^{t_1}\big|f\big(t,x_1(t,x_0)\big)-f\big(t,x_0(t,x_0)\big)\big|dt+$$

$$+\frac{t_1}{T}\int_{t_1}^T\big|f\big(t,x_1(t,x_0)\big)-f\big(t,x_0(t,x_0)\big)\big|dt\Big] \le \Big[|k_1|+\Big|\frac{(k_2-k_1)T}{t_1}\Big|\Big]|HA_1|K\cdot$$

$$\cdot\left[\left(1-\frac{t_1}{T}\right)\int_0^{t_1}(\alpha_1(t)M+\beta_1(x_0))dt+\frac{t_1}{T}\int_{t_1}^{T}(\alpha_1(t)M+\beta_1(x_0))dt\right]+$$

$$+K\left[\left(1-\frac{t}{T}\right)\int_0^{t}(\alpha_1(t)M+\beta_1(x_0))dt+\frac{t}{T}\int_{t}^{T}(\alpha_1(t)M+\beta_1(x_0))dt\right].$$

Applying Lemma 2.2 of [9, p. 31] we get

$$|x_2(t,x_0)-x_1(t,x_0)|\le K\left[\alpha_2(t)M+\alpha_1(t)\beta_1(x_0)\right]+$$

$$+G\left[\alpha_2(t_1)M+\alpha_1(t_1)\beta_1(x_0)\right],$$

where

$$\alpha_2(t)\le\frac{T}{3}\alpha_1(t)\quad\text{and}\quad\alpha_1(t)\le\frac{T}{2},$$

consequently

$$|x_2(t,x_0)-x_1(t,x_0)|\le K\left[\frac{T}{3}M+\beta_1(x_0)\right]\alpha_1(t)+G\left[\frac{T}{3}M+\beta_1(x_0)\right]\alpha_1(t_1),$$

thus

$$|x_2(t,x_0)-x_1(t,x_0)|\le Q\beta(x_0), \tag{16}$$

with

$$Q=\frac{T}{2}[K+G],\quad \beta(x_0)=\frac{T}{3}M+\beta_1(x_0).$$

Equality (11) immediately gives

$$x_{m+1}(t,x_0)-x_m(t,x_0)=-\left[k_1+\frac{t}{t_1}(k_2-k_1)\right]HA_1\int_0^{t_1}\Big[\left[f\left(t,x_m(t,x_0)\right)-\right.$$

$$\left.-f\left(t,x_{m-1}(t,x_0)\right)\right]-\frac{1}{T}\int_0^T\left[f\left(s,x_m(s,x_0)\right)-f\left(s,x_{m-1}(s,x_0)\right)\right]ds\Big]dt+$$

$$+\int_0^t\Big[\left[f\left(t,x_m(t,x_0)\right)-f\left(t,x_{m-1}(t,x_0)\right)\right]-\frac{1}{T}\int_0^T\left[f\left(s,x_m(s,x_0)\right)-\right.$$

$$\left.-f\left(s,x_{m-1}(s,x_0)\right)\right]ds\Big]dt$$

and

$$\left|x_{m+1}(t,x_0)-x_m(t,x_0)\right|\le G\left[\left(1-\frac{t_1}{T}\right)\int_0^{t_1}\left|x_m(t,x_0)-x_{m-1}(t,x_0)\right|dt+\right.$$

$$(17)\quad \begin{aligned} &+\frac{t_1}{T}\int_{t_1}^{T}\left|x_m(t,x_0)-x_{m-1}(t,x_0)\right|dt\Bigg]+\\ &+K\left[\left(1-\frac{t}{T}\right)\int_0^t\left|x_m(t,x_0)-x_{m-1}(t,x_0)\right|dt+\right.\\ &\left.+\frac{t}{T}\int_t^T\left|x_m(t,x_0)-x_{m-1}(t,x_0)\right|dt\right].\end{aligned}$$

Using induction and (15) and (16) it can be shown that

$$(18)\qquad \left|x_{m+1}(t,x_0)-x_m(t,x_0)\right|\le Q^m\beta(x_0),$$

supposing the validity of inequality

$$(19)\qquad \left|x_m(t,x_0)-x_{m-1}(t,x_0)\right|\le Q^{m-1}\beta(x_0).$$

In fact, using inequalities (17) and (19) we get

$$\left|x_{m+1}(t,x_0)-x_m(t,x_0)\right|\le G\left[\left(1-\frac{t_1}{T}\right)\int_0^{t_1}Q^{m-1}\beta(x_0)dt+\right.$$

$$\left.+\frac{t_1}{T}\int_{t_1}^{T}Q^{m-1}\beta(x_0)dt\right]+K\left[\left(1-\frac{t}{T}\right)\int_0^{t}Q^{m-1}\beta(x_0)dt+\right.$$

$$\left.+\frac{t}{T}\int_t^T Q^{m-1}\beta(x_0)dt\right]=GQ^{m-1}\beta(x_0)\left[\left(1-\frac{t_1}{T}\right)\int_0^{t_1}dt+\frac{t_1}{T}\int_{t_1}^{T}dt\right]+$$

$$+KQ^{m-1}\beta(x_0)\left[\left(1-\frac{t}{T}\right)\int_0^{t}dt+\frac{t}{T}\int_t^T dt\right]=$$

$$=Q^{m-1}\beta(x_0)\left[G\alpha_1(t_1)+K\alpha_1(t)\right]\le Q^{m-1}\beta(x_0)\left[\frac{T}{2}(G+K)\right]=Q^m\beta(x_0).$$

Introducing the notation

$$r_{m+1}(t)=\left|x_{m+1}(t,x_0)-x_m(t,x_0)\right|$$

and using (18),

$$(20)\qquad \left|x_{m+j}(t,x_0)-x_m(t,x_0)\right|\le\sum_{i=1}^{j}r_{m+i}(t)\le Q^m\sum_{i=0}^{j-1}Q^i\beta(x_0).$$

From (7) we get

$$\sum_{i=0}^{j-1}Q^i\le\sum_{i=0}^{\infty}Q^i=(E-Q)^{-1},\qquad \lim_{m\to\infty}Q^m\to 0,$$

where E is the unit matrix. Hence from (20) one can easily get that

$\{x_m(t,x_0)\}$ is a Cauchy-sequence, therefore it uniformly converges to a continuous limit function $x^*(t,x_0)$:

$$\lim_{m\to\infty} x_m(t,x_0) = x^*(t,x_0).$$

It is evident that taking the limit $(m \to \infty)$ in (11) the limit function $x^*(t,x_0)$ is a solution of the integral equation (12). Since all the elements of sequences (11) satisfy the boundary-value conditions (2), therefore so does the limit function too.

From (12) it is easily seen that $x^*(0,x_0) = z_0(x_0, x^*(t,x_0))$ and $x^*(t,x_0)$ is a solution of the perturbed boundary-value problem (13), which is equivalent to the integral equation (12).

It is easy to see that taking the $(j \to \infty)$ limit in (20) the inequality (14) holds for the deviation of the limit function from its m^{th} iteration. $\lozenge$

3. Some properties of the limit function

It is demonstrated how the right-hand side of the system of differential equations can be modified in such a way that the solution of the Cauchy-problem belonging to the newly constructed equation satisfies the given boundary value condition.

Theorem 2. *If the conditions of* Th. 1 *are satisfied then in an arbitrary point* $x_0 \in \mathcal{D}_\beta$ *a unique regulating parameter* $\mu = (\mu_1, \dots, \mu_n)$ *of the form*

$$\mu = \frac{1}{t_1}(k_2 - k_1)Hd(x_0, x^*(t,x_0)) - \frac{1}{T}\int_0^T f(t, x^*(t,x_0))dt, \tag{21}$$

can be constructed, where $x^*(t,x_0)$ *is the limit function of the sequence of functions* $\{x_m(t,x_0)\}$ *given by* (11). *Under these conditions the solution* $x = x(t) = x^*(t,x_0)$ *of the Cauchy-problem*

$$\dot{x} = f(t,x) + \mu, \qquad x(0) = z_0(x_0) \tag{22}$$

$$z_0(x_0) = z_0\big(x_0, x^*(t,x_0)\big) = x_0 + k_1 H\Bigg[d - (A + A_1 + C)x_0 -$$

$$-A_1\int_0^{t_1}\left[f\big(t,x^*(t,x_0)\big) - \frac{1}{T}\int_0^T f\big(s,x^*(s,x_0)\big)ds\right]dt\Bigg]$$

satisfies the boundary value conditions (2) *i.e. it is a solution of the perturbed boundary value problem* (13) *with* $\Delta(x_0) = \mu$.

Proof. Th. 1 implies that the function $x(t) = x^*(t, x_0)$ is a solution for both the integral equation (12) and the Cauchy-problem

$$\begin{aligned}
&\dot{x} = f(t,x) + \frac{1}{t_1}(k_2 - k_1)Hd\big(x, x^*(t,x_0)\big) - \\
&\quad -\frac{1}{T}\int_0^T f\big(t, x^*(t,x_0)\big)dt, \\
(23)\quad & x(0) = z_0(x_0) = z_0\big(x_0, x^*(t,x_0)\big) = x_0 + k_1 H\Big[d - (A + A_1 + C)x_0 - \\
&\quad -A_1 \int_0^{t_1}\Big[f\big(t, x^*(t,x_0)\big) - \frac{1}{T}\int_0^T f\big(s, x^*(s,x_0)\big)ds\Big]dt\Big]
\end{aligned}$$

and, in addition, $x^*(t, x_0)$ satisfies the boundary value conditions (2). This means that we have found the parameter μ of the form (21) for which the function $x(t) = x^*(t, x_0)$ is a solution of the initial value problem (23). It can be shown that this parameter value is unique, since for any other value of the parameter μ (not of the form (21)) x^* is a solution of the Cauchy-problem (22) but does not satisfy the boundary conditions (2).

Let us suppose that the statement above is not true. Then there exist two such values μ', μ'', $\mu' \neq \mu''$ that the solutions of the Cauchy-problem (22) $x(t, x_0, \mu')$ and $x(t, x_0, \mu'')$ with $\mu = \mu'$ and $\mu = \mu''$ satisfy even the boundary value conditions (2). Then using (12) the following identity for the difference of these solutions is obtained

$$\begin{aligned}
x(t,x_0,\mu'') - x(t,x_0,\mu') = &\int_0^t \Big[\big[f\big(t, x(t,x_0,\mu'')\big) - f\big(t, x(t,x_0,\mu')\big)\big] - \\
&-\frac{1}{T}\int_0^T \big[f\big(s, x(s,x_0,\mu'')\big) - f\big(s, x(s,x_0,\mu')\big)\big]ds\Big]dt + \\
&+\frac{T}{t_1}(k_2 - k_1)Hd(x_0, x(t,\mu'')) - \frac{T}{t_1}(k_2 - k_1)Hd(x_0, x(t,\mu')) - \\
&-k_1 H\Big[A_1 \int_0^{t_1}\Big[\big(f(t, x(t,x_0,\mu'')) - f(t, x(t,x_0,\mu'))\big) - \\
&-\frac{1}{T}\int_0^T \big(f(s, x(s,x_0,\mu'')) - f(s, x(s,x_0,\mu'))\big)ds\Big]dt\Big].
\end{aligned}$$

Supposing $|x(t,x_0,\mu'') - x(t,x_0,\mu')| = r(t)$ and using Lemma 2.1 of [9,

p. 31],

$$r(t) \le K\left[\left(1-\frac{t}{T}\right)\int_0^t r(s)ds + \frac{t}{T}\int_t^T r(s)ds\right] + \left|\frac{T}{t_1}(k_2-k_1)H\right||A_1|\cdot$$

$$\cdot\left|\int_0^{t_1}\left[(f(t,x(t,x_0,\mu''))-f(t,x(t,x_0,\mu'))) - \frac{1}{T}\int_0^T (f(s,x(s,x_0,\mu''))-\right.\right.$$

$$\left.\left.-f(s,x(s,x_0,\mu')))ds\right]dt\right| + |k_1H||A_1|\left|\int_0^{t_1}\left[(f(t,x(t,x_0,\mu''))-\right.\right.$$

$$\left.\left.-f(t,x(t,x_0,\mu'))) - \frac{1}{T}\int_0^T (f(s,x(s,x_0,\mu''))-f(s,x(s,x_0,\mu')))ds\right]dt\right| \le$$

$$\le \left[|k_1| + \left|\frac{(k_2-k_1)T}{t_1}\right|\right]|HA_1|K\left[\left(1-\frac{t_1}{T}\right)\int_0^{t_1} r(t)dt + \frac{t_1}{T}\int_{t_1}^T r(t)dt\right] +$$

$$+K\left[\left(1-\frac{t}{T}\right)\int_0^t r(t)dt + \frac{t}{T}\int_t^T r(t)dt\right],$$

$$r(t) \le G\left[\left(1-\frac{t_1}{T}\right)\int_0^{t_1} r(t)dt + \frac{t_1}{T}\int_{t_1}^T r(t)dt\right] +$$

$$+K\left[\left(1-\frac{t}{T}\right)\int_0^t r(t)dt + \frac{t}{T}\int_t^T r(t)dt\right].$$

Let $|r(t)|_0 = \left(\sup_t |r_1(t)|, \dots, \sup_t |r_n(t)|\right)$. We have

$$r(t) \le [G\alpha_1(t_1) + K\alpha_1(t)]\,|r(t)|_0 \le Q|r(t)|_0,$$

$$r(t) \le G\left[\left(1-\frac{t_1}{T}\right)\int_0^{t_1} Q|r(t)|_0 dt + \frac{t_1}{T}\int_{t_1}^T Q|r(t)|_0 dt\right] +$$

$$+K\left[\left(1-\frac{t}{T}\right)\int_0^t Q|r(t)|_0 dt + \frac{t}{T}\int_t^T Q|r(t)|_0 dt\right] \le$$

$$\le Q\left[G\alpha_1(t_1) + K\alpha_1(t)\right]|r(t)|_0 \le Q^2|r(t)|_0, \dots$$

$$r(t) \le Q^m|r(t)|_0, \quad \text{i.e.} \quad |r(t)|_0 \le Q^m|r(t)|_0.$$

Since all the eigenvalues of the matrix Q are within the circle of unit radius, therefore the last inequality holds only if $|r(t)|_0 = 0$, i.e.

$\mu' = \mu''$. This is a contradiction, thus there exists only one parameter value μ. $\Diamond$

The following statement gives a necessary and sufficient condition for the existence of the solution of the boundary value problem (1), (2).

Theorem 3. *Let us consider the initial value problem*

$$(24)\qquad \begin{cases} \dot{x} = f(t,x) \\ x(0) = x_0^* + k_1 H\Big[d - (A + A_1 + C)x_0^* - \\ -A_1 \int_0^{t_1}\Big[f\big(t, x^*(t,x_0^*)\big) - \frac{1}{T}\int_0^T f\big(s, x^*(s,x_0^*)\big)ds\Big]dt\Big] \end{cases}$$

connected to the given differential equation. If the conditions of Th. 1 *are fulfilled, then a solution of* (24) $x = x^*(t)$ *is a solution of the original boundary value problem* (1), (2) *if and only if the determining function* $\Delta(x_0)$ *assumes the value zero at point* x_0^**:*

$$(25)\quad \Delta(x_0^*) = \frac{1}{t_1}(k_2 - k_1)H\, d(x_0^*, x^*(t,x_0^*)) - \frac{1}{T}\int_0^T f(t, x^*(t,x_0^*))dt = 0,$$

where $x^*(t,x_0^*)$ *is the limit function of the sequence of function* $x_m(t,x_0^*)$. *In this case* $x^*(t) = x^*(t,x_0^*)$ *and the deviation of* $x^*(t)$ *from its* $x_m(t,x_0^*)$ m^{th} *approximation is determined by inequality* (14).

Proof. Since the function $x^*(t,x_0)$ is a solution of the Cauchy-initial value problem (23) and satisfies the boundary-value conditions (2), therefore if inequality (25) holds, then the problems (24) and (23) are equivalent at value $x_0 = x_0^*$. In such a way we proved that (25) is a sufficient condition.

The necessity of the condition (25) is a consequence of the fact that if $x = x^*(t)$ is a solution of the boundary value problem (1), (2) with the initial value

$$x^*(0) = x_0^* + k_1 H\left[d - (A + A_1 + C)x_0^* - A_1 \int_0^{t_1}\left[f(t, x^*(t,x_0^*)) - \right.\right.$$

$$\left.\left. -\frac{1}{T}\int_0^T f(s, x^*(s,x_0^*))ds\right]dt\right],$$

then the solution $x = x(t, x_0^*, \mu)$ of the Cauchy-initial value problem satisfies the initial value conditions (2) exactly at $\mu = \Delta(x_0^*) = 0$. Then equality $x(t,x_0^*,\mu) = x^*(t)$ holds and according to Th. 2 the parameter $\mu = \Delta(x_0^*) = 0$ is unique. Thus $x^*(t) = x^*(t,x_0^*)$ and the following inequality holds

$$|x^*(t) - x_m(t, x_0^*)| \leq Q^m(E - Q)^{-1}\beta(x_0^*). \quad \diamond$$

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ON A PROBLEM OF R. SCHILLING I.

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Abstract: Studies of a physical problem (cf. [4]) led to the functional equation

(1) $$f(qx) = \frac{1}{4q}(f(x+1) + f(x-1) + 2f(x)) \quad \text{for all } x \in \mathbb{R}$$

with the boundary condition

(2) $$f(x) = 0 \quad \text{for all } x \text{ with } |x| > Q := \frac{q}{1-q}$$

where $q \in]0,1[$ is a fixed real number. In this paper the general solution of (1) with unbounded support is given. It can be shown that in the case of unbounded support a function on a special interval can be chosen arbitrarily and then uniquely extended to a solution of (1). Furthermore, investigations are done on the continuity, differentiability, measurability and integrability of such solutions.

Studies of a physical problem (cf. [4]) led Prof. R. Schilling to the functional equation given below. It was known that in the case $q = \frac{1}{2}$ there is a continuous solution with bounded support. Now the question arose to find all the solutions of this equation. At the moment the problem is far from being solved completely, but in the sequel there will be given some partial answers:

Let the functional equation

(1) $$f(qx) = \frac{1}{4q}\big(f(x+1) + f(x-1) + 2f(x)\big) \quad \text{for all } x \in \mathbb{R} .$$

and the boundary condition

$$(2) \qquad f(x) = 0 \quad \text{for all } x \text{ with } \quad |x| > Q := \frac{q}{1-q}$$

be given, where $q \in]0,1[$ is a fixed real number. First of all we conduct some investigations on the boundary condition (2). As in our considerations the set $\{x \mid f(x) \neq 0\}$ plays a more important role than the support $\operatorname{supp}(f)$, which denotes the closure of this set, we abbreviate $S(f) := \{x \mid f(x) \neq 0\}$.

I. The boundary condition

In this chapter we show that the boundary condition (2) is natural in some sense. This is done in the subsequent theorem. Therefore, let $q \in]0,1[$ be a fixed real number, $Q = \frac{q}{1-q}$. First we give a short lemma and start with a remark:

Remark 1. $q(Q+1) = Q$, which can easily be verified by direct computation.

Lemma 1. *Let f be a solution of (1) whose support is contained in the interval $]-\infty, b]$ for some $b \in \mathbb{R}$. Then the following holds:*

(i) *If $b \geq Q$, then $\operatorname{supp}(f) \subseteq]-\infty, Q]$; moreover, if $q \neq \frac{1}{4}$, then $S(f) \subseteq]-\infty, Q[$.*

(ii) *If $b < Q$, then f is identically 0.*

Proof. Let $\operatorname{supp}(f) \subseteq]-\infty, b]$. As the case $b = Q$ is evident, we only have to deal with the other two possibilities:

(i): Suppose that $b > Q$. Then $b(1-q) > q$ and therefore $b > q(b+1) > > q(Q+1) = Q$. Now let $x > b+1$. Then $x+1 > x > x-1 > b$, and thus we have $f(x+1) = f(x) = f(x-1) = 0$, which implies that $f(qx) = 0$ by equation (1). Thus in this case we have $\operatorname{supp}(f) \subseteq]-\infty, q(b+1)]$. Define a sequence (b_n) by $b_0 := b$, $b_{n+1} := q(b_n+1)$. As shown above, for $b_0 > Q$ this sequence is strictly decreasing and has the lower bound Q. Furthermore, by induction one can immediately see that $\operatorname{supp}(f) \subseteq]-\infty, b_n]$ for any $n \in \mathbb{N}$. Therefore the sequence (b_n) is convergent, the limit B fulfills $B \geq Q$ and $B = q(B+1)$, which implies that $B = Q$, and we have $\operatorname{supp}(f) \subseteq]-\infty, Q]$.

In the case $\operatorname{supp}(f) \subseteq]-\infty, Q]$ we have

$$f(Q) = f(q(Q+1)) = \frac{1}{4q}(f(Q) + f(Q+2) + 2f(Q+1)) = \frac{1}{4q}f(Q).$$

Therefore, if $q \neq \frac{1}{4}$, we have $f(Q) = 0$ and $S(f) \subseteq]-\infty, Q[$.

(ii): Suppose that $b < Q$. Then $b(1-q) < q$, and therefore $\frac{b}{q} - 1 < b$.

We first assume $b > 0$. In this case for $x > \frac{b}{q} > b$ we have $f(x+1) = f(x) = f(qx) = 0$, and therefore by equation (1) we get $\operatorname{supp}(f) \subseteq]-\infty, \frac{b}{q} - 1]$. Define the sequence (b_n) by $b_0 := b$, $b_{n+1} := = b_n/q - 1$. This sequence is decreasing, and by induction we get $\operatorname{supp}(f) \subseteq]-\infty, b_n]$ for each n where $b_{n-1} > 0$. On the other hand, this sequence (b_n) cannot have a lower bound, because this bound would be the limit, fulfilling $\frac{B}{q} - 1 = B$, i.e. $B = Q$, a contradiction. Thus there is an n such that $b_n \leq 0$, and $\operatorname{supp}(f) \subseteq]-\infty, 0]$.

Now we assume $b \leq 0$: For $x > b$ we have $x + 1 > x > b$ and $qx > b$. Therefore $f(x) = f(qx) = f(x+1) = 0$, which implies that $f(x-1) = 0$. Thus $\operatorname{supp}(f) \subseteq]-\infty, b-1]$, and by induction we get $\operatorname{supp}(f) = \emptyset$. $\Diamond$

A similar lemma can be proved in the same way for supports bounded from below:

Lemma 2. *Let f be a solution of* (1) *whose support is contained in the interval $[a, \infty[$ for some $a \in \mathbb{R}$. Then the following holds:*

(i) *If $a \leq -Q$, then* $\operatorname{supp}(f) \subseteq [-Q, \infty[$*; moreover, if $q \neq \frac{1}{4}$, then $S(f) \subseteq]-Q, \infty[$.*

(ii) *If $a > -Q$, then f is identically* 0.

Combining these two lemmata, we get the following

Theorem 1. *Let f be a nonvanishing solution of* (1)*, then $S(f)$ is contained in exactly one of the following intervals, and it is not contained in any proper subintervals:*

(a) *for $q \neq \frac{1}{4}$: $]-Q, Q[$ or $]-\infty, Q[$ or $]-Q, \infty[$ or $\mathbb{R}$;*

(b) *for $q = \frac{1}{4}$: $]-Q, Q[$ or $]-\infty, Q[$ or $]-Q, \infty[$ or $\mathbb{R}$ or $[-Q, Q]$ or $]-Q, Q]$ or $[-Q, Q[$ or $]-\infty, Q]$ or $[-Q, \infty[$.*

Proof. In Lemma 1 it was shown that a nonempty support bounded from above has Q as its least upper bound, Lemma 2 gave the answer for bounds from below. The restriction to open intervals for $S(f)$, except for the case $q = \frac{1}{4}$, was also shown in these two lemmata. Later on it will be shown that all these cases really can occur. $\Diamond$

II. Solutions with unbounded support

a) General results

In this chapter we give some general results on the solutions of equation (1) and also present general solutions with unbounded supports. First we start with a uniqueness theorem (cf. [2]):

Theorem 2. *Let $f, g \colon \mathbb{R} \to \mathbb{R}$ be solutions of* (1) *which coincide on the half-open interval* $[-1, 1[$. *Then they are identical.*

Proof. We give this proof by induction and show that f coincides with g on any interval $[-n, n[$, where n is a positive integer. For $n = 1$ this is true by assumption. Now suppose that f and g coincide on the interval $[-n, n[$, and let $x \in [-(n+1), n+1[\setminus [-n, n[$. Then either $n \le x < n+1$ or $-n-1 \le x < -n$. In the first case choose $y := x - 1 \in [-n, n[$. Then $y, y-1, qy \in [-n, n[$, and by (1) we have $g(x) = g(y+1) = 4qg(qy) - g(y-1) - 2g(y) = 4qf(qy) - f(y-1) - 2f(y) = f(y+1) = f(x)$. Similarly, choose $z := x + 1$ in the second case. $\Diamond$

Next we give a theorem how to get all the solutions in the case of unbounded support. By Th. 2 it is sufficient to give the restriction of the solution to the interval $[-1, 1[$.

Theorem 3 (cf. [2]). *Let $h \colon [-1, 1[\to \mathbb{R}$ be an arbitrary function. Then there exists exactly one solution of* (1) *such that the restriction of this solution to the interval* $[-1, 1[$ *coincides with h.* In other words: *Any function $h \colon [-1, 1[\to \mathbb{R}$ can be uniquely extended to a solution of* (1).

Proof. Let $h \colon [-1, 1[\to \mathbb{R}$ be given. We first extend h by induction to the intervals $[-1, n[$ for each natural number n and then to the intervals $[-n, \infty[$:

Let $f_1 := h \colon [-1, 1[\to \mathbb{R}$. Suppose that f_n is given on $[-1, n[$ for some nonnegative integer n. We define f_{n+1} on the interval $[-1, n+1[$ by

$$f_{n+1}(x) := \begin{cases} f_n(x) & \text{for } x \in [-1, n[\\ 4qf_n(q(x-1)) - f_n(x-2) - 2f_n(x-1) & \text{otherwise} \end{cases}$$

(it is easy to see that for $n \le x < n+1$ we have $q(x-1), x-2, x-1 \in [-1, n[$). As — by definition — any two functions f_m, f_n coincide on the intersection of their domains, this family of functions uniquely defines a function F_1 on the interval $[-1, \infty[$. We continue like before: Suppose that F_n is given on the interval $[-n, \infty[$. We define F_{n+1} on the interval $[-(n+1), \infty[$ by

$$F_{n+1}(x) := \begin{cases} F_n(x) & \text{for } x \in [-n, \infty[\\ 4qF_n(q(x+1)) - F_n(x+2) - 2F_n(x+1) & \text{otherwise.} \end{cases}$$

Like before, this family of functions uniquely defines a function f on the whole real line. We only have to show that f is a solution of (1):

Let $x \in \mathbb{R}$. (α) If $x < 0$, then there is an $n \in \mathbb{N}$ such that $-(n+1) \le x - 1 < -n$. By definition, f coincides with F_{n+1} on the interval $[-(n+1), \infty[$. Thus — by definition of F_{n+1} — we have $f(x-1) = F_{n+1}(x-1) = 4qF_n(qx) - F_n(x+1) - 2F_n(x) = 4qf(qx) - - f(x+1) - 2f(x)$, which is nothing else but equation (1).

(β) If $x \geq 0$, then there is an $n \in \mathbb{N}$ such that $n \leq x+1 < n + + 1$. Like before, we have $f(x+1) = f_{n+1}(x+1) = 4qf_n(qx) - f_n(x-1) - -2f_n(x) = 4qf(qx) - f(x-1) - 2f(x)$, and (1) is fulfilled, too. ◊

The next two theorems deal with solutions whose support is bounded from above. First we give a uniqueness theorem.

Theorem 4. *Let f, g be solutions of (1) whose supports are contained in the interval $]-\infty, Q]$. Then $f = g$ iff the restrictions of f and g to the interval $]Q-1, qQ]$ coincide and $f(Q) = g(Q)$. (The second condition is necessary only in the case $q = \frac{1}{4}$.)*

Proof. We define a sequence (x_n) by $x_0 := Q - 1$, $x_{n+1} := q(x_n + 1)$. As $x_0 < Q$, we have $x_n < x_{n+1} < Q$ for any $n \in \mathbb{N}$, and $\lim_{n\to\infty} x_n = Q$. Now suppose that f and g coincide on the interval $]x_n, x_{n+1}]$ for some nonnegative integer n, and let $x \in]x_{n+1}, x_{n+2}]$. By definition of the sequence (x_n) we have $x = q(y+1)$ for some $y \in]x_n, x_{n+1}]$, which also implies that $y + 2 > y + 1 > Q$. Using equation (1) for the value $y+1$, we get $g(x) = g(q(y+1)) = \frac{1}{4q}(g(y) + g(y+2) + 2g(y+1)) = \frac{1}{4q}g(y) = = \frac{1}{4q}f(y) = f(x)$. Thus by induction we get the result that f and g coincide on the interval $]Q-1, Q[$ and — by assumption — their values at the point Q are identical, thus they coincide on the interval $]Q-1, Q]$ and, therefore, on the interval $]Q-1, \infty[$.

Now let $y_0 := Q - 1$ and $y_{n+1} := \frac{1}{q}y_n - 1$. The sequence (y_n) is strictly decreasing and unbounded, thus there is a nonnegative integer k such that $y_k < 0$, $y_{k-1} \geq 0$. We will show that f and g coincide on the interval $]y_k, \infty[$. Let $0 \leq m < k$, and suppose that f and g coincide on $]y_m, \infty[$. For $x \in]y_{m+1}, y_m]$ we have $x + 2 > x + 1 > q(x+1) > y_m$ and can derive from equation (1) that $f(x) = g(x)$, i.e., f and g coincide on $]y_{m+1}, \infty[$. A usual induction argument shows that f and g coincide on $]y_k, \infty[$ and, therefore, on $[0, \infty[$.

We finish the proof by one more induction process: Suppose that f and g coincide on $[-n, \infty[$ for some nonnegative integer n. Then for $x \in [-(n+1), -n[$ we have $x + 2 > x + 1 \geq -n$, $q(x+1) \geq -n$, thus equation (1) gives $f(x) = g(x)$, and f and g coincide on $[-(n+1), \infty[$. Thus $f = g$. ◊

We can use the same ideas to give all the solutions of equation (1) under the assumption that the support is bounded from above:

Theorem 5. *Let $h\colon]Q-1, qQ] \to \mathbb{R}$ be an arbitrary function, and α a real number which is arbitrary in the case $q = \frac{1}{4}$ and 0 otherwise. Then there exists a unique solution f of (1) such that the restriction of f to the interval $]Q-1, qQ]$ is identical to h, $f(Q) = \alpha$ and* $\operatorname{supp}(f) \subseteq$ $\subseteq]-\infty, Q]$.

Proof. The uniqueness has been shown in the preceding theorem. For the existence, we will make an extension of h: As in Th. 4, let (x_n) be the sequence given by $x_0 := Q-1$, $x_{n+1} := q(x_n+1)$. Let $h_0 := h$, and h_n defined on the interval $]x_0, x_{n+1}]$ by induction:

$$h_{n+1}(x) := \begin{cases} h_n(x) & \text{for } x \in]x_0, x_{n+1}] \\ \frac{1}{4q} h_n(y) & \text{for } x = q(y+1) \in]x_{n+1}, x_{n+2}]. \end{cases}$$

As any two of the functions h_n, h_m coincide on the intersection of their domains, they uniquely define a function $h_\infty\colon]x_0, Q[\to \mathbb{R}$. Next we extend to the interval $]x_0, \infty[$ by the formula

$$g_0(x) := \begin{cases} h_\infty(x) & \text{for } x \in]x_0, Q[\\ \alpha & \text{for } x = Q \\ 0 & \text{for } x > Q. \end{cases}$$

Now we use the sequence (y_n) defined as in Th. 4 by $y_0 := Q-1$, $y_{n+1} := \frac{1}{q} y_n - 1$, which is strictly decreasing and unbounded, thus there is a nonnegative integer k such that $y_k < 0$, $y_{k-1} \geq 0$. Let m be an integer, $0 \leq m < k$, and suppose that g_m is defined on $]y_m, \infty[$. For $x \in]y_{m+1}, y_m]$ we have $x+2 > x+1 > q(x+1) > y_m$, thus we may define g_{m+1} on $]y_{m+1}, \infty[$ by

$$g_{m+1}(x) := \begin{cases} g_m(x) & \text{for } x \in]y_m, \infty[\\ 4q g_m(q(x+1)) - g_m(x+2) - 2g_m(x+1) & \text{for } x \in]y_{m+1}, y_m]. \end{cases}$$

By this process we get an extension of h to the interval $]y_k, \infty[$, which we call $f_0\colon]y_k, \infty[\to \mathbb{R}$.

Now suppose that $f_n\colon]y_k - n, \infty[\to \mathbb{R}$ is defined for a nonnegative integer n. Then we define $f_{n+1}\colon]y_k - n - 1, \infty[\to \mathbb{R}$ by

$$f_{n+1}(x) := \begin{cases} f_n(x) & \text{for } x \in]y_k - n, \infty[\\ 4q f_n(q(x+1)) - f_n(x+2) - 2f_n(x+1) & \text{otherwise.} \end{cases}$$

(As $y_k < 0$, it is easy to check that the numbers $x+2$, $x+1$, $q(x+1)$ are greater than $y_k - n$ for $x \in]y_k - n - 1, y_k - n]$.)

By the same arguments as before the family of functions (f_n) uniquely defines a function $f\colon \mathbb{R} \to \mathbb{R}$. We only have to check that this function f fulfills equation (1). From the construction it is evident that $f(Q) = \alpha$, $\operatorname{supp}(f) \subseteq]-\infty, Q]$ and f coincides with h on $]Q-1, qQ]$. Let $x \in \mathbb{R}$:

(α) $x \le y_k + 1$: There is a nonnegative integer n such that $y_k - n - 1 < x - 1 \le y_k - n$. As f coincides with f_{n+1} on $]y_k - n - 1, \infty[$, and from the definition of f_{n+1} (the formula given above defines the value at $x-1$) we immediately get that (1) is fulfilled.

(β) $y_k + 1 < x \le y_0 + 1 = Q$: Once more the definition of the functions g_m shows that (1) is fulfilled.

(γ) $Q < x \le Q+1$: Here we can use the definition of the functions h_n to show that equation (1) is fulfilled.

(δ) $x > Q+1$: As $\operatorname{supp}(f) \subseteq]-\infty, Q]$, equation (1) is trivially fulfilled. $\Diamond$

From equation (1) it is evident that in any case when f is a solution of (1), then also the function $x \to f(-x)$ is a solution of (1). Therefore, without giving any new proofs we can reformulate Ths. 4 and 5 for the case that $\operatorname{supp}(f) \subseteq [-Q, \infty[$:

Theorem 6. *Let f, g be solutions of (1) whose supports are contained in the interval $[-Q, \infty[$. Then $f = g$ iff the restrictions of f and g to the interval $[-qQ, 1-Q[$ coincide and $f(-Q) = g(-Q)$. (The second condition is only necessary in the case $q = \frac{1}{4}$.)*

Theorem 7. *Let $h\colon [-qQ, 1-Q[\to \mathbb{R}$ be an arbitrary function, and α a real number which is arbitrary in the case $q = \frac{1}{4}$ and 0 otherwise. Then there exists a unique solution f of (1) such that the restriction of f to the interval $[-qQ, 1-Q[$ is identical to h, $f(-Q) = \alpha$ and $\operatorname{supp}(f) \subseteq [-Q, \infty[$.*

Remark 2. Ths. 5 and 7 show that *in the case $q = \frac{1}{4}$ really both cases $S(f) \subseteq]-\infty, Q[$ and $Q \in S(f) \subseteq]-\infty, Q]$ (resp. $S(f) \subseteq]-Q, \infty[$ and $-Q \in S(f) \subseteq [-Q, \infty[$) can occur.*

Next we conduct investigations on the solutions of (1) under special conditions like continuity, differentiability, measurability, integrability. With respect to the remark before Th. 6, we may restrict ourselves to the cases $S(f) \subseteq \mathbb{R}$ and $S(f) \subseteq]-\infty, Q]$. The main question will be: Which conditions have to be imposed on the defining function h (cf. Th. 3 resp. 5) in order that the solution f has the desired property?

b) Continuous solutions

It is evident that in this case h has to be continuous. The answer concerning the necessity of further properties on h is given below:

Theorem 8 (The case $S(f) \subseteq \mathbb{R}$). *Let* $h\colon [-1,1[\to \mathbb{R}$ *be continuous. Then the unique solution* f *of* (1) *defined by* h (unique extension by Th. 3) *is continuous iff* $\lim_{x\to 1} h(x) = (4q-2)h(0) - h(-1)$.

Proof. We use the notations f_n and F_n of Th. 3.

"only if": $\lim_{x\to 1} h(x) = f(1) = (4q-2)h(0) - h(-1)$ by equation (1).

"if": The construction of f given in Th. 3 is very useful: $f_1 := := h\colon [-1,1[\to \mathbb{R}$. If f_n is given on $[-1,n[$ for some nonnegative integer n, then f_{n+1} is defined on $[-1,n+1[$ by

$$f_{n+1}(x) := \begin{cases} f_n(x) & \text{for } x\in[-1,n[\\ 4qf_n(q(x-1)) - f_n(x-2) - 2f_n(x-1) & \text{otherwise.} \end{cases}$$

As f_n is supposed to be continuous (induction hypothesis), we only have to show that f_{n+1} is continuous at the point n (in the neighbourhoods of any other point f_{n+1} is given as a composition of continuous functions). To be more precise: We only have to show that $\lim_{x\nearrow n} f_{n+1}(x) = f_{n+1}(n)$, i.e.,

$$(*) \qquad \lim_{x\nearrow n} f_n(x) = 4qf_n(q(n-1)) - f_n(n-2) - 2f_n(n-1).$$

$n=1$: $(*)$ is fulfilled because of our assumption on h.
$n>1$: By definition of f_n we have

$$\begin{aligned} \lim_{x\nearrow n} f_n(x) &= \lim_{x\nearrow n} \big(4qf_{n-1}(q(x-1)) - f_{n-1}(x-2) - 2f_{n-1}(x-1)\big) = \\ &= \lim_{x\nearrow n} \big(4qf_n(q(x-1)) - f_n(x-2) - 2f_n(x-1)\big) = \\ &\qquad \text{(because } f_n \text{ coincides with } f_{n-1} \text{ on } [-1,n-1[) \\ &= \lim_{x\nearrow n-1} \big(4qf_n(q(x)) - f_n(x-1) - 2f_n(x)\big) = \\ &= 4qf_n(q(n-1)) - f_n(n-2) - 2f_n(n-1) = f_{n+1}(n) \\ &\qquad \text{(because } n-1 \text{ is an interior point of } [-1,n[\text{, and} \\ &\qquad\qquad f_n \text{ is continuous on } [-1,n[). \end{aligned}$$

Thus the function F_1 of Th. 3 is continuous on $[-1,\infty[$. We proceed once more by induction, showing that each F_n is continuous. These functions are inductively defined by

$$F_{n+1}(x) := \begin{cases} F_n(x) & \text{for } x\in[-n,\infty[\\ 4qF_n(q(x+1)) - F_n(x+2) - 2F_n(x+1) & \text{otherwise.} \end{cases}$$

Here we have to show that the function F_{n+1} is continuous at the point $-n$, to be more precise, we have to show:

$$\lim_{x \nearrow -n} F_{n+1}(x) = F_n(-n).$$

But

$$\lim_{x \nearrow -n} F_{n+1}(x) = \lim_{x \nearrow -n} \big(4qF_n(q(x+1)) - F_n(x+2) - 2F_n(x+1)\big) =$$
$$= \lim_{x \nearrow -n+1} \big(4qF_n(q(x)) - F_n(x+1) - 2F_n(x)\big) =$$
$$= \big(4qF_n(q(-n+1)) - F_n(-n+2) - 2F_n(-n+1)\big) =$$
$$= 4qF_{n-1}(q(-n+1)) - F_{n-1}(-n+2) - 2F_{n-1}(-n+1) = F_n(-n) \text{ for } n \geq 2.$$

For $n = 1$ we compute like before

$$\lim_{x \nearrow -n} F_{n+1}(x) = 4qF_n(q(-n+1)) - F_n(-n+2) - 2F_n(-n+1) =$$
$$= 4qF_1(0) - F_1(1) - 2F_1(0) = F_1(-1) \quad \text{by the condition on } h.$$

Thus each F_n is continuous, and therefore the solution f is continuous. ◊

Before we deal with the case $S(f) \subseteq]-\infty, Q]$, we introduce some notation. This will be useful to make the theorems on this case $S(f) \subseteq$ $\subseteq]-\infty, Q]$ more easily readable — and the same notation is also useful to treat the question of differentiable solutions.

Definition 1. Let $q \in]0,1[$.

$$E(q) := \left\{ \frac{Q - a_0 - a_1 q^1 - \ldots - a_m q^m}{q^m} \,\middle|\, m, a_i \in \mathbb{Z},\ m \geq 0,\ a_i \geq 1 \right\} \cup \{Q\}$$

(*exceptional points*). For real x and integers $m \geq 1$ we define the set $M(q, x, m)$ by

$$M(q,x,m) := \big\{(l_1, \ldots, l_m) \,\big|\, q^m x + l_m q^m + \ldots + l_1 q = Q,$$
$$l_i \in \mathbb{Z},\ l_1, \ldots, l_m > 0\big\}$$

(*m-tuples*). Furthermore, let $S(q, x, m)$ denote the sum

$$S(q,x,m) := \sum_{(l_1,\ldots,l_m) \in M(q,x,m)} (-1)^{l_1 + \ldots + l_m} \cdot l_1 \ldots l_m.$$

(As usual $S(q, x, m) = 0$ whenever $M(q, x, m) = \emptyset$.) Finally, let $C(q)$ denote the set

$$C(q) := \big\{x \in \mathbb{R} \,\big|\, \lim_{m \to \infty} S(q,x,m) = 0\big\}$$

(*points of continuity*, as we will see later on).

Proposition 1.

(a) *For $x \in \mathbb{R}$ the following relation holds: $x \in E(q)$ iff $M(q,x,m) \neq \emptyset$ for some $m \in \mathbb{N}$.*

(b) $\{1\} \times M(q,x,m) \subseteq M(q,x,m+1)$ *for any m and for any x.*

(c) *For any $x \in \mathbb{R}$ there exists a natural number m_0 such that for any $m \geq m_0$ the relations $M(q,x,m+1) = \{1\} \times M(q,x,m)$ and $S(q,x,m+1) = -S(q,x,m)$ hold. Furthermore, the sets $M(q,x,m)$ are finite.*

(d) *The set $E(q)$ only contains isolated points, and for any compact interval J the set $J \cap E(q)$ is finite.*

(e) $\mathbb{R} \backslash E(q) \subseteq C(q)$.

(f) *Recursion formula for $S(q,x,m)$:*

$$S(q,x,m+1) = \sum_{l=1}^{u} (-1)^l l \cdot S\Big(q, x + \frac{l-1}{q^m}, m\Big),$$

where u denotes the greatest integer with $u \leq Q + 1 - q^m x$.

Proof. (a) First suppose that $x \in E(q)$.

Case 1: $x = Q$. Then $qx + q = q(Q+1) = Q$, which implies that $M(q,x,1) \neq \emptyset$.

Case 2: $x = \frac{Q - a_0 - a_1 q^1 - \ldots - a_m q^m}{q^m}$, where $m, a_i \in \mathbb{Z}$, $m \geq 0$, $a_i \geq 1$. Then

$$q^{m+1}x + (a_0+1)q^1 + a_1 q^2 + \ldots + a_m q^{m+1} =$$

$$= q\big(Q+1-1-a_0-a_1q^1-\ldots-a_m q^m\big) + (a_0+1)q^1 + a_1q^2 + \ldots + a_m q^{m+1} =$$

$$= q(Q+1) = Q.$$

Thus $M(q,x,m+1) \neq \emptyset$. Now suppose that $M(q,x,m) \neq \emptyset$ for some $m \geq 1$: Then there are integers $l_1, \ldots, l_m \geq 1$ such that $q^m x + l_m q^m + \ldots + l_1 q = Q$, which implies that

$$x = \frac{Q - l_1 q^1 - \ldots - l_m q^m}{q^m} = \frac{q(Q+1) - l_1 q^1 - \ldots - l_m q^m}{q^m}.$$

Cancellation of the factor q gives

$$x = \frac{Q - (l_1 - 1) - \ldots - l_m q^{m-1}}{q^{m-1}}.$$

If $l_1 > 1$, this expression shows that $x \in E(q)$, otherwise we proceed cancelling like before until we get the desired expression.

(b) Let $(l_1, \dots, l_m) \in M(q, x, m)$. Then $q^m x + l_m q^m + \ldots + l_1 q = Q$, which implies that

$$q^{m+1}x + l_m q^{m+1} + \ldots + l_1 q^2 + 1q = qQ + q = q(Q+1) = Q,$$

in other words, $(1, l_1, \dots, l_m) \in M(q, x, m+1)$.

(c) *Case* 1: $x > Q$. In this case $M(q, x, m) = \emptyset$ for any m, because the relation $q(Q+1) = Q$ immediately implies that $q^m Q + q^m + \ldots + q^1 = Q$. Therefore, as $x > Q$ and $l_1, \dots, l_m \geq 1$, we immediately get $q^m x + l_m q^m + \ldots + l_1 q > q^m Q + q^m + \ldots + q^1 = Q$. Thus it is impossible to find elements belonging to $M(q, x, m)$.

Case 2: $x = Q$. From the computation of Case 1 it follows immediately that $M(q, Q, m) = \{(1, \dots, 1)\}$.

Case 3: $x < Q$, $x \notin E(q)$. Then $M(q, x, m) = \emptyset$ for any m.

Case 4: $x < Q$, $x \in E(q)$. Then $0 < q(Q-x) = q((Q+1)-(x+1)) = Q - q(x+1)$ and $Q - q(x+1) = q(Q-x) < Q - x$, which implies that $x < q(x+1) < Q$, and a usual induction argument shows that $x < qx + q < q^2 x + q^2 + q < \ldots < q^m x + q^m + \ldots + q < \ldots < Q$, and this strictly increasing sequence tends to Q. Now choose an integer n such that

$$Q - q < q^n x + q^n + \ldots + q < Q.$$

Let $m \geq n$ be an arbitrary integer and $(l_1, \dots, l_m) \in M(q, x, m)$. Suppose that $l_1 \geq 2$, then

$$Q = q^m x + l_m q^m + \ldots + l_1 q \geq q^m x + q^m + \ldots + q + q > Q - q + q = Q,$$

a contradiction. Thus the only possiblity is that $l_1 = 1$. From this fact we deduce that $Q = q(Q+1) = q^m x + l_m q^m + \ldots + 1q$. Cancellation of the factor q gives $Q = q^{m-1}x + l_m q^{m-1} + \ldots + l_2 q$, which implies that $(l_2, \dots, l_m) \in M(q, x, m-1)$, in other words: We have shown that $M(q, x, m) \subseteq \{1\} \times M(q, x, m-1)$.

Thus in any case $M(q, x, m+1) = \{1\} \times M(q, x, m)$ for $m \geq m_0$ holds. Furthermore, for any fixed natural number m the set $M(q, x, m)$ is finite, because all the numbers $q, q^2, \dots, q^m$ are positive. The formula for the sum $S(q, x, m)$ is a trivial consequence of the equation for $M(q, x, m)$ given above.

(d) From the proof of (c) we see that $E(q)$ is a subset of the interval $[-\infty, Q]$. Thus we only have to show that the intersection of $E(q)$ with any interval $[a, Q[$ (for $a < Q$) is finite. Let $a < Q$, and as in the proof of (c) choose an n such that $Q - q < q^n a + q^n + \ldots + q < q$. Now suppose that $x \in [a, Q[$. Then $Q - q < q^n x + q^n + \ldots + q < Q$, and combining (a) and the computation in the proof of (c) we immediately see that $x \in E(q)$ iff $M(q,x,n) \neq \emptyset$. A simple computation also immediately gives that the intersection of $M(q,x,n)$ and $M(q,y,n)$ is empty, if $x \neq \neq y$. If $M(q,x,n)$ is nonempty, then there is an n-tuple $(l_1, \ldots, l_n)$ such that $Q = q^n x + l_n q^n + \ldots + l_1 q$. Thus

$$Q - q < q^n a + q^n + \ldots + q \leq q^n a + l_n q^n + \ldots + l_1 q \leq q^n x + l_n q^n + \ldots + l_1 q = Q.$$

As the numbers $q, q^2, \ldots, q^n$ are positive, there are only finitely many n-tuples $(l_1, \ldots, l_n)$ which fulfill the inequality $q^n a + l_n q^n + \ldots + l_1 q \leq \leq Q$. Thus there are only finitely many points $x \in [a, Q[$ such that $M(q,x,n) \neq \emptyset$.

(e) By (a), for any $x \in \mathbb{R} \backslash E(q)$ we have $M(q,x,m) = \emptyset$ for any m. Thus for any m and any such x we get $S(q,x,m) = 0$.

(f)
$$M(q,x,m+1) := \\ := \left\{ (l_1, \ldots, l_m, l_{m+1}) \,\middle|\, {}^{q^{m+1}x + l_{m+1}q^{m+1} + l_m q^m + \ldots + l_1 q = Q,}_{l_i \in \mathbb{Z}, l_1, \ldots, l_{m+1} > 0} \right\}.$$

Fixing l_1, the condition can be written as

$$q^{m+1}x + l_1 q + l_{m+1}q^{m+1} + \ldots + l_2 q^2 = Q = q(Q+1) \quad \text{or}$$
$$q^{m+1}x + (l_1 - 1)q + l_{m+1}q^{m+1} + \ldots + l_2 q^2 = qQ \quad \text{or}$$
$$q^m \left(x + \frac{l_1 - 1}{q^m} \right) + l_{m+1}q^m + \ldots + l_2 q^1 = Q.$$

From the last condition we immediately get

$$M(q,x,m+1) = \bigcup_{l=1}^{\infty} \{l\} \times M\left(q, x + \tfrac{l-1}{q^m}, m\right).$$

The formula for $S(q,x,m+1)$ is a trivial consequence, because the union above is disjoint and $M(q,y,m)$ is empty for $y > Q$. $\Diamond$

We will need the set $C(q)$ in order to describe continuity resp. differentiability properties. As the description of the set $E(q)$ is much easier to handle than the definition of $C(q)$, we try to find a relation "easy to handle" between the sets $\mathbb{R} \backslash E(q)$ and $C(q)$. In Prop. 1(c) it has been shown that $\mathbb{R} \backslash E(q) \subseteq C(q)$. Thus the question arises whether this inclusion is proper or not. At the moment only a partial answer can be given:

Proposition 2.

(a) *If $q \in]0,1[$ is transcendental over the field $\mathbb{Q}$, then $C(q) = \mathbb{R} \backslash E(q)$.*

(b) *If $q \in]0,1[$ is algebraic over the field $\mathbb{Q}$, it has a minimal polynomial in the algebra $\mathbb{Q}[Z]$. We normalize this polynomial not as* *usual (leading coefficient* $= 1$*), but with integer coefficients whose g.c.d. is equal to* 1 *(this polynomial is unique up to the factor* ± 1*). Let this polynomial be denoted by* $p(z)$*.*

(b1) *If* $p(1)$ *is even (i.e., the sum of the coefficients is even), then* $C(q) = \mathbb{R}\backslash E(q)$*.*

(b2) *If* $q \leq \frac{1}{3}$ *and* $p(z) = a - bz^k$*, where* a, b, k *are positive integers and* $a + b$ *is odd, then* $\mathbb{R}\backslash E(q)$ *is a proper subset of* $C(q)$*.*

Proof. Let $x \in E(q)$. Suppose that $(l_1, l_2, \ldots, l_m) \in M(q, x, m)$ and $(l'_1, l'_2, \ldots, l'_m) \in M(q, x, m)$. Then

$$0 = Q - Q = (q^m x + l_m q^m + \ldots + l_1 q) - (q^m x + l'_m q^m + \ldots + l'_1 q) = \\ = (l_m - l'_m) q^m + \ldots + (l_1 - l'_1) q.$$

(a) Let q be transcendental over the field $\mathbb{Q}$. Then no nonzero polynomial with integer coefficients can have q as a zero. Therefore, $M(q, x, m)$ contains exactly one element, and $S(q, x, m) \neq 0$.

(b1) For 2 elements $(l_1, l_2, \ldots, l_m) \in M(q, x, m)$ and $(l'_1, l'_2, \ldots, l'_m) \in M(q, x, m)$ the minimal polynomial $p(z)$ of q is a divisor of the polynomial $(l_m - l'_m) z^m + \ldots + (l_1 - l'_1) z$ (by Gauss' lemma from elementary algebra). Thus the even number $p(1)$ is a divisor of $(l_m - l'_m) + \ldots + (l_1 - l'_1)$, in other words: $l_1 + \ldots + l_m$ and $l'_1 + \ldots + l'_m$ are either both even or both odd. Therefore, all the terms in the sum $S(q, x, m)$ have the same sign, which implies that $S(q, x, m) \neq 0$.

(b2) In order to show that in this case $\mathbb{R}\backslash E(q)$ is a proper subset of $C(q)$, we give an element $x \in E(q)$ with $x \in C(q)$: Let r be the smallest positive integer such that

$$\frac{q - q^{k+r}}{1 - q} + (3a - 1) q^r < 1,$$

and let $m := k + r$. (Such an integer r exists, because $Q = \frac{q}{1-q} < 1$.) Now define $(l_1, \ldots, l_m)$ by

$$l_i := \begin{cases} 2a, & \text{if } i = r \\ b & \text{if } i = m \\ 1 & \text{otherwise.} \end{cases}$$

and let $x := \frac{Q - l_1 q^1 - \ldots - l_m q^m}{q^m}$. We determine the set $M(q, x, m)$. Of course, $(l_1, \ldots, l_m) \in M(q, x, m)$. Now let $(l'_1, \ldots, l'_m) \in M(q, x, m)$. Then $l'_i > 0$, and the polynomial $(l'_m - l_m) z^m + \ldots + (l'_1 - l_1) z$ is a multiple of $a - bz^k$ in the ring $\mathbb{Z}[Z]$, i.e., there is a polynomial $s(z)$ with integer coefficients such that $(l'_m - l_m)z^n + \ldots + (l'_1 - l_1)z = (a - bz^k) \cdot$ $\cdot s(z)$. This fact implies that the first nonvanishing difference $l'_i - l_i$ is an integer multiple of a. Now suppose that $j < r$ and $l_1 = l'_1, \ldots, l_{j-1} =$ $= l'_{j-1}$, $l_j \neq l'_j$. Then $l'_j \geq l_j + a = 1 + a$. As the sums $l'_m q^m + \ldots + l'_1 q$ are equal for all $(l'_1, \ldots, l'_m)$ in $M(q, x, m)$, we have

$$1 > l'_m q^m + \ldots + l'_1 q \geq q^m + \ldots + q + aq^j = \frac{q - q^m}{1 - q} + q^j(a + q^{m-j}).$$

As $q \leq \frac{1}{3}$, we have $q(3a - 1) < q \cdot 3a < a + q^{m-j}$, and therefore,

$$1 > l'_m q^m + \ldots + l'_1 q \geq \frac{q - q^m}{1 - q} + q^{j+1}(3a - 1).$$

According to the minimality of r we must have $j \geq r - 1$. On the other hand, $\deg((a - bz^k) \cdot s(z)) = k + r$, which implies that $\deg(s) \leq r$. Thus the only possibilities are that $s(z) = z^{r-1} \cdot (\alpha + \beta z)$ with integer coefficients α, β.

Case 1: $k > 1$. Then $l'_{m-1} = 1 - \alpha b > 0$ and $l'_{r-1} = l_{r-1} + a\alpha \geq 0$, which implies that $\alpha = 0$.

Case 2: $k = 1$. If $r = 1$, then clearly $\alpha = 0$ (the left-hand-side polynomial has no constant term). If $r > 1$, then

$$\begin{aligned} l'_m &= b - \beta b > 0, \quad \text{which implies that } \beta \leq 0; \\ l'_{r-1} &= 1 + \alpha a > 0, \quad \text{which implies that } \alpha \geq 0; \\ l'_r &= 2a - \alpha b + \beta a > 0, \quad \text{i.e. } (2 + \beta)a > \alpha b. \end{aligned}$$

As $q = \frac{a}{b} \leq \frac{1}{3}$, we immediately get $\alpha = 0$.

Thus in any case we have $\alpha = 0$ and, therefore,

$$l'_i = \begin{cases} (2 + \beta)a, & \text{if } i = r \\ (1 - \beta)b, & \text{if } i = m \text{ for some integer } \beta \\ 1 & \text{otherwise.} \end{cases}$$

The condition $l'_i > 0$ implies that the only possible values for β are 0 and -1. Thus the set $M(q, x, m)$ contains exactly two elements, namely,

$$M = \{(1, \ldots, 1, 2a, 1, \ldots, 1, b), (1, \ldots, 1, a, 1, \ldots, 1, 2b)\}.$$

As $1. \ldots .1.2a.1. \ldots .1.b = 1. \ldots .1.a.1. \ldots .1.2b$ and $(1 + \ldots + 1 + 2a +$ $+ 1 + \ldots + 1 + b) - (1 + \ldots + 1 + a + 1 + \ldots + 1 + 2b) = a - b$ is odd, the sum $S(q, x, m)$ is equal to 0. As $(3a - 1)q^r < 1$, we have $(3a - 1)q^{r+1} < 1 + q^{m+1}$, which implies that

$$Q(1 - q) - q = 0 < 1 + (1 - 2a)q^{r+1} + (1 - b)q^{m+1} \quad \text{resp.}$$

$$Q - q < q^{m+1}x + q + \ldots + q^{m+1}.$$

As we had seen in the proof of Prop. 1(c), this condition guarantees that for any $n \geq (m+1)-1$ we have $S(q,x,n+1) = -S(q,x,n)$. Thus $S(q,x,n) = 0$ for any $n \geq m$, which implies $x \in C(q)$. ◊

Theorem 9 (The case $S(f) \subseteq]-\infty, Q]$). *Let* $h\colon]Q-1, qQ] \to \mathbb{R}$ *be a function, and* α *a real number which fulfills* $\alpha = 0$ *in the case* $q \neq \frac{1}{4}$. *Then the unique solution* f *of* (1) *which coincides with* h *on* $]Q-1, qQ]$ *and fulfills* $f(Q) = \alpha$ *and* $S(f) \subseteq]-\infty, Q]$ *is continuous iff* $\alpha = 0$ *and* h *fulfills the following condition:*

(i) *case* $q \leq \frac{1}{4}$: $h \equiv 0$ (in other words: *in this case the zero function is the only continuous solution);*

(ii) *case* $q > \frac{1}{4}$: *h is continuous and* $\lim\limits_{x \searrow Q-1} h(x) = 4qh(qQ)$.

Proof. We use the notatios x_n, h_n, y_n, g_n, f_n of Th. 5.

First suppose that f is continuous. As $f(Q) = \lim\limits_{x \searrow Q} f(x) = \lim\limits_{x \searrow Q} 0 = 0$, we must have $\alpha = 0$, furthermore, h must be continuous. Also $\lim\limits_{x \searrow Q-1} h(x) = \lim\limits_{x \searrow Q-1} f(x) = f(Q-1) = 4qf(qQ) - f(Q+1) - -2f(Q) = 4qf(qQ) = 4qh(qQ)$. Now let $z \in]Q-1, qQ]$ be arbitrary, and define a sequence (z_n) by $z_0 := z$, $z_{n+1} := q(z_n+1)$. Then $\lim\limits_{n\to\infty} z_n = Q$, (z_n) is strictly increasing, and because $z_n + 2 > z_n + 1 > Q$, we have

$$f(z_{n+1}) = f(q(z_n+1)) = \frac{1}{4q}\big(f(z_n) + f(z_n+2) + 2f(z_n+1)\big) = \frac{1}{4q} f(z_n).$$

Thus, $f(z_n) = \left(\frac{1}{4q}\right)^n f(z)$, and therefore

$$0 = f(Q) = \lim_{n\to\infty} f(z_n) = \lim_{n\to\infty} \left(\frac{1}{4q}\right)^n f(z).$$

The right-hand-side limit exists and is equal to 0 iff $q > \frac{1}{4}$ or $f(z) = 0$.

For the reverse direction we may suppose that $q > \frac{1}{4}$, $\alpha = 0$, $\lim\limits_{s \searrow Q-1} h(x) = 4qh(qQ)$ and h continuous. We use the construction of the solution f of Th. 5. Let (x_n) be the sequence given by $x_0 := Q-1$, $x_{n+1} := q(x_n+1)$. Let $h_0 := h$, and h_n be defined on the interval $]x_0, x_{n+1}]$ by induction:

$$h_{n+1}(x) := \begin{cases} h_n(x) & \text{for } x \in]x_0, x_{n+1}] \\ \frac{1}{4q} h_n(y) & \text{for } x = q(y+1) \in]x_{n+1}, x_{n+2}]. \end{cases}$$

If h_n is continuous on $]x_0, x_{n+1}]$, then by definition it is evident that h_{n+1} is continuous in $]x_0, x_{n+2}] \setminus \{x_{n+1}\}$. To prove continuity at the point x_{n+1} it is only necessary to compute $\lim_{x \searrow x_{n+1}} h_{n+1}(x)$:

$$\lim_{x \searrow x_{n+1}} h_{n+1}(x) = \lim_{x \searrow x_{n+1}} \frac{1}{4q} h_n\left(\frac{x}{q} - 1\right) =$$
$$= \lim_{x \searrow x_n} \frac{1}{4q} h_n(x) = \frac{1}{4q} h_n(x_n) = h_{n+1}(x_{n+1}).$$

Thus the function $h_\infty \colon]Q-1, Q[\to \mathbb{R}$ is continuous, and for the continuity of the function

$$g_0 :]Q-1, \infty[\to \mathbb{R} \colon x \to \begin{cases} h_\infty(x) & \text{for } x \in]Q-1, Q[\\ 0 & \text{for } x \geq Q \end{cases}$$

we only have to show that $\lim_{x \nearrow Q} h_\infty(x) = 0$. By assumption h is bounded on $]Q-1, qQ]$, let us say, by a constant M. But then we have h_∞ bounded on $]x_n, x_{n+1}]$ by $\left(\frac{1}{4q}\right)^n M$, which immediately implies that g_0 is continuous at Q. The next extension is done via the sequence (y_n) of Th. 5, defined by $y_0 := Q - 1$, $y_{n+1} := \frac{1}{q} y_n - 1$, and the nonnegative integer k such that $y_k < 0$, $y_{k-1} \geq 0$. For g_m defined on $]y_m, \infty[$, $0 \leq m < k$, we define g_{m+1} on $]y_{m+1}, \infty[$ by

$$g_{m+1}(x) :=$$
$$:= \begin{cases} g_m(x) & \text{for } x \in]y_m, \infty[\\ 4q g_m(q(x+1)) - g_m(x+2) - 2g_m(x+1) & \text{for } x \in]y_{m+1}, y_m]. \end{cases}$$

Once more using an induction argument, we have to show that g_{m+1} is continuous under the assumption that g_m is continuous. According to the definition, the only critical point is the point y_m:

$m=0$: $\lim_{x \searrow y_0} g_1(x) = \lim_{x \searrow y_0} g_0(x) = \lim_{x \searrow y_0} h(x) = 4qh(qQ) =$
$= 4q g_0(q(y_0+1)) - g_0(y_0+2) - 2g_0(y_0+1) = g_1(y_0)$.

$m > 0$: We use the fact that g_m is continuous at the interior points of its domain:

$$\lim_{x \searrow y_m} g_{m+1}(x) = \lim_{x \searrow y_m} \left(4q g_m(q(x+1)) - g_m(x+2) - 2g_m(x+1)\right) =$$
$$= 4q g_m(q(y_m+1)) - g_m(y_m+2) - 2g_m(y_m+1) = g_{m+1}(y_m).$$

Thus the function $f_0 \colon]y_k, \infty[\to \mathbb{R}$ is continuous. Now, if $f_n \colon]y_k - n, \infty[\to \mathbb{R}$ is continuous, f_{n+1} is defined on $]y_k - n - 1, \infty[$ by the formula

$$f_{n+1}(x) := \begin{cases} f_n(x) & \text{for } x \in]y_k - n, \infty[\\ 4qf_n(q(x+1)) - f_n(x+2) - 2f_n(x+1) & \text{otherwise.} \end{cases}$$

Thus it is continuous at any point except possibly the point $y_k - n$:
$n = 0$, $k = 0$:

$$\lim_{x \searrow y_k} f_1(x) = \lim_{x \searrow y_k} f_0(x) = \lim_{x \searrow y_k} g_0(x) = \lim_{x \searrow y_0} h(x) = 4qh(qQ) =$$
$$= 4qf_0(q(y_0+1)) - f_0(y_0+2) - 2f_0(y_0+1) = f_1(y_0).$$

$n = 0$, $k > 0$:

$$\lim_{x \searrow y_k} f_1(x) = \lim_{x \searrow y_k} f_0(x) = \lim_{x \searrow y_k} g_k(x) =$$

$$= \lim_{x \searrow y_k} \big(4qg_{k-1}(q(x+1)) - g_{k-1}(x+2) - 2g_{k-1}(x+1)\big) =$$
$$= \lim_{x \searrow y_k} \big(4qg_k(q(x+1)) - g_k(x+2) - 2g_k(x+1)\big) =$$
$$= 4qg_k(q(y_k+1)) - g_k(y_k+2) - 2g_k(y_k+1) =$$
$$= 4qf_0(q(y_k+1)) - f_0(y_k+2) - 2f_0(y_k+1) = f_1(y_k).$$

$n > 0$:

$$\lim_{x \searrow y_k - n} f_{n+1}(x) + \lim_{x \searrow y_k - n} \big(4qf_n(q(x+1)) - f_n(x+2) - 2f_n(x+1)\big) =$$

$$= 4qf_n(q(y_k - n + 1)) - f_n(y_k - n + 2) - 2f_n(y_k - n + 1) = f_{n+1}(y_k - n).$$

This fact proves that the resulting solution f of (1) is continuous everywhere. ◊

Of course, it was necessary to have h continuous in the preceding theorem in order to get a continuous solution. And — together with the boundary condition $\lim_{x \searrow Q-1} h(x) = 4qh(qQ)$ — this is also sufficient in the case $q > \frac{1}{4}$. The question arises: What can be said about solutions f in the case $q \le \frac{1}{4}$, if the defining function h is continuous and fulfills this boundary condition?

Theorem 10 (the case $S(f) \subseteq]-\infty, Q]$). *Let $q \le \frac{1}{4}$, $\alpha \in \mathbb{R}$, $\alpha = 0$ if $q < \frac{1}{4}$, $h\colon]Q-1, qQ] \to \mathbb{R}$ be continuous and nonvanishing, $\lim_{x \searrow Q-1} h(x) = 4qh(qQ)$, and let f be the unique solution of* (1) *which extends h and fulfills $S(f) \subseteq]-\infty, Q]$ and $f(Q) = \alpha$. Then the set of points where f is continuous coincides with $C(q)$.*

Proof. We use the notations of Ths. 5 and 9. The proof in Th. 9 shows that f is continuous on the set $]Q-1, Q[$. Furthermore, if one chooses a point $z \in]Q-1, qQ]$ such that $h(z) \neq 0$, then the sequence (z_n) defined by $z_0 := z$, $z_{n+1} := q(z_n + 1)$ tends to Q, and the values are given by $f(z_n) = \left(\frac{1}{4q}\right)^n f(z)$. This sequence tends to infinity in the case $q < \frac{1}{4}$, and it has a constant value, different from 0, in the case $q = \frac{1}{4}$. Thus the function $g_0 :]Q-1, \infty[\to \mathbf{R}$ has exactly one point of discontinuity, namely the point $x = Q$.

Now for the solution f the equation

$$f(x) = 4qf(q(x+1)) - f(x+2) - 2f(x+1)$$

holds. By usual induction argument from this equation we can derive the formula

$$f(x) = 4q\sum_{l=1}^{k}(-1)^{l+1}l.f(q(x+1)) + (-1)^k\big((k+1)f(x+k) + kf(x+k+1)\big)$$

for any natural number k: In the case $k = 1$ this formula is nothing else but equation (1), and using equation (1) for the expression $f(x+k)$ we get

$$f(x) =$$

$$= 4q\sum_{l=1}^{k}(-1)^{l+1}l.f(q(x+1)) + (-1)^k\big((k+1)f(x+k) + kf(x+k+1)\big) =$$

$$= 4q\sum_{l=1}^{k}(-1)^{l+1}l.f(q(x+1)) + (-1)^k kf(x+k+1) +$$

$$+(-1)^k(k+1)\big(4q.f(q(x+k+1)) - f(x+k+2) - 2f(x+k+1)\big) =$$

$$= 4q\sum_{l=1}^{k+1}(-1)^{l+1}l.f(q(x+l)) +$$

$$+(-1)^{k+1}((k+2)f(x+k+1) + (k+1)f(x+k+2)).$$

Now suppose that $x < Q$, and let k be an integer such that $x + k > Q$. As $S(f) \subseteq]-\infty, Q]$, we immediately get

$$f(x) = 4q\sum_{l=1}^{k}(-1)^{l+1}l.f(q(x+1)).$$

Repeating this formula for the arguments $q(x+l)$ we immediately get

$$f(x)=4q\sum_{l_1=1}^{k}(-1)^{l_1+1}.l_1.4q\sum_{l_2=1}^{k}(-1)^{l_2+1}.l_2.f(q^2x+l_1q^2+l_2q),$$

and by a usual induction argument, for any natural number m we get

$$f(x)=(-4q)^m\cdot\sum_{l_1,\dots,l_m=1}^{k}(-1)^{l_1+\dots+l_m}.l_1\dots l_m.f(q^mx+l_mq^m+\dots+l_1q).$$

Now we may choose m large enough such that $q^mx+q^m+\dots+q>$ $>Q-1$. As f is continuous in the interval $]Q-1,\infty[$, except at the point Q, f can be discontinuous at x only, if at least one of the values $q^mx+l_mq^m+\dots+l_1q$ is equal to Q, because otherwise we can find a whole neighbourhood U of x such that $q^my+l_mq^m+\dots+l_1q\neq Q$, for any $y\in U$. Thus f is continuous on the set $\mathbb{R}\setminus E(q)$.

For a detailed description of the points of continuity of f now let $x\in E(q)$, and let m be chosen large enough such that $q^my+q^m+\dots+$ $+q>Q-1$ in a neighbourhood U of x. Furthermore, we choose k large enough such that $y+k>Q$ for $y\in U$ and $M(q,x,m)\subseteq\{1,2,\dots,k\}^m$ (the last condition makes sense because $M(q,x,m)$ is a finite set). As m,k are fixed, let us abbreviate $M(q,x,m)$ by M and use the notation P for the set $P:=\{1,2,\dots,k\}^m\setminus M$. Then $\{1,2,\dots,k\}^m$ is the disjoint union of M and P. Thus for $y\in U$

$$f(y)=(-4q)^m\cdot\sum_{l_1,\dots,l_m=1}^{k}(-1)^{l_1+\dots+l_m}.l_1\dots l_m.f(q^my+l_mq^m+\dots+l_1q)=$$

$$=(-4q)^m\cdot\sum_{(l_1,\dots,l_m)\in M}(-1)^{l_1+\dots+l_m}.l_1\dots l_m.f(q^my+l_mq^m+\dots+l_1q)+$$

$$+(-4q)^m\cdot\sum_{(l_1,\dots,l_m)\in P}(-1)^{l_1+\dots+l_m}.l_1\dots l_m.f(q^my+l_mq^m+\dots+l_1q).$$

As P is a finite set, we can choose a neighbourhood V of x such that $x\in V\subseteq U$ and $q^my+l_mq^m+\dots+l_1q\neq Q$ for any $(l_1,\dots,l_m)\in P$, $y\in V$. Then the sum $\sum_{(l_1,\dots,l_m)\in P}(-1)^{l_1+\dots+l_m}.l_1\dots l_m.f(q^my+l_mq^m+\dots+l_1q)$ describes a continuous function on V, because f is a continuous on the set $]Q-1,\infty[\setminus\{Q\}$. On the other hand, for $(l_1,\dots,l_m)\in M$ we have $f(q^my+l_mq^m+\dots+l_1q)=f(q^my-q^mx+q^mx+l_mq^m+\dots+l_1q)=$ $=f(q^m(y-x)+Q)$ by the definition of $M(q,x,m)$. Thus we have

$$\sum_{(l_1,\dots,l_m)\in M} (-1)^{l_1+\dots+l_m}.l_1\dots l_m.f(q^m y + l_m q^m + \dots + l_1 q) =$$
$$= \sum_{(l_1,\dots,l_m)\in M} (-1)^{l_1+\dots+l_m}.l_1\dots l_m.f(q^m(y-x)+Q) =$$
$$= f(q^m(y-x)+Q)\cdot \sum_{(l_1,\dots,l_m)\in M} (-1)^{l_1+\dots+l_m}.l_1\dots l_m =$$
$$= f(q^m(y-x)+Q).S(q,x,m).$$

As f is discontinuous at the point Q, f is continuous at x (in the neighbourhood V), if and only if $S(q,x,m)=0$. These arguments hold for any m large enough, therefore, we may conclude that f is continuous at x iff $x \in C(q)$. $\Diamond$

In general, it is not easy to decide for a point $x \in E(q)$ whether it belongs to the set $C(q)$ or not. A special case is the case $q = \frac{1}{4}$. In this case a complete description of the set $C(q)$ can be given. As a consequence, in this case the points of (dis-)continuity of the solution f can be given explicitely. The following theorem will give this description, and an example will illustrate this fact.

Theorem 11. *Let $q = \frac{1}{4}$. For any integer $p \geq 0$ let the sequences $\alpha(p) = (\alpha_0, \alpha_1, \alpha_2, \dots)$ and $\beta(p) = (\beta_0, \beta_1, \dots)$ be defined as follows: $\alpha_0 := p(\operatorname{mod} 8)$ (the remainder term of the division by 8), $\beta_0 := \frac{p-\alpha_0}{8}$, and the next terms are defined by induction $\alpha_{i+1} := \beta_i(\operatorname{mod} 4)$, $\beta_{i+1} := := \frac{\beta_i - \alpha_{i+1}}{4}$ for $i \geq 0$. (The sequence $\alpha(p)$ is constructed like the 4-adic expansion of p, except the first element α_0.) Now let L denote the set*

$$L := \left\{ p \in \mathbb{Z} \;\middle|\; \begin{matrix} p \geq 0, \text{ and the sequence } \alpha(p) \text{ fulfills the condition:} \\ \alpha_0 = 7, \text{ or there is an } i \geq 1 \text{ such that } \alpha_i = 3 \end{matrix} \right\}.$$

Then $E(\frac{1}{4}) = \{Q - p | p \in \mathbb{Z}, p \geq 0\}$ and $E(\frac{1}{4}) \cap C(\frac{1}{4}) = \{Q - p | p \in L\}$.

Proof.

$$E(q) = \left\{ \frac{Q - a_0 - a_1 q^1 - \dots - a_m q^m}{q^m} \;\middle|\; m, a_i \in \mathbb{Z}, m \geq 0, a_i \geq 1 \right\} \cup \{Q\} =$$
$$= \left\{ Qq^{-m} - a_0 q^{-m} - \dots - a_m \;\middle|\; m, a_i \in \mathbb{Z}, m \geq 0, a_i \geq 1 \right\} \cup \{Q\}.$$

Now $q = \frac{1}{4}$ and $Q = \frac{1}{3}$, thus $q^{-1} = 4$ and, therefore,

$$E\Big(\frac{1}{4}\Big)\{Q - p | p \in \mathbb{Z}, p \geq 0\}.$$

In order to find $C(\frac{1}{4})$ we compute the values $S(\frac{1}{4}, x, m)$ for $x \in E(\frac{1}{4})$. For the sake of simplicity let us denote $S(p,m) := S(\frac{1}{4}, Q - p, m)$

for integers $m, p \geq 1$. (From the proof of Prop. 1(c) we know that $S(q, Q, m) = (-1)^m$ for any m, any q.) Of course, we use the recursion formula from Prop. 1(f):

$$S(q, x, m+1) = \sum_{l=1}^{u} (-1)^l l.S\Big(q, x + \frac{l-1}{q^m}, m\Big),$$

where u denotes the largest integer with $u \leq Q+1-q^m x$. In the special case $q = \frac{1}{4}$ and $x = Q - p$ this formula reads as

$$S(p, m{+}1) = \sum_{l=1}^{u} (-1)^l l.S\Big(\frac{1}{4}, Q{-}p{+}\frac{l-1}{q^m}, m\Big) = \sum_{l=1}^{u} (-1)^l l.S(p{-}4^m(l{-}1), m).$$

What is the upper bound u? By definition, we have to look for all l such that there is an m-tuple $(l_2, \ldots, l_{m+1})$ with the property

$$q^{m+1}x + l_{m+1}q^{m+1} + \ldots + l_2 q^2 + lq = Q$$

resp.

$$Qq^{-m-1} = x + l_{m+1} + \ldots + l_2 q^{1-m} + lq^{-m} =$$
$$= Q - p + l_{m+1} + \ldots + l_2 q^{1-m} + lq^{-m}.$$

Thus the equation

$$\frac{1}{3}(4^{m+1} - 1) + p = l_{m+1} + \ldots + l_2 4^{m-1} + l.4^m$$

should have a solution, which is possible if

$$l.4^m \leq \frac{4^{m+1} - 1}{3} + p - \frac{4^m - 1}{3} = p + 4^m.$$

Therefore, $\frac{p}{4^m} + 1$ is an upper bound for l — let us denote by $u(p, m)$ the greatest integer less or equal to $\frac{p}{4^m} + 1$.

Now we start computing the values $S(p, m)$:

$m = 1$: We have to find all the solutions for the equation $qx + l_1 q = Q = q(Q+1)$, which is equivalent to $Q - p + l_1 = Q + 1$. The only possible choice is $l_1 = p + 1$, therefore

$$S(p, 1) = (-1)^{p+1}(p+1).$$

$m = 2$: Suppose that $p = 8r + \alpha$, where $r \in \mathbb{Z}$, $r \geq 0$, and $\alpha \in \{0, 1, 2, \ldots, 7\}$. Then the upper bound $u(p, 1)$ is $2r + 1$ for $\alpha \in \{0, 1, 2, 3)$ and $2r + 2$ for $\alpha \in (4, 5, 6, 7)$.

$\alpha \in \{0,1,2,3\}$:

$$S(p,2) = \sum_{l=1}^{2r+1} (-1)^l . l . (8r + \alpha - 4(l-1) + 1) = (-\alpha - 1)(r+1).$$

$\alpha \in \{4,5,6,7\}$:

$$S(p,2) = \sum_{l=1}^{2r+2} (-1)^l . l . (8r + \alpha - 4(l-1) + 1) = (\alpha - 7)(r+1).$$

Thus

$$S(p,2) = (r+1)\varphi(\alpha),$$

where

$$\varphi(\alpha) = \begin{cases} -\alpha - 1 & \text{for } \alpha \in \{0,1,2,3\} \\ \alpha - 7 & \text{for } \alpha \in \{4,5,6,7\}. \end{cases}$$

$m = 3$: As $p = 8r + \alpha$, we now suppose that $r = 4s + \delta$ ($s \in \in \mathbb{Z}$, $s \geq 0$, $\delta \in \{0,1,2,3\}$). Then $u(p,2) = 2s + 1$ for $\delta \in \{0,1\}$, and $u(p,2) = 2s + 2$ for $\delta \in \{2,3\}$.

$\delta \in \{0,1\}$:

$$\begin{aligned} S(p,3) &= \sum_{l=1}^{2s+1} (-1)^l . l . S(p - 16(l-1), 2) = \\ &= \sum_{l=1}^{2s+1} (-1)^l . l . S(8r + \alpha - 8.2(l-1), 2) = \\ &= \sum_{l=1}^{2s+1} (-1)^l . l . (r - 2(l-1) + 1)\varphi(\alpha) = \\ &= \sum_{l=1}^{2s+1} (-1)^l . l . (4s + \delta + 3 - 2l)\varphi(\alpha) = -(\delta + 1)(s+1)\varphi(\alpha). \end{aligned}$$

$\delta \in \{2,3\}$:

$$\begin{aligned} S(p,3) &= \sum_{l=1}^{2s+2} (-1)^l . l . S(p - 16(l-1), 2) = \\ &= \sum_{l=1}^{2s+2} (-1)^l . l . (4s + \delta + 3 - 2l)\varphi(\alpha) = (\delta - 3)(s+1)\varphi(\alpha). \end{aligned}$$

Thus

$$S(m,3) = (s+1)\varphi(\alpha)\psi(\delta),$$

where ψ is given by

$$\psi(\delta) = \begin{cases} -\delta - 1 & \text{for } \delta \in \{0,1\} \\ \delta - 3 & \text{for } \delta \in \{2,3\} \end{cases}.$$

Using the sequences $\alpha(p)$ and $\beta(p)$ defined in the statement of this theorem, we have

$$S(p,1) = (-1)^{m+1}(m+1), \quad S(p,2) = (\beta_0 + 1)\varphi(\alpha_0)$$

$$S(p,3) = (\beta_1 + 1)\psi(\alpha_1)\varphi(\alpha_0).$$

Now we may proceed by induction:

$$S(p, k+2) = (\beta_k + 1)\psi(\alpha_k)\dots\psi(\alpha_2)\psi(\alpha_1)\varphi(\alpha_0) \quad \text{for } k \geq 1.$$

This formula is true for $k = 1$, and $k \to k+1$: By definition of the sequences $\alpha(p)$ and $\beta(p)$ the bound $u = u(p, k+2)$ is given by $2\beta_{k+1}+1$ in the case $\alpha_{k+1} \in \{0,1\}$ and by $2\beta_{k+1} + 2$ in the case $\alpha_{k+1} \in \{2,3\}$. Thus we have

$$S(p, k+3) = \sum_{l=1}^{u}(-1)^l.l.S(p - 4^{k+2}(l-1), k+2) =$$

$$= \sum_{l=1}^{u}(-1)^l.l.(4\beta_{k+1} + 3 + \alpha_{k+1} - 2l)\psi(\alpha_k)\dots\psi(\alpha_2)\psi(\alpha_1)\varphi(\alpha_0) =$$

(by the same computation as before)

$$= (\beta_{k+1} + 1)\psi(\alpha_{k+1})\psi(\alpha_k)\dots\psi(\alpha_2)\psi(\alpha_1)\varphi(\alpha_0).$$

From the last formula it follows that $S(p,m) = 0$, if and only if $\varphi(\alpha_0) = 0$ or $\psi(\alpha_i) = 0$ for some $i \geq 1$. The first is fulfilled iff $\alpha_0 = 7$, the latter is fulfilled iff $\alpha_i = 3$ for some $i \geq 1$, which proves the theorem. $\Diamond$

Remark 3. The first elements of the set L in the preceding theorem are given by $L = \{7, 15, 23, 24, 25, 26, 27, 28, 29, 30, 31, 39, 47, 55, 56, 57, 58, 59, 60, 61, 62, 63, 71, 79, 87, 88, 89, 90, 91, 92, 93, 94, 95, 96, 97, 98, 99, \dots\}$.

The following example is very easy to construct and shows in the case $q = \frac{1}{4}$ that the point $Q - 7$ really is a point of continuity, though the defining function h (and the value α) at the beginning do not look as if this were true.

Example. Let $q = \frac{1}{4}$, then $Q = \frac{1}{3}$, and let h be defined on $]Q-1, qQ] =]-\frac{2}{3}, \frac{1}{12}]$ to be constant equal to 1, and let $\alpha \in \mathbb{R}$ be arbitrary. It is evident that h fulfills the conditions of Th. 11. By the construction of Th. 5 f is given by

$$f(x) = \begin{cases} 0 & \text{for } x \in]Q, \infty[\\ \alpha & \text{for } x = Q \\ 1 & \text{for } x \in]Q-1, Q[\\ 1-2\alpha & \text{for } x = Q-1 \\ -1 & \text{for } x \in]Q-2, Q-1[\\ -1+3\alpha & \text{for } x = Q-2 \\ 2 & \text{for } x \in]Q-3, Q-2[\\ 2-4\alpha & \text{for } x = Q-3 \\ -2 & \text{for } x \in]Q-4, Q-3[\\ -2+3\alpha & \text{for } x = Q-4 \\ 1 & \text{for } x \in]Q-5, Q-4[\\ 1-2\alpha & \text{for } x = Q-5 \\ -1 & \text{for } x \in]Q-6, Q-5[\\ -1+\alpha & \text{for } x = Q-6 \\ 0 & \text{for } x \in]Q-7, Q-6[\\ 0 & \text{for } x = Q-7 \\ 0 & \text{for } x \in]Q-8, Q-7[\\ 2\alpha & \text{for } x = Q-8 \\ 2 & \text{for } x \in]Q-9, Q-8[\\ \dots\dots & \dots \\ \dots\dots & \dots \end{cases}$$

It is clear that f is continuous at the point $Q-7$, but not continuous at the points Q, $Q-1$, $Q-2$, $Q-3$, $Q-4$, $Q-5$, $Q-6$, $Q-8$.

After this example we close this section on continuous solutions and turn over to

c) Differentiable solutions

Like in the continuous case, the solutions without any boundary conditions are much easier to handle, and for the proofs we again use the constructions of the solutions given in Ths. 3 and 5.

Theorem 12 (The case $S(f) \subseteq \mathbb{R}$). *Let p be a positive integer,*

$h\colon [-1,1[\to \mathbb{R}$ an arbitrary function, and let f be the unique solution of (1) which coincides with h on $[-1,1[$. Then f is p-times differentiable (resp. p-times continuously differentiable) iff h is continuous, continuously extendable to $H\colon [-1,1] \to \mathbb{R}$ and H is p-times differentiable (resp. p-times continuously differentiable) and fulfills the following system of equations:

$$\begin{cases} 4q.H(0) = H(-1) + H(1) + 2H(0) \\ 4q.q.H'(0) = H'(-1) + H'(1) + 2H'(0) \\ 4q.q^2.H''(0) = H''(-1) + H''(1) + 2H''(0) \\ \dots\dots\dots\dots \\ 4q.q^p.H^{(p)}(0) = H^{(p)}(-1) + H^{(p)}(1) + 2H^{(p)}(0). \end{cases}$$

Remark: As H is defined on the closed interval $[-1,1]$, differentiability is to be understood in the following sense: At any point in the open interval $]-1,1[$ the derivative $H'(x)$ exists, at the point 1 the left derivative of H exists, and at the point -1 the right derivative of H exists. In the case of continuous differentiability this function H' has to be continuous on the whole interval $[-1,1]$ and similarly for derivatives of higher order.

Proof. We use the notations f_n and F_n of Th. 3.

"only if": f is a continuous solution, which coincides with h on $[-1,1[$. Therefore h has to be continuously extendable to the function $H\colon [-1,1] \to \mathbb{R}$. Furthermore, f is p-times differentiable (resp. p-times continuously differentiable) and fulfills the equation

$$4q.f(qx) = f(x-1) + f(x+1) + 2f(x) \quad \text{for all} \quad x \in \mathbb{R}.$$

Differentiating this equation with respect to x up to p times and putting $x = 0$ gives the system of equations for h. As H is the restriction of f to the interval $[-1,1]$ it is clear that H has to be p-times differentiable (resp. p-times continuously differentiable).

"if": Suppose that h fulfills the conditions of the theorem. Th. 8 guarantees that the unique solution f is continuous (from the first equation of the system for H). Thus we only have to show that this solution is p-times differentiable (resp. p-times continuously differentiable). Now let $f_1 := h\colon [-1,1[\to \mathbb{R}$. If f_n is given on $[-1,n[$ for some nonnegative integer n, then f_{n+1} is defined on $[-1,n+1[$ by

$$f_{n+1}(x) := \begin{cases} f_n(x) & \text{for } x \in [-1, n[\\ 4qf_n(q(x-1)) - f_n(x-2) - 2f_n(x-1) & \text{otherwise.} \end{cases}$$

As f_n is supposed to be p-times differentiable (resp. p-times continuously differentiable) by induction hypothesis, we only have to show that f_{n+1} is p-times differentiable (resp. p-times continuously differentiable) at the point n (in the neighbourhood of any other point f_{n+1} is given as a composition of p-times differentiable (resp. p-times continuously differentiable) functions):

Case $n = 1$. Left side: Here we have $f_2'(1) = H'(1), \dots, f_2^{(k)}(1) = H^{(k)}(1)$, because f_2 is continuous and coincides with h on $[-1, 1[$ and therefore with H on $[-1, 1]$. Right side: We have to take the right derivatives of the defining expression:

$$f_2^{(k)}(1) = 4q.q^k.f_1^{(k)}(0) - f_1^{(k)}(-1) - 2f_1^{(k)}(0) =$$

$$= 4q.q^k.H^{(k)}(0) - H^{(k)}(-1) - 2H^{(k)}(0) = H^{(k)}(1).$$

Thus in this case the right and left derivatives are identical. In the case of p-times continuous differentiability we have to show that $f_2^{(p)}$ is continuous at the point 1. From the left: $H^{(p)}$ is continuous on the left at 1, and therefore also $f_2^{(p)}$. From the right: As $H^{(p)}$ is continuous on the right at 0 and -1, the definition of f_2 shows that $f_2^{(p)}$ is also continuous on the right at 1.

Case $n \geq 2$. The crucial point is that for $x < n$ and $x \geq n$ we have two different expressions defining the value of $f_{n+1}(x)$, but we can very well use the fact that any two of the functions (f_n) coincide on the intersection of their domains:

$n \leq x < n+1$: $f_{n+1}(x) = 4qf_n(q(x-1)) - f_n(x-2) - 2f_n(x-1)$;
$n-1 < x < n$: $f_{n+1}(x) = f_n(x) = 4qf_{n-1}(q(x-1)) - f_{n-1}(x-2) - 2f_{n-1}(x-1) = 4qf_n(q(x-1)) - f_n(x-2) - 2f_n(x-1)$.

As these two expressions are identical and f_n is p-times differentiable (resp. p-times continuously differentiable) on its domain, we immediately get that f_{n+1} is also p-times differentiable (resp. p-times continuously differentiable) at the point n.

Thus each of the functions f_n is p-times differentiable (resp. p-times continuously differentiable) and, therefore, the resulting function $F_1: [-1, \infty[\to \mathbb{R}$ is p-times differentiable (resp. p-times continuously differentiable).

The next step is dealing with the functions F_n defined on $[-n, \infty[$ which are given by

$$F_{n+1}(x) := \begin{cases} F_n(x) & \text{for } x \in [-n, \infty[\\ 4qF_n(q(x+1)) - F_n(x+2) - 2F_n(x+1) & \text{otherwise.} \end{cases}$$

Like in the case of the functions f_n here we have to verify that F_{n+1} is p-times differentiable (resp. p-times continuously differentiable) at the point $-n$.

Case $n = 1$. Right side: $F_2^{(k)}(-1) = F_1^{(k)}(-1) = H^{(k)}(-1)$. Left side: $F_1(-1) = H(-1) = 4qH(0) - H(1) - 2H(0)$. For $-2 < x < -1$ we have

$$F_2(x) = 4qF_1(q(x+1)) - F_1(x+2) - 2F_1(x+1) = \\ = 4qH(q(x+1)) - H(x+2) - 2H(x+1).$$

Thus it is possible to compute the left derivatives

$$F_2^{(k)}(-1) = 4q.q^k.H^{(k)}(0) - H^{(k)}(1) - 2H^{(k)}(0) = H^{(k)}(-1).$$

As in the case of f_2 the continuity of $F_2^{(p)}$ at the point -1 in the case of p-times continuous differentiability immediately follows by the same arguments.

Case $n \geq 2$.

$-n-1 < x < -n$: $F_{n+1}(x) = 4qF_n(q(x+1)) - F_n(x+2) - 2F_n(x+1)$.

$-n \leq x < -n+1$: $F_{n+1}(x) = F_n(x) = 4qF_{n-1}(q(x+1)) - F_{n-1}(x+2) - 2F_{n-1}(x+1) = 4qF_n(q(x+1)) - F_n(x+2) - 2F_n(x+1)$.

Thus we have one expression for all arguments x such that $-n-1 < x < -n+1$, which is p-times differentiable (resp. p-times continuously differentiable) because F_n is supposed to be p-times differentiable (resp. p-times continuously differentiable).

Thus the solution defined by h is p-times differentiable (resp. p-times continuously differentiable). $\Diamond$

The question of differentiable solutions in the case that $S(f) \subseteq]-\infty, Q]$ is similar to handle. An answer in this case can be given, the proofs are very similar to the case of continuous solutions.

Theorem 13. *Let* $h\colon]Q-1, qQ] \to \mathbb{R}$ *be an arbitrary function, and* α *a real number which is arbitrary in the case* $q = \frac{1}{4}$ *and* 0 *otherwise, and let* f *be the unique solution which extends* h *and fulfills* $f(Q) = \alpha$, $S(f) \subseteq]-\infty, Q]$. *Furthermore, let* r *be a natural number. Then* f *is* r*-times differentiable (resp.* r*-times continuously differentiable) on the set* $\mathbb{R} \setminus E(q)$ *if and only if the function* h *fulfills the following conditions:*

(a) $h\colon]Q-1, qQ] \to \mathbb{R}$ *is continuous;*

(b) h *is continuously extendable to a function* $H\colon [Q-1, qQ] \to \mathbb{R}$, *where* (b1) *$H$ is r-times differentiable (resp. r-times continuously differentiable),*

(b2) *H fulfills the following conditions:*

$$\begin{cases} H(Q-1) = 4q.H(qQ) \\ H'(Q-1) = 4q^2.H'(qQ) \\ \quad \cdots\cdots \\ H^{(r)}(Q-1) = 4q^{r+1}.H^{(r)}(qQ). \end{cases}$$

Remark. As H is defined on the closed interval $[Q-1, qQ]$, differentiability is to be understood in the following way: At any point in the open interval $]Q-1, qQ[$ the derivative $H'(x)$ exists, at the point qQ the left derivative of H exists, and at the point $Q-1$ the right derivative of H exists. In the case of continuous differentiability this function H' has to be continuous on the whole interval $[Q-1, qQ]$ — and similarly for derivatives of higher order.

Proof. "only if": Suppose that f is r-times differentiable (resp. r-times continuously differentiable) on the set $\mathbb{R} \setminus E(q)$. As this set contains the interval $]Q-1, Q[$, we immediately get:

(a) $h = f \,|]Q-1, qQ]$ is continuous.

(b) $\lim\limits_{x \searrow Q-1} h(x) = \lim\limits_{x \searrow Q-1} f(x) = \lim\limits_{x \searrow Q-1} 4q.f(q(x+1)) = \lim\limits_{x \searrow qQ} 4q.f(x) = 4q.f(qQ) = 4q.h(qQ)$. Thus h can be extended continuously to a function h on $[Q-1, qQ]$ which fulfills $H(Q-1) = 4qH(qQ)$.

(b1) We have $H(x) = 4q.f(q(x+1))$ and $q(x+1) \in [qQ, q(qQ+1)] \subseteq]Q-1, Q[$ for each $x \in [Q-1, qQ]$. As f is r-times differentiable (resp. r-times continuously differentiable) on the interval $]Q-1, Q[$, it is evident that the same holds for H.

(b2) From the formula $H(x) = 4q.f(q(x+1))$ we immediately get $H'(Q-1) = 4q^2.f'(qQ) = 4q^2.H'(qQ)$, and by induction for any integer k, $1 \le k \le r$: $H^{(k)}(Q-1) = 4q^{k+1}.f^{(k)}(qQ) = 4q^{k+1}.H^{(k)}(qQ)$.

"if": Suppose that h fulfills conditions (a) and (b). Like in Th. 9 first we show that the solution f is r-times differentiable (resp. r-times continuously differentiable) on the interval $]Q-1, Q[$. Let (as in Th. 5 resp. Th. 9) $x_0 := Q-1$, $x_{n+1} := q(x_n+1)$. From Th. 9 it follows that f is continuous on the interval $]Q-1, Q[$, and from the construction it is evident that f is r-times differentiable (resp. r-times continuously differentiable) on each of the open intervals $]x_n, x_{n+1}[$ for $n \geq 0$. Furthermore, the conditions (b2) immediately show that the left derivatives (of order k, $1 \leq k \leq r$) at the point qQ coincide with the right derivatives at this point. If H is r-times continuously differentiable, the left and the right limits of $f^{(r)}(x)$ coincide with $H^{(r)}(qQ) = f^{(r)}(qQ)$ when x tends to qQ. Thus f has the desired properties on the interval $]x_0, x_2[$. Now we may proceed by induction: Suppose that f is r-times differentiable (resp. r-times continuously differentiable) on the interval $]x_0, x_n[$ $(n \geq 2)$. Then f is given on the interval $]x_1, x_{n+1}[$ by the formula $f(x) = \frac{1}{4q}.f\left(\frac{x}{q} - 1\right)$, where the right hand side uses arguments of the interval $]x_0, x_n[$. Thus f is r-times differentiable (resp. r-times continuously differentiable) on the interval $]x_1, x_{n+1}[$, and as the intersection of $]x_0, x_n[$ and $]x_1, x_{n+1}[$ is nonvoid, f has this property on the interval $]x_0, x_{n+1}[$. We may conclude that f is r-times differentiable (resp. r-times continuously differentiable) on the interval $]Q-1, Q[$ and, therefore, on the set $]Q-1, \infty[\backslash\{Q\}$. For further investigations on f we use the formula derived in Th. 10: Let be $x < 0$, let $k \in \mathbb{N}$ be such that $x + k > Q$, and let $m \in \mathbb{N}$ be such that $q^m x + q^m + \ldots + q > Q - 1$. Then

$$f(x) = (-4q)^m \cdot \sum_{l_1, \ldots, l_m = 1}^{k} (-1)^{l_1 + \ldots + l_m}.l_1 \ldots l_m.f(q^m x + l_m q^m + \ldots + l_1 q).$$

The right-hand-side expression is a finite sum of terms, where each argument depends continuously on x and is contained in the interval $]Q-1, \infty[$. Thus if none of these arguments is equal to Q, this property holds in a whole neighbourhood of x, and f is given in this neighbourhood as a finite sum of r-times differentiable (resp. r-times continuously differentiable) expressions. Therefore, f is r-times differentiable (resp. r-times continuously differentiable) in this neighborhood. On the other hand, from Prop. (1) we know that the set where at least one of these arguments in the right-hand-side expression is equal to Q is the set $E(q)$, which proves the statement that the solution f is r-times differentiable (resp. r-times continuously differentiable) on the set $\mathbb{R}\backslash E(q)$. $\Diamond$

The preceding theorem gives two possibilities to make the set of points where f is not r-times differentiable "small":

— f is r-times differentiable at Q or

— the sum of coefficients at $f(Q)$ (used in the proof of Th. 13 in the sum expression for $f(x)$) is equal to 0.

A precise answer will be given in the following two theorems.

Theorem 14. *Let h and α be as in* Th. 13 *and suppose that h fulfills conditions* (a) *and* (b). *Then the solution f is r-times differentiable (resp. r-times continuously differentiable) at the point Q, if and only if h is identically* 0 *and $\alpha = 0$, or q fulfills the condition $4q^{r+1} > 1$.*

Proof. We may assume that h is nonvanishing, and we give the proof by induction.

$r = 1$: First suppose that f is differentiable at Q. Then f is continuous at Q, and we have $\alpha = 0$ and $4q > 1$ by Th. 9. Let $z \in]Q-1, qQ]$ be arbitrary such that $h(z) \neq 0$. Then the sequence $z_0 := z$, $z_{n+1} := q(z_n + 1)$ tends to Q, and from the right derivative at Q we have $f'(Q) = 0$. Thus

$$0 = f'(Q) = \lim_{n\to\infty} \frac{f(z_n) - f(Q)}{z_n - Q}.$$

Now

$$f(z_{n+1}) - f(Q) = f(z_{n+1}) = f(q(z_n+1)) = \frac{1}{4q} f(z_n) = \frac{1}{4q}(f(z_n) - f(Q))$$

and

$$z_{n+1} - Q = q(z_n + 1) - q(Q+1) = q(z_n - Q),$$

which implies

$$\frac{f(z_n) - f(Q)}{z_n - Q} = \left(\frac{1}{4q^2}\right)^n \cdot \frac{f(z) - f(Q)}{z - Q}.$$

This sequence tends to 0 iff $4q^2 > 1$.

Now suppose that $4q^2 > 1$. Then $4q > 1$, which implies that f is continuous at Q. We have to show that the left derivative of f at Q is equal to 0 and — in the case of continuous differentiability — that $\lim_{x \nearrow Q} f'(x) = 0$. As in Th. 5, let $x_0 = Q-1$, $x_{n+1} = q(x_n+1)$, and let M be a bound for the function h on $]Q-1, qQ] =]x_0, x_1]$. (Such a bound exists because h is continuously extendable to the compact interval $[x_0, x_1]$.) Then f is bounded by $\left(\frac{1}{4q}\right)^n . M$ on the interval $]x_n, x_{n+1}]$. Now suppose that $(z_n)_{n\in\mathbb{N}}$ is an arbitrary, strictly increasing sequence tending to Q. Without loss of generality we may assume that $z_0 > Q - 1$. Then for each $n \in \mathbb{N}$ there is a unique $m \in \mathbb{N}$ such that $z_n \in]x_m, x_{m+1}]$. Let us denote this m by $m(n)$. Then $\lim_{n\to\infty} m(n) = \infty$. Furthermore,

$$|z_n - Q| \geq |x_{m(n)+1} - Q| = q^{m(n)}|x_1 - Q|,$$

and

$$|f(z_n) - f(Q)| = |f(z_n)| \leq \left(\frac{1}{4q}\right)^{m(n)} M.$$

Thus

$$\left|\frac{f(z_n) - f(Q)}{z_n - Q}\right| \leq \frac{\left(\frac{1}{4q}\right)^{m(n)} M}{q^{m(n)}|x_1 - Q|} = \left(\frac{1}{4q^2}\right)^{m(n)} \cdot \frac{M}{Q - x_1},$$

which goes to 0 when n tends to infinity. Further, if H is continuously differentiable then H' is bounded by some constant N on the interval $[x_0, x_1]$. As f fulfills the equation $f(x) = 4qf(q(x+1))$ in the interval $]x_0, Q[$ we immediately get $f'(x) = 4q^2 f'(q(x+1))$ in this interval. Thus f' is bounded by $\left(\frac{1}{4q^2}\right)^n N$ on the interval $[x_n, x_{n+1}]$. By the same arguments as before we can conclude that $\lim_{x \nearrow Q} f'(x) = 0$.

Now the step $r \to r+1$: Suppose that f is $(r+1)$-times differentiable in $]Q-1, \infty[$. Then f is r-times differentiable at Q, which implies $4q > 4q^2 > \ldots > 4q^{r+1} > 1$. As the function H fulfills $H^{(k)}(Q-1) = 4q^{k+1} H^{(k)}(qQ)$ for any k, $0 \leq k \leq r$, each of these functions $H^{(k)}$ is either identically 0 or nonconstant. Nonconstant differentiable functions have a nonvanishing derivative, and, therefore, if h is nonvanishing, the only possibility is that $H^{(r)}$ is nonconstant. Thus there is a point $z \in]x_0, x_1]$ such that $f^{(r)}(z) = H^{(r)}(z) \neq 0$. Once more we use the sequence $z_0 := z$, $z_{n+1} := q(z_n + 1)$. From $f(x) = 4qf(q(x+1))$ we derive $f^{(r)}(x) = 4q^{r+1} f^{(r)}(q(x+1))$, especially $f^{(r)}(z_n) = 4q^{r+1} f^{(r)}(z_{n+1})$, which implies

$$f^{(r)}(z_n) = \left(\frac{1}{4q^{r+1}}\right)^n \cdot f^{(r)}(z).$$

Comparing the right and left derivative of $f^{(r)}$ at the point Q, we get

$$0 = f^{(r+1)}(Q) = \lim_{n \to \infty} \frac{f^{(r)}(z_n) - f^{(r)}(Q)}{z_n - Q} = \lim_{n \to \infty} \left(\frac{1}{4q^{r+2}}\right)^n \cdot \frac{f^{(r)}(z)}{z - Q}.$$

As $f^{(r)}(z) \neq 0$, this limit is equal to 0 iff $4q^{r+2} > 1$. On the other hand, suppose that $4q^{r+2} > 1$. Then $4q^{r+1} > 1$, and as H is $(r+1)$-times differentiable in $[Q-1, qQ]$, the r-th derivative $H^{(r)}$ is continuous on $[Q-1, qQ]$ and therefore bounded by a constant M. With the same arguments as before we can conclude that $f^{(r)}$ is bounded by $\left(\frac{1}{4q^{r+1}}\right)^n M$ on the interval $]x_n, x_{n+1}]$. Now suppose that $(z_n)_{n \in \mathbb{N}}$ is an arbitrary strictly increasing sequence tending to Q. Like before, we may assume that $z_0 > Q-1$, and denote by $m(n)$ the unique $m \in \mathbb{N}$ such that $z_n \in]x_m, x_{m+1}]$. Then

$$\left|\frac{f^{(r)}(z_n) - f^{(r)}(Q)}{z_n - Q}\right| \leq \frac{\left(\frac{1}{4q^{r+1}}\right)^{m(n)} M}{q^{m(n)}|x_1 - Q|} = \left(\frac{1}{4q^{r+2}}\right)^{m(n)} \cdot \frac{M}{Q - x_1}.$$

As $4q^{r+2} > 1$ we immediately get that $f^{(r+1)}(Q)$ exists and is equal to 0.

Further, if H is $(r+1)$-times continuously differentiable then $H^{(r+1)}$ is bounded by some constant N on the interval $[x_0, x_1]$. Like before we get the equation $f^{(r+1)}(x) = 4q^{r+2} f^{(r+1)}(q(x+1))$ in the interval $]x_0, Q[$. Thus $f^{(r+1)}$ is bounded by $\left(\frac{1}{4q^{r+2}}\right)^n N$ on the interval $[x_n, x_{n+1}]$. From this we can conclude that $\lim\limits_{x \nearrow Q} f^{(r+1)}(x) = 0$. ◊

Corollary 1. *The only solution f of equation* (1) *which fulfills $S(f) \subseteq$ $\subseteq] -\infty, Q]$ and which is C^∞ on $\mathbb{R}$ is the zero function.*

Proof. The preceding theorem shows that for a nonvanishing C^∞-solution the inequality $4q^{r+1} > 1$ has to be fulfilled for any natural number r. But this is impossible because $0 < q < 1$. ◊

Corollary 2. *Let h and α be as in* Th. 13 *and suppose that h fulfills conditions* (a) *and* (b). *Then the solution f is r-times differentiable (resp. r-times continuously differentiable) on the whole real line, if and only if h is identically* 0 *and $\alpha = 0$, or q fulfills the condition $4q^{r+1} >$ > 1.*

Proof. First suppose that f is r-times differentiable on $\mathbb{R}$. Then f is r-times differentiable at Q, which implies (by Th. 14) that $4q^{r+1} > 1$. On the other hand, suppose that $4q^{r+1} > 1$. By Th. 14, f is r-times differentiable (resp. r-times continuously differentiable) in the interval $]Q-1, \infty[$. Thus from the formula

$$f(x) = (-4q)^m \cdot \sum_{l_1, \ldots, l_m = 1}^{k} (-1)^{l_1 + \ldots + l_m} . l_1 \ldots l_m . f(q^m x + l_m q^m + \ldots + l_1 q)$$

of Th. 13, where $x < Q$, $x + k > Q$, and m is chosen large enough such that the arguments on the right side are greater than $Q - 1$, we immediately get that f is r-times differentiable (resp. r-times continuously differentiable) on the whole real line. ◊

Theorem 15. *Let h and α be as in* Th. 13, *and suppose that h fulfills conditions* (a) *and* (b) *and $4q^{r+1} \leq 1$. Then the solution f is r-times* *differentiable (resp. r-times continuously differentiable) on the set $C(q)$, and not r-times differentiable at the points of the set $E(q) \setminus C(q)$.*

Proof. In Th. 13 it was proved that f is r-times differentiable (resp. r-times continuously differentiable) in the set $\mathbb{R} \setminus E(q)$. Now let $x < Q$, $x \in E(q)$, and choose m large enough such that $q^m x + q^m + \ldots + q > Q - 1$. Furthermore, choose an integer k such that $x + k > Q$ and $M(q, x, m) \subseteq \{1, \ldots, k\}^m$ as in Th. 10. As in the stated theorem, let $M := M(q, x, m)$, $P := \{1, 2, \ldots, k\}^m \setminus M$. As in Th. 10, in a neighbourhood of x the formula

$$f(y) = (-4q)^m \sum_{l_1, \ldots, l_m = 1}^{k} (-1)^{l_1 + \ldots + l_m} . l_1 \ldots l_m . f(q^m y + l_m q^m + \ldots + l_1 q) =$$

$$= (-4q)^m . \sum_{(l_1, \ldots, l_m) \in M} (-1)^{l_1 + \ldots + l_m} . l_1 \ldots l_m . f(q^m y + l_m q^m + \ldots + l_1 q) +$$

$$+ (-4q)^m . \sum_{(l_1, \ldots, l_m) \in P} (-1)^{l_1 + \ldots + l_m} . l_1 \ldots l_m . f(q^m y + l_m q^m + \ldots + l_1 q) =$$

$$= (-4q)^m . \sum_{(l_1, \ldots, l_m) \in P} (-1)^{l_1 + \ldots + l_m} . l_1 \ldots l_m . f(q^m y + l_m q^m + \ldots + l_1 q) +$$

$$+ (-4q)^m . f\big(q^m (y - x) + Q\big) . S(q, x, m)$$

holds. The first sum gives f in this neighbourhood of x as a finite sum of r-times differentiable (resp. r-times continuously differentiable) terms, thus the second summand makes the decision whether f is r-times differentiable (resp. r-times continuously differentiable) at the point x.

Case 1: $x \in C(q)$. In this case $S(q, x, m) = 0$, thus f is r-times differentiable (resp. r-times continuously differentiable) at x.

Case 2: $x \notin C(q)$. In this case $S(q, x, m) \neq 0$. As f is not r-times differentiable at Q, it cannot be r-times differentiable at x. $\Diamond$

After these investigations on differentiable solutions we turn over to measurable and integrable solutions. However, one question has not been discussed because it is still unsolved: Do there exist analytic solutions (of course, only in the case $S(f) \subseteq \mathbb{R}$)?

d) Measurable solutions

In this section we deal with measurability in the sense of Borel or Lebesgue (i.e., the term "measurable" is to be understood in this sense). The results are very simple:

Theorem 16 (the case $S(f) \subseteq \mathbb{R}$). *Let $h\colon [-1,1[\to \mathbb{R}$ be an arbitrary function, and let f be the unique solution of equation* (1) *which coincides with h on $[-1,1[$. Then f is measurable if and only if h is measurable.*

Proof. "only if" is obvious. "if": Suppose that h is measurable.Then from the construction given in Th. 3 and from the σ-additivity of the measure it follows immediately that f is measurable. $\Diamond$

A similar result holds for the case $S(f) \subseteq]-\infty, Q]$:

Theorem 17 (the case $S(f) \subseteq]-\infty, Q]$). *Let $h\colon]Q-1, qQ] \to \mathbb{R}$ be an arbitrary function and α a real number which is arbitrary in the case $q = \frac{1}{4}$ and* 0 *otherwise, and let f be the unique solution of equation* (1) *which coincides with h on $]Q-1, qQ]$ and fulfills $f(Q) = \alpha$, $S(f) \subseteq]-\infty, Q]$. Then f is measurable if and only if h is measurable.*

Proof. Like that of Th. 16; the construction of f from h has been given in Th. 5. $\Diamond$

More interesting than these "trivial results" on measurable solutions are the following about

e) Integrable solutions

It will be shown that the vector space of integrable solutions for a given number q is at most of dimension 1 over the field of reals. Furthermore, the very interesting result is that any integrable solution has bounded support, in other words, for any integrable solution $S(f) \subseteq [-Q, Q]$ holds. Thus the result on the dimension of this space of solutions follows immediately from the theorem of Baron and Volkmann [1]. We prepare the results by a lemma:

Lemma 3. (α) *Let f be a Lebesgue (resp. Borel)-integrable solution of equation* (1). *Then the function $F(x) := \int_{]-\infty,x]} f\, d\lambda$ (λ represents the usual Borel resp. Lebesgue measure) is well defined and has the properties:*

(i) *F is continuous,*

(ii) $\lim_{x \to -\infty} F(x) = 0$,

(iii) $\lim_{x \to \infty} F(x) = \int_{\mathbb{R}} f\, d\lambda \in \mathbb{R}$,

(iv) $F(qx) = \frac{1}{4}(F(x+1) + F(x-1) + 2F(x))$ *for any* $x \in \mathbb{R}$.

(β) *Let f be a solution of equation (1) whose improper Riemann integral over $\mathbb{R}$ exists. Then the function $F(x) := \int_{-\infty}^{x} f(t)\,dt$ is well defined and has the properties:*

(i) *F is continuous,*

(ii) $\lim_{x\to-\infty} F(x) = 0$,

(iii) $\lim_{x\to\infty} F(x) = \int_{-\infty}^{\infty} f(t)\,dt \in \mathbb{R}$,

(iv) $F(qx) = \frac{1}{4}(F(x+1) + F(x-1) + 2F(x))$ *for any* $x \in \mathbb{R}$.

Proof. (α): (i), (ii), (iii) are well-known from elementary integration theory (e.g., cf. Hewitt–Stromberg [3]). (iv) can be derived as follows:

$$\frac{1}{4}(F(x+1) + F(x-1) + 2F(x)) =$$

$$= \frac{1}{4}\Big(\int_{]-\infty,x+1]} f\,d\lambda + \int_{]-\infty,x-1]} f\,d\lambda + 2\int_{]-\infty,x]} f\,d\lambda\Big) =$$

$$= \frac{1}{4}\Big(\int_{]-\infty,x]} f(\xi+1)\,d\lambda(\xi) + \int_{]-\infty,x]} f(\xi-1)\,d\lambda(\xi) + 2\int_{]-\infty,x]} f(\xi)\,d\lambda(\xi)\Big) =$$

$$= \frac{1}{4}\int_{]-\infty,x]} (f(\xi+1) + f(\xi-1) + 2f(\xi))d\lambda(\xi) =$$

$$= \frac{1}{4}\int_{]-\infty,x]} 4qf(q\xi)\,d\lambda(\xi) = \int_{]-\infty,qx]} f(\xi)\,d\lambda(\xi) = F(qx).$$

(β): (i) is well-known from elementary analysis, (ii) and (iii) are immediate consequences of the definition of the Riemann integral from $-\infty$ to ∞. (iv) can be computed similarly to the case (α):

$$F(qx) = \int_{-\infty}^{qx} f(t)\,dt = \int_{-\infty}^{x} f(qt)\,q dt =$$

$$= q\int_{-\infty}^{x} \frac{1}{4q}(f(t+1) + f(t-1) + 2f(t))dt =$$

$$= \frac{1}{4}\Big(\int_{-\infty}^{x+1} f(t)\,dt + \int_{-\infty}^{x-1} f(t)\,dt + 2\int_{-\infty}^{x} f(t)\,dt\Big) =$$

$$= \frac{1}{4}(F(x+1) + F(x-1) + 2F(x)). \quad \Diamond$$

Theorem 18. *Let f be a solution of equation (1) which is Riemann (resp. Borel- resp. Lebesgue-) integrable and let the value of this integral* *be* 0. *Then* f *vanishes almost everywhere (in the Borel resp. Lebesgue case) resp.* f *is equal to* 0 *except on a zero set (in the Riemann case).*

Proof. We use the function F defined in Lemma 3. Then it follows that F fulfills the following conditions:

(a) F is continuous,

(b) $\lim_{x\to-\infty} F(x) = 0 = \lim_{x\to\infty} F(x)$,

(c) $F(qx) = \frac{1}{4}(F(x+1) + F(x-1) + 2F(x))$ for any $x \in \mathbb{R}$.

As F is continuous and tends to 0 when x tends to $\pm\infty$, it has a maximum value M at some point x_0. Now let $x_0 = qy_0$. Then

$$M = F(x_0) = F(qy_0) = \frac{1}{4}(F(y_0+1) + F(y_0-1) + 2F(y_0)).$$

As M is maximal, this equality can only hold if

$$F(y_0+1) = F(y_0-1) = F(y_0) = F(qy_0) = F(x_0) = M.$$

Case 1: $x_0 \neq 0$. Define a sequence (x_n) by $x_n =: qx_{n+1}$. Then $y_0 = x_1$, and repeating the above argument by induction, we get that the sequence $(F(x_n))$ is constant with value M. As $\lim_{n\to\infty} x_n = \pm\infty$ we immediately get that $M = 0$.

Case 2: $x_0 = 0$. The computation given above shows that $F(1) = M$, and we may proceed with the value $x_0 = 1$ like in Case 1.

Thus in any case we get $M = 0$. The same arguments show that also the minimum of F must be equal to 0, and therefore F vanishes identically. As a trivial consequence the assertion of the theorem holds. ◊

Corollary 3. *The set of Riemann integrable solutions of equation* (1) *as well as the set of Lebesgue (resp. Borel) integrable solutions is at most of dimension* 1.

Proof. Integration is a linear mapping from the set of all integrable solutions into the one-dimensional space $\mathbb{R}$. By the preceding theorem, the kernel of this mapping contains only the zero function. Thus the dimension of the space of integrable solutions cannot exceed 1. ◊

Theorem 19. *Let* f *be an integrable solution of equation* (1) *(in the Riemann or Borel resp. Lebesgue sense). Then* f *vanishes almost everywhere outside the interval* $[-Q, Q]$.

Proof. We use the function F of Lemma 3: Let G be the value of the integral of f, i.e., $\lim_{x\to\infty} F(x) = G \in \mathbb{R}$. Now let $\varepsilon > 0$ be arbitrarily chosen. There exists a number z such that $|F(x) - G| < \varepsilon$ for any $x > z$. Without loss of generality we may assume that $z > Q$. Thus for any $x > z + 1$ we have that $F(x-1), F(x), F(x+1) < G + \varepsilon$ and, similarly, $F(x-1), F(x), F(x+1) > G - \varepsilon$, which implies that $G - \varepsilon < F(qx) < G + \varepsilon$. In other words, the inequality $|f(x) - G| < \varepsilon$ holds for any $x > q(z+1)$. Repeating this argument, we immediately get $|F(x) - G| < \varepsilon$ for any $x > Q$, because the sequence z, $q(z+1)$, $q(q(z+1)+1)$, ... tends to Q. As ε was chosen arbitrarily, we may conclude that $F(x) = G$ for any $x > Q$. Similarly, using the same arguments in the other direction, we conclude that $F(x) = 0$ for any $x < -Q$. As a trivial consequence, f must be 0 a.e. (resp. except on a zero set outside the interval $[-Q, Q]$. $\Diamond$

After these results on solutions with unbounded support we make a short break. A paper on solutions with bounded support will follow.

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RELATIONSHIPS BETWEEN DISTANCE DOMINATION PARAMETERS

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Abstract: For any integer $n \geq 2$ a set D of vertices of a graph G of order p is defined to be a $P_{\leq n}$-dominating set (total $P_{\leq n}$-dominating set) of G if every vertex in $V(G) - D$ (respectively $V(G)$) is at distance at most $n-1$ from some vertex in D other than itself. The $P_{\leq n}$-domination number, $\gamma_n(G)$ (total $P_{\leq n}$-domination number $\gamma_n^t(G)$) is the minimum cardinality among all $P_{\leq n}$-dominating sets (total $P_{\leq n}$-dominating sets) of G. It is shown that if G is a connected graph on $p \geq 2n$ vertices, then $\gamma_n(G) + \gamma_n^t(G) \leq 2p/n$. A set I of vertices in a graph G is $P_{\leq n}$-independent if the distance between every two vertices of I is at least n. A $P_{\leq n}$-dominating set that is also $P_{\leq n}$-independent is called a $P_{\leq n}$-independent dominating set. The minimum cardinality among all $P_{\leq n}$-independent dominating sets in a graph G is the $P_{\leq n}$-independent domination number of G and is denoted by $i_n(G)$. It is shown that if G is a connected graph of order $p \geq n$, then $i_n(G) + (n-1)\gamma_n(G) \leq p$.

The terminology and notation of [2] will be used throughout. Recall that a *dominating set* (*total dominating set*) D of a graph G is a set of vertices of G such that every vertex of $V(G)-D$ (respectively, $V(G)$) is adjacent to some vertex of D. The *domination number* (*total domination number*) of G is the minimum cardinality of a dominating set (total dominating set) of G. Further, the *distance* $d(u,v)$ between two vertices u and v of G is the length of a shortest $u-v$ path if one exists, otherwise $d(u,v)=\infty$. In [5] generalizations of the above-mentioned domination parameters are defined and studied. For an integer $n \geq 2$, a set D of vertices of a graph G is defined to be a $P_{\leq n}$*-dominating set* (*total* $P_{\leq n}$*-dominating set*) of G if every vertex in $V(G)-D$ (respectively $V(G)$) is at distance at most $n-1$ from some vertex in D other than itself. The $P_{\leq n}$*-domination number* $\gamma_n(G)$ (*total* $P_{\leq n}$*-domination number* $\gamma_n^t(G)$) is the minimum cardinality of a $P_{\leq n}$-dominating set (total $P_{\leq n}$-dominating set) of G. Hence $\gamma_2(G)=\gamma(G)$ and $\gamma_2^t(G)=\gamma_t(G)$.

In [5] sharp bounds for the $P_{\leq n}$-domination number and total $P_{\leq n}$-domination number of a graph are established. In particular the following two results were obtained.

Theorem A. *If G is a connected graph of order $p \geq n$, then* $\gamma_n(G) \leq$ $\leq p/n$.

Theorem B. *If G is a connected graph of order $p \geq 2$, then*

$$\gamma_n^t(G) = 2 \qquad \text{for } 2 \leq p \leq 2n-1$$

and

$$\gamma_n^t(G) \leq \frac{2p}{2n-1} \qquad \text{for } p \geq 2n-1.$$

We now investigate relationships between these two generalized domination parameters. Observe that if G is a connected graph on p vertices with $2 \leq p \leq 2n-1$, then $\mathrm{rad}(G) \leq n-1$ and so $\gamma_n(G)+$ $+\gamma_n^t(G)=3$. We thus consider graphs of order $p \geq 2n$. Allan, Laskar and Hedetniemi [1] showed that, if G is a connected graph of order $p \geq 3$, then $\gamma(G)+\gamma_t(G) \leq p$. The following theorem generalizes this result.

Theorem 1. *For an integer $n \geq 2$, if G is a connected graph of order $p \geq 2n$, then*

$$\gamma_n(G)+\gamma_n^t(G) \leq 2p/n.$$

Proof. Let $n \geq 2$ be an integer. If T is a spanning tree of a connected graph G of order at least $2n$ and $\gamma_n(T)+\gamma_n^t(T) \leq 2p(G)/n$, then

$\gamma_n(G) + \gamma_n^t(G) \leq \gamma_n(T) + \gamma_n^t(T) \leq 2p(G)/n$. Hence we shall prove the theorem by establishing its validity for a tree G. We proceed by induction on the order of a tree of order at least $2n$.

Let T be a tree of order $2n$. Then $\operatorname{diam} T \leq 2n - 1$, and so $\operatorname{rad} T \leq n - 1$ or T is bicentral with $\operatorname{rad} T \leq n$. If $\operatorname{rad} T \leq n - 1$, then a central vertex of T is within distance $n - 1$ from every vertex of T, while a central vertex, together with any other vertex of T, forms a total $P_{\leq n}$-dominating set of T. Hence in this case, $\gamma_n(T) + \gamma_n^t(T) = 3 < < 2p(T)/n$. If, however, $\operatorname{rad} T = n$, then the central vertices of T form a total $P_{\leq n}$-dominating set (and hence certainly a $P_{\leq n}$-dominating set) of T and so $\gamma_n(T) + \gamma_n^t(T) = 4 = 2p(T)/n$. Hence the theorem is true for a tree of order $2n$.

Assume that $\gamma_n(T') + \gamma_n^t(T') \leq 2p(T')/n$ for all trees T' with $2n \leq p(T') < k$, and let T be a tree of order k. If $\operatorname{diam} T \leq 2n - 1$, then $\gamma_n(T) + \gamma_n^t(T) \leq 4 < 2p(T)/n$. So we may assume that $\operatorname{diam} T \geq 2n$.

Suppose that there exists an edge e of T such that both components of $T - e$ are of order at least $2n$. Let T_1 and T_2 be the components of $T - e$. Then $2n \leq p(T_i) < k$ and so, by the induction hypothesis, for $i \in \{1, 2\}$, T_i has a $P_{\leq n}$-dominating set D_i and a total $P_{\leq n}$-dominating set D_i' with $|D_i| + |D_i'| = \gamma_n(T_i) + \gamma_n^t(T_i) \leq 2p(T_i)/n$. Then $D_1 \cup D_2$ is a $P_{\leq n}$-dominating set of T and $D_1' \cup D_2'$ is a total $P_{\leq n}$-dominating set of T with $\gamma_n(T) + \gamma_n^t(T) \leq |D_1 \cup D_2| + |D_1' \cup D_2'| \leq 2p(T)/n$. For the remainder of the proof we shall therefore assume that, for each edge e of T, at least one of the (two) components of $T - e$ is of order less than $2n$. In particular, we note that $2n \leq \operatorname{diam} T \leq 4n - 2$. Let $\operatorname{diam} T = d$ and let u, v be two vertices of T such that $d(u, v) = d \geq 2n$. Let the $u - v$ path in T be denoted by $P : u = u_0, u_1, \dots, u_d = v$. To complete the proof we consider four lemmas.

Lemma 1. *If* $2n < p(T) \leq 3n - 2$, *then* $\gamma_n(T) + \gamma_n^t(T) < 2p(T)/n$.

Proof. Let T_1, T_2 and T_3 denote the components of $T - u_{n-1}u_n$, $T - - u_{d-n}u_{d-n+1}$ and $T - \{u_{n-1}u_n, u_{d-n}u_{d-n+1}\}$, respectively, containing u, v and u_n respectively. Since $p(T) \leq 3n - 2$, it follows that $d \leq 3n - - 3$; so $d(u_{n-1}, u_{d-n+1}) = d + 2 - 2n \leq n - 1$. Moreover, since P is a longest path in T, the vertex u_{n-1} (u_{d-n+1}) is at distance at most $n - 1$ from every vertex in T_1 (T_2, respectively). As $p(T_3) = p(T) - - (p(T_1) + p(T_2)) \leq 3n - 2 - 2n = n - 2$, every vertex of T_3 is within distance $n - 2$ from both u_{n-1} and u_{d-n+1} in T. It follows that $\gamma_n(T) = = \gamma_n^t(T) = \big|\{u_{n-1}, u_{d-n+1}\}\big| = 2$; so $\gamma_n(T) + \gamma_n^t(T) = 4 < 2p(T)/n$. This completes the proof of Lemma 1. $\Diamond$

Lemma 2. *If* $p(T) \geq 3n-1$ *and* $2n \leq d \leq 3n-3$, *then* $\gamma_n(T)+\gamma_n^t(T) \leq$ $\leq 2p(T)/n$.

Proof. Let T_1, T_2 and T_3 be defined as in the proof of Lemma 1. Since $d \leq 3n-3$, $d(u_{n-1}, u_{d-n+1}) \leq n-1$. Moreover, as P is a longest path in T, $u_{n-1}(u_{d-n+1})$ is at distance at most $n-1$ from every vertex in T_1 (T_2, respectively).

If $p(T_3) \leq n-1$, then every vertex of T_3 is within distance $n-1$ from both u_{n-1} and u_{d-n+1}; consequently, $\gamma_n(T)+\gamma_n^t(T) = 4 <$ $< 2p(T)/n$.

Suppose that $n \leq p(T_3) \leq 2n-1$. Then $p(T) \geq 3n$ and $\operatorname{diam} T_3 \leq$ $\leq 2n-2$; so $\operatorname{rad} T_3 \leq n-1$. We show that there exists a central vertex of T_3 that is distance at most $n-1$ from u_{n-1} or u_{d-n+1}. If this is not the case, then, for w a central vertex of T_3, w is at distance $n-1$ from both u_n and u_{d-n}. Since $d(u_n, u_{d-n}) = d-2n \leq n-3$, w is not a vertex of the $u_n - u_{d-n}$ path. Let $Q : v = w_0, w_1, \ldots, w_s$ be the shortest path from w to a vertex of the $u_n - u_{d-n}$ path. Then, necessarily, $w_s = u_j$ for some $j \in \{n+1, \ldots, d-n-1\}$ and $V(Q) \cap V(P) = \{u_j\}$. Let T' and T'' denote the components of $T_3 - ww_1$ containing w_1 and w respectively. Since the $w_1 - u_n$ path (of order $n-1$) does not contain the vertex u_{d-n}, we observe that $p(T') \geq n$. Further, if $p(T'') \leq n-1$, then it follows that w_1 is a central vertex of T_3 at distance $n-1$ from both u_{n-1} and u_{d-n+1}, which contradicts our assumption. Hence $p(T'') \geq$ $\geq n$, and so $p(T_3) \geq 2n$, which again produces a contradiction. Hence there exists a central vertex w (say) of T_3 that is at distance at most $n-1$ from u_{n-1} or u_{d-n+1}, and from each vertex of T_3. Thus $D =$ $= \{u_{n-1}, u_{d-n+1}, w\}$ is a total $P_{\leq n}$-dominating set (and so certainly a $P_{\leq n}$-dominating set) of T; so $\gamma_n(T) + \gamma_n^t(T) \leq 6 \leq 2p(T)/n$.

If $p(T_3) \geq 2n$, then it follows from the induction hypothesis that T_3 has a $P_{\leq n}$-dominating set D' and a total $P_{\leq n}$-dominating set D'' with $|D'| + |D''| = \gamma_n(T_3) + \gamma_n^t(T_3) \leq 2p(T_3)/n$. So $D_1 =$ $= D' \cup \{u_{n-1}, u_{d-n+1}\}$ is a $P_{\leq n}$-dominating set of T and $D_2 = D'' \cup$ $\cup \{u_{n-1}, u_{d-n+1}\}$ is a total $P_{\leq n}$-dominating set of T with $\gamma_n(T) +$ $+ \gamma_n^t(T) \leq |D_1| + |D_2| + 4 \leq 2p(T_3)/n + 2(p(T_1) + p(T_2))/n = 2p(T)/n$. This completes the proof of Lemma 2. $\Diamond$

Lemma 3. *If* $3n-2 \leq d \leq 4n-3$, *then* $\gamma_n(T) + \gamma_n^t(T) \leq 2p(T)/n$.

Proof. Necessarily there exists an integer i, $1 \leq i \leq d-1$, such that the components of $T - u_{i-1}u_i$ and $T - u_iu_{i+1}$ containing u are, respectively, of order less than $2n$ and of order at least $2n$. From the assumption that, for every edge e of T, $T-e$ contains a component of order at most $2n-1$, it follows that $d-2n+1 \le i \le 2n-1$.

Let T_1' and T_2' be the components of $T-u_i$ containing u and v, respectively. We note that T_1' and T_2' are both of order less than $2n$. Further, let $\deg u_i = r$ and denote by $T_1', T_2', \dots, T_r'$ the components of $T-u_i$ and by w_i the vertex in T_i' adjacent to u_i in $T (i = 1, 2 \dots, r)$. We note that $w_1 = u_{i-1}$ and $w_2 = u_{i+1}$. If $r \ge 3$, then for $j \in \{3, \dots, r\}$ we observe that, since one component of $T - u_i w_j$ contains P and is therefore of order at least $2n$, the component T_j' is of order at most $2n-1$.

We consider two possibilities.

Case 1: Suppose that $i = 2n-1$ or $i = d-2n+1$. Without loss of generality, we may assume (relabelling the path P by $v = u_0, u_1, \dots, u_d = u$ if necessary) that $i = 2n-1$. Since $p(T_1') \le 2n-1$, $T_1' \cong P_{2n-1}$ and $\{u_{n-1}\}$ is a $P_{\le n}$-dominating set of T_1'. We consider two possibilities.

Case 1.1: Suppose that $d = 3n-2$. Then $u_{2n-1} = u_{d-n+1}$ and every vertex of T_2' is within distance $n-1$ from u_{2n-1}. Consequently, if $r = 2$, then $\gamma_n(T) + \gamma_n^t(T) \le |\{u_{n-1}, u_{2n-1}\}| + |\{u_{n-1}, u_{2n-2}, u_{2n-1}\}| = 5 \le 2(3n-1)/n \le 2p(T)/n$. We now consider the case where $r \ge 3$. Let $\{3, \dots, r\} = I = I_1 \cup I_2 \cup I_3$ where

$$I_1 = \{j \in I \mid p(T_j') \le n-1\},$$
$$I_2 = \{j \in I \mid n \le p(T_j') \le 2n-2\},$$
$$I_3 = \{j \in I \mid p(T_j') = 2n-1\}.$$

If $j \in I_1$, then u_{2n-1} is within distance $n-1$ from every vertex of T_j'. If $j \in I_2$, then since $p(\langle V(T_j') \cup \{u_{2n-1}\}\rangle) \le 2n-1$, T_j' contains a vertex z_j such that $\{z_j\}$ is a $P_{\le n}$-dominating set of T_j' and $d(u_{2n-1}, z_j) \le n-1$. If $j \in I_3$, then $\operatorname{rad} T_j' \le n-1$. Let x_j be a central vertex of T_j'. It follows, therefore, that $\gamma_n(T) \le |\{u_{n-1}, u_{2n-1}\}| + |\bigcup_{j \in I_2} \{z_j\}| + |\bigcup_{j \in I_3} \{x_j\}| = 2 + |I_2| + |I_3|$ and $\gamma_n^t(T) \le |\{u_{n-1}, u_{2n-2}, u_{2n-1}\}| + |\bigcup_{j \in I_2} \{z_j\}| + |\bigcup_{j \in I_3} \{x_j, w_j\}| = 3 + |I_2| + 2|I_3|$; so $\gamma_n(T) + \gamma_n^t(T) \le 5 + 2|I_2| + 3|I_3|$. However, $p(T) \ge d+1+n|I_2| + (2n-1)|I_3| = 3n-1+n|I_2|+(2n-1)|I_3|$. Hence $2p(T)/n \ge 6 - 2/n + 2|I_2| + (4-2/n)|I_3| \ge 5 + 2|I_2| + 3|I_3| \ge \gamma_n(T) + \gamma_n^t(T)$.

Case 1.2: Suppose that $3n-1 \le d \le 4n-3$. Then $d-n+1 > 2n-1$ and so $u_{d-n+1} \in V(T_2')$. Further, since $p(T_2') \le 2n-1$,

$\{u_{d-n+1}\}$ is a $P_{\leq n}$-dominating set of T_2'. Since $d \leq 4n-3$, we observe that $d(u_{d-n+1}, u_{2n-1}) = d - 3n + 2 \leq n - 1$.

If $r = 2$, then

$$\gamma_n(T) + \gamma_n^t(T) \leq |\{u_{n-1}, u_{d-n+1}\}| + |\{u_{n-1}, u_{2n-2}, u_{2n-1}, u_{d-n+1}\}| = \\ = 6 \leq 2(3n)/n \leq 2p(T)/n.$$

If $r \geq 3$, then let $I = \{3, \ldots, r\} = I_1 \cup I_2 \cup I_3 \cup I_4$ where

$$I_1 = \{j \in I \mid p(T_j') \leq 4n - d - 3\},$$
$$I_2 = \{j \in I \mid 4n - d - 2 \leq p(T_j') \leq n - 1\},$$
$$I_3 = \{j \in I \mid n \leq p(T_j') \leq 2n - 2\},$$
$$I_4 = \{j \in I \mid p(T_j') = 2n - 1\}.$$

If $j \in I_1$, then, since $d(u_{d-n+1}, u_{2n-1}) = d - 3n + 2$, it follows that u_{d-n+1} is within distance $n-1$ from every vertex of T_j'. If $j \in I_2$, then u_{2n-1} is within distance $n-1$ from every vertex of T_j'. If $j \in I_3$, then T_j' contains a vertex z_j such that $\{z_j\}$ is a $P_{\leq n}$-dominating set of T_j' and $d(u_{2n-1}, z_j) \leq n - 1$. If $j \in I_4$, then $\operatorname{rad} T_j' \leq n - 1$. Let x_j be a central vertex of T_j'. We now consider two possibilities.

Case 1.2.1: Suppose that $|I_2| \geq 1$. Then it follows that $\gamma_n(T) \leq \leq |\{u_{n-1}, u_{2n-1}, u_{d-n+1}\}| + |\bigcup_{j \in I_3} \{z_j\}| + |\bigcup_{j \in I_4} \{x_j\}| = 3 + |I_3| + |I_4|$ and $\gamma_n^t(T) \leq |\{u_{n-1}, u_{2n-2}, u_{2n-1}, u_{d-n+1}\}| + |\bigcup_{j \in I_3} \{z_j\}| + |\bigcup_{j \in I_4} \{x_j, w_j\}| = = 4 + |I_3| + 2|I_4|$; so $\gamma_n(T) + \gamma_n^t(T) \leq 7 + 2|I_3| + 3|I_4|$. However, $p(T) \geq (d+1) + (4n-d-2)|I_2| + n|I_3| + (2n-1)|I_4| \geq 4n - 1 + n|I_3| + + (2n-1)|I_4|$. Hence $2p(T)/n \geq 8 - 2/n + 2|I_3| + (4 - 2/n)|I_4| \geq 7 + + 2|I_3| + 3|I_4| \geq \gamma_n(T) + \gamma_n^t(T)$.

Case 1.2.2: Suppose that $|I_2| = 0$. Then it follows that $\gamma_n(T) \leq \leq |\{u_{n-1}, u_{d-n+1}\}| + |I_3| + |I_4| = 2 + |I_3| + |I_4|$ and $\gamma_n^t(T) \leq 4 + |I_3| + 2|I_4|$; so $\gamma_n(T) + \gamma_n^t(T) \leq 6 + 2|I_3| + 3|I_4|$. However, $p(T) \geq d + 1 + n|I_3| + + (2n-1)|I_4| \geq 3n + n|I_3| + (2n-1)|I_4|$. Hence $2p(T)/n \geq 6 + 2|I_3| + + 3|I_4| \geq \gamma_n(T) + \gamma_n^t(T)$.

Case 2: Suppose that $d - 2n + 2 \leq i \leq 2n - 2$. Then, since $d \geq \geq 3n - 2$, $n \leq d - 2n + 2 \leq i \leq 2n - 2 \leq d - n$. Hence $u_{n-1}(u_{d-n+1})$ is a vertex of T_1' (T_2', respectively). In fact, as P is a longest path in T and as $p(T_i') \leq 2n - 1$ $(1 \leq i \leq 2)$, $\{u_{n-1}\}$ $(\{u_{d-n+1}\})$ is a $P_{\leq n}$-dominating set of T_1' (T_2', respectively). Furthermore, since $i \leq \leq 2n - 2$, $d(u_{n-1}, u_i) = i - n + 1 \leq n - 1$ and since $i \geq d - 2n + + 2$, $d(u_{d-n+1}, u_i) = d - n + 1 - i \leq n - 1$. Consequently, if $r = 2$, then $\gamma_n(T) + \gamma_n^t(T) \le |\{u_{n-1}, u_{d-n+1}\}| + |\{u_{n-1}, u_i, u_{d-n+1}\}| = 5 \le \le 2(3n-1)/n \le 2p(T)/n$.

If $r \ge 3$, then let $I = \{3, \dots, r\} = I_1 \cup I_2 \cup I_3 \cup I_4$, where

$$I_1 = \{j \in I \mid p(T_j') < \max(2n-i-1, 2n+i-d-1)\},$$
$$I_2 = \{j \in I \mid \max(2n-i-1, 2n+i-d-1) \le p(T_j') \le n-1\},$$
$$I_3 = \{j \in I \mid n \le p(T_j') \le 2n-2\},$$
$$I_4 = \{j \in I \mid p(T_j') = 2n-1\}.$$

If $j \in I_1$, then $p(T_j') \le 2n-i-2$ or $p(T_j') \le 2n+i-d-2$. If $p(T_j') \le 2n-i-2$, then since $d(u_{n-1}, u_i) = i-n+1$, it follows that u_{n-1} is within distance $n-1$ from every vertex of T_j'. If $p(T_j') \le 2n + +i-d-2$, then, since $d(u_{d-n+1}, u_i) = d-n+1-i$, it follows that u_{d-n+1} is within distance $n-1$ from every vertex of T_j'. If $j \in I_2$, then u_i is within distance $n-1$ from every vertex of T_j'. If $j \in I_3$, then T_j' contains a vertex z_j such that $\{z_j\}$ is a $P_{\le n}$-dominating set of T_j' and $d(u_i, z_j) \le n-1$. If $j \in I_4$, then $\operatorname{rad} T_j' \le n-1$. Let x_j be a central vertex of T_j'. We now consider two possibilities.

Case 2.1: Suppose that $|I_2| \ge 1$. Then it follows that $\gamma_n(T) \le \le |\{u_{n-1}, u_i, u_{d-n+1}\}| + |\bigcup_{j\in I_3} \{z_j\}| + |\bigcup_{j \in I_4} \{x_j\}| = 3 + |I_3| + |I_4|$ and $\gamma_n^t(T) \le |\{u_{n-1}, u_i, u_{d-n+1}\}| + |\bigcup_{j\in I_3} \{z_j\}| + |\bigcup_{j\in I_4} \{x_j, w_j\}| = 3 + |I_3| + + 2|I_4|$; so $\gamma_n(T) + \gamma_n^t(T) \le 6 + 2|I_3| + 3|I_4|$.

If $\max(2n-i-1, 2n+i-d-1) = 2n-i-1$, then $p(T) \ge \ge (d+1) + (2n-i-1)|I_2| + n|I_3| + (2n-1)|I_4| \ge 2n+d-i+n|I_3| + + (2n-1)|I_4| \ge 3n + n|I_3| + (2n-1)|I_4|$, since $d-i \ge n$. Hence $2p(T)/n \ge 6 + 2|I_3| + (4-2/n)|I_4| \ge 6 + 2|I_3| + 3|I_4| \ge \gamma_n(T) + \gamma_n^t(T)$.

If $\max(2n-i-1, 2n+i-d-1) = 2n+i-d-1$, then $p(T) \ge \ge (d+1) + (2n+i-d-1)|I_2| + n|I_3| + (2n-1)|I_4| \ge 2n+i+ + n|I_3| + (2n-1)|I_4| \ge 3n + n|I_3| + (2n-1)|I_4|$, since $i \ge n$. Hence $2p(T)/n \ge 6 + 2|I_3| + 3|I_4| \ge \gamma_n(T) + \gamma_n^t(T)$.

Case 2.2: Suppose that $|I_2| = 0$. Then it follows that $\gamma_n(T) \le \le |\{u_{n-1}, u_{d-n+1}\}| + |I_3| + |I_4| = 2 + |I_3| + |I_4|$ and $\gamma_n^t(T) \le 3 + |I_3| + + 2|I_4|$; so $\gamma_n(T) + \gamma_n^t(T) \le 5 + 2|I_3| + 3|I_4|$. However, $p(T) \ge d+1+ + n|I_3| + (2n-1)|I_4| \ge 3n-1+n|I_3| + (2n-1)|I_4|$. Hence $2p(T)/n \ge \ge 6 - 2/n + 2|I_3| + (4-2/n)|I_4| \ge 5 + 2|I_3| + 3|I_4| \ge \gamma_n(T) + \gamma_n^t(T)$.

This completes the proof of Lemma 3. $\Diamond$

Lemma 4. *If $d = 4n-2$, then $\gamma_n(T) + \gamma_n^t(T) \le 2p(T)/n$.*

Proof. Suppose that $d = 4n-2$. Then, using the notation introduced in the first two paragraphs of the proof of Lemma 3, it follows that $i = 2n-1$. Furthermore, since $p(T_i') \leq 2n-1$, we therefore have $T_i' \cong P_{2n-1}$ $(1 \leq i \leq 2)$ and so $\{u_{n-1}\}$ $(\{u_{3n-1}\})$ is a $P_{\leq n}$-dominating set of T_1' $(T_2'$, respectively). We observe, however, that u_{2n-1} is at distance n from both u_{n-1} and u_{3n-1}. Consequently, if $r = 2$, then $\gamma_n(T)+\gamma_n^t(T) = \left|\{u_{n-1}, u_{2n-1}, u_{3n-1}\}\right| + \left|\{u_{n-1}, u_{2n-2}, u_{2n}, u_{3n-1}\}\right| = 7 \leq 2(4n-1)/n = 2p(T)/n$.

If $r \geq 3$, then let $I = \{3, \ldots, r\} = I_1 \cup I_2 \cup I_3 \cup I_4$, where

$$I_1 = \{j \in I \mid p(T_j') \leq n-2\},$$
$$I_2 = \{j \in I \mid p(T_j') = n-1\},$$
$$I_3 = \{j \in I \mid n \leq p(T_j') \leq 2n-2\},$$
$$I_4 = \{j \in I \mid p(T_j') = 2n-1\}.$$

If $j \in I_1$, then every vertex of T_j' is within distance $n-1$ from the vertices u_{2n-2}, u_{2n-1} and u_{2n}. If $j \in I_2$, then u_{2n-1} is within distance $n-1$ from every vertex of T_j'. If $j \in I_3$, then T_j' contains a vertex z_j such that $\{z_j\}$ is a $P_{\leq n}$-dominating set of T_j' and $d(z_j, u_{2n-1}) \leq n-1$. If $j \in I_4$, then $\operatorname{rad} T_j' \leq n-1$. Let x_j be a central vertex of T_j'. We now consider two possibilities.

Case 1: Suppose that $|I_2| \geq 1$. Then it follows that $\gamma_n(T) \leq \left|\{u_{n-1}, u_{2n-1}, u_{3n-1}\}\right| + \left|\bigcup_{j\in I_3}\{z_j\}\right| + \left|\bigcup_{j \in I_4}\{x_j\}\right| = 3 + |I_3| + |I_4|$ and $\gamma_n^t(T) \leq \left|\{u_{n-1}, u_{2n-2}, u_{2n-1}, u_{2n}, u_{3n-1}\}\right| + \left|\bigcup_{j\in I_3}\{z_j\}\right| + \left|\bigcup_{j\in I_4}\{w_j, x_j\}\right|$; so $\gamma_n(T) + \gamma_n^t(T) \leq 8 + 2|I_3| + 3|I_4|$. However, $p(T) \geq 4n-1+(n-1)|I_2| + n|I_3| + (2n-1)|I_4| \geq 5n-2+n|I_3|+(2n-1)|I_4|$. Hence $2p(T)/n \geq 10-2/n+2|I_3|+(4-2/n) > 8+2|I_3|+3|I_4| \geq \gamma_n(T)+\gamma_n^t(T)$.

Case 2: Suppose that $I_2| = 0$. Then, if $|I_3| \geq 1$, it follows that $\gamma_n(T) \leq \left|\{u_{n-1}, u_{3n-1}\}\right| + \left|\bigcup_{j\in I_3}\{z_j\}\right| + \left|\bigcup_{j\in I_4}\{x_j\}\right| = 2 + |I_3| + |I_4|$ and $\gamma_n^t(T) \leq \left|\{u_{n-1}, u_{2n-2}, u_{2n-1}, u_{2n}, u_{3n-1}\}\right| + |I_3| + 2|I_4| \leq 5 + |I_3| + 2|I_4|$; so $\gamma_n(T) + \gamma_n^t(T) \leq 7 + 2|I_3| + 3|I_4|$. However, $p(T) \geq 4n-1+n|I_3| + (2n-1)|I_4|$. Hence $2p(T)/n \geq 8 - 2/n + 2|I_3| + (4-2/n)|I_4| \geq 7 + 2|I_3| + 3|I_4| \geq \gamma_n(T) + \gamma_n^t(T)$.

If $|I_3| = 0$, then it follows that $\gamma_n(T) \leq \left|\{u_{n-1}, u_{2n-1}, u_{3n-1}\}\right| + \left|\bigcup_{j\in I_4}\{x_j\}\right| = 3+|I_4|$ and $\gamma_n^t(T) \leq \left|\{u_{n-1}, u_{2n-2}, u_{2n}, u_{3n-1}\}\right| + 2|I_4| = 4 + 2|I_4|$; so $\gamma_n(T) + \gamma_n^t(T) \leq 7 + 3|I_4|$. However, $p(T) \geq 4n-1+$

$+(2n-1)|I_4|$. Hence $2p(T)/n \geq 8-2/n+(4-2/n)|I_4| \geq 7+3|I_4| \geq$ $\geq \gamma_n(T)+\gamma_n^t(T)$.

This completes the proof of Lemma 4 and thus of Th. 1. ◊

That the bound in Th. 1 is best possible may be seen as follows: Let G be obtained from a connected graph H by attaching a path of length $n-1$ to each vertex of H. (The graph G is shown in Fig. 1.) Then $\gamma_n(G)+\gamma_n^t(G)=2p(H)=2p(G)/n$.

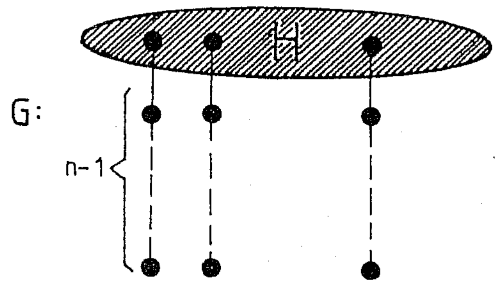

Fig. 1.

The fact that every maximal independent set of vertices in a graph is also a dominating set motivated Cockayne and Hedetmiemi [3] in 1974 to initiate the study of another domination parameter. A dominating set of vertices in a graph that is also an independent set is called an *independent dominating set.* The minimum cardinality among all independent dominating sets of a graph G is called the *independent domination number* of G and is denoted by $i(G)$.

The independent domination number of a graph and the distance domination parameters introduced earlier suggest yet another distance domination parameter. A set I of vertices in a graph G is *$P_{\leq n}$-independent* in G if every two vertices of I are at distance at least n apart in G. A $P_{\leq n}$-independent set of vertices in a graph that is also a $P_{\leq n}$-dominating set is called a *$P_{\leq n}$-independent dominating set.* The minimum cardinality among all $P_{\leq n}$-independent dominating sets of a graph G is called the *$P_{\leq n}$-independent domination number* of G and is denoted by $i_n(G)$. Hence $i_2(G)=i(G)$.

Before investigating relationships between the distance domination parameter i_n and the distance domination parameters γ_n and γ_n^t we need some additional concepts. A set of vertices $X \subset V(G)$ has *property* $\pi_n (n \geq 2)$ if and only if every nontrivial path of length $\ell \leq$ $\leq n-1$ in G contains at least ℓ vertices of X. A set of vertices with property π_n is called a *$P_{\leq n}$-cover* of G. So a $P_{\leq 2}$-cover of G is simply a cover of G. The minimum cardinality among all $P_{\leq n}$-covers of G is called the *$P_{\leq n}$-covering number* of G and is denoted by $\alpha_n(G)$. The maximum cardinality among all $P_{\leq n}$-independent sets is called the *$P_{\leq n}$-independence number* of G and is denoted by $\beta_n(G)$. Hence $\alpha_2(G)$ is simply the covering number $\alpha(G)$ and $\beta_2(G)$ is the independence number $\beta(G)$. The next Gallai-type result generalizes a well-known relationship between the covering number and independence number of a graph [4].

Theorem 2. *If G is a connected graph of order $p \geq n$, then*

$$\alpha_n(G) + \beta_n(G) = p.$$

Proof. We note that X is a $P_{\leq n}$-cover if and only if $V(G) - X$ is a $P_{\leq n}$-independent set of vertices. So if X is a $P_{\leq n}$-cover of cardinality $\alpha_n(G)$, then $\alpha_n(G) = |X|$ and $|V(G) - X| = p - \alpha_n(G) \leq \beta_n(G)$. Similarly if Y is a $P_{\leq n}$-independent set of vertices of cardinality $\beta_n(G)$, $p - \beta_n(G) = |V(G) - Y| \geq \alpha_n(G)$. Thus $\alpha_n(G) + \beta_n(G) = p$. $\Diamond$

Allan, Laskar and Hedetniemi [1] showed that if G is a graph of order p that has no isolated vertices, then $\gamma(G) + i(G) \leq p$. We now present a generalization of this result.

Theorem 3. *If G is a connected graph of order $p \geq n \geq 2$, then*

$$i_n(G) + (n-1)\gamma_n(G) \leq p.$$

Proof. Let X be a $P_{\leq n}$-cover such that $\langle X \rangle$ contains as few components as possible of order less than $n-1$. We show that $\langle X \rangle$ has no components of order less than $n-1$. Suppose $\langle X \rangle$ has a component G_1 of order $p_1 \leq n-2$. Since G is connected, and $p \geq n$, there is a vertex $s \in S = V(G) - X$ that is adjacent with a vertex y in G_1 and a vertex z in $V(G) - V(G_1)$. Since S is $P_{\leq n}$-independent, z must belong to some component $G_2 \neq G_1$ of $\langle X \rangle$. Note that s is the only vertex of S which is adjacent to a vertex (or vertices) in G_1, for if t is any other vertex of S that is adjacent to a vertex of G $d(t,s) \leq n-1$, which is not possible since S is $P_{\leq n}$-independent.

Now if $p(G_1) = 1$, let $S' = (S - \{s\}) \cup \{y\}$. Otherwise if $p(G_1) \geq$ ≥ 2, let $x \neq y$ be a vertex of G_1 which is not a cut-vertex of G_1 and set $S' = (S - \{s\}) \cup \{x\}$. Then S' is a $P_{\leq n}$-independent set of cardinality $|V(G) - X|$. Since X is a $P_{\leq n}$-cover of cardinality $\alpha_n(G)$, it follows from Th. 2, that $|V(G) - X| = p - \alpha_n(G) = \beta_n(G)$, i.e., $|S'| = \beta_n(G)$. However, then $X' = V(G) - S'$ is a $P_{\leq n}$-cover of G of cardinality $\alpha_n(G)$

such that $\langle X' \rangle$ contains fewer components of order less than $n-1$ than $\langle X \rangle$. This contradicts our choice of X. Hence $\langle X \rangle$ has no components of order less than $n-1$.

Since G is connected, every vertex in $V(G) - X$ is adjacent with a vertex in X and, consequently

$$\gamma_n(G) \leq \gamma_{n-1}(\langle X \rangle).$$

Since $\langle X \rangle$ has no component of order smaller than $n-1$, it follows from Th. A that

$$\gamma_n(G) \leq \frac{p(\langle X \rangle)}{n-1} = \frac{|X|}{n-1} = \frac{\alpha_n(G)}{n-1}.$$

The fact that $\beta_n(G) = |V(G) - X| \geq i_n(G)$ and Th. 2 now imply that

$$i_n(G) + (n-1)\gamma_n(G) \leq \alpha_n(G) + \beta_n(G) = p. \ \Diamond$$

The bound given in Th. 3 is best possible as we now see. Let G be the graph shown in Fig. 1. Then $i_n(G) = \gamma_n(G) = p(H)$ and $i_n(G) + (n-1)\gamma_n(G) = np(H) = p(G)$. It is shown in [6] that if T is a tree of order $p \geq 2n-1$, then $i_n(T) + (n-1)\gamma_n^t(T) \leq p$.

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w-JORDAN NEAR-RINGS II

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Abstract: In previous papers [3, 4] we have studied near-rings with invariant series of ideals. In the present paper we study near-rings with invariant series whose factors are without proper subnear-rings and we characterize those whose zero-symmetric part is an ideal. Moreover we continue the study of near-rings with invariant series whose factors are of prime order and we provide a complete characterisation for those of length 2.

Introduction

In previous papers we have studied near-rings with invariant series of ideals. In [3] we observed that numerous results, particularly concerning closure problems, do not depend on the near-ring structure and therefore are valid in a more general ambit, that is for universal algebras or, at least, for Ω-groups. In [4] we considered several classes of near-rings: simple (S_0), simple and strongly monogenic (S_1), N_0-simple (S_2), without proper subnear-rings (S_3), of prime order (S_4), and we studied near-rings with an invariant series whose factors belong to S_w ($w \in \{0, 1, 2, 3, 4\}$), called *w-Jordan near-rings.* While in [4] we turned our attention to the zero-symmetric case, here we present the results of the study of finite 3-Jordan near-rings, completely characterizing those whose zero-symmetric part is an ideal. Moreover, we continue the study of the 4-Jordan near-rings, begun in [4], and we provide a complete characterization for those of length 2.

Hereafter N will indicate a left near-ring and we refer to [10] without mentioning this explicitely. In particular, we shall use the term *"mixed"* to describe a near-ring N, with $N_0 \neq \{0\}$ and $N_c \neq \{0\}$. In general N_0 is a right ideal of N and N_c is an invariant subnear-ring. Furthermore, a zero-symmetric near-ring without proper N-subgroups H such that $HN = \{0\}$ is called *A-simple*. A near-ring is *N-simple* if it is without proper N-subgroups, that is if the additive group N^+ does not contain proper subgroups which are proper right ideals of the multiplicative semigroup $N^{\cdot}$. N is *N_0-simple* if it is without proper N_0-subgroups. In [7] a near-ring N is called *p-singular* if the order of N is divisible by a prime number p but the order of every proper subnear-ring of N is not divisible by p. $A_s(N) = \{x \in N/xA = \{0\}\}$, $(A_d(N) = \{x \in N/Ax = \{0\})$ denote the left (right) annihilator of N and $A(N) = A_s(N) \cap A_d(N)$, the annihilator of N; $r(n) = \{x \in N/nx = 0\}$ denotes the annihilator of $n \in N$; $r(n)$ is always a right ideal. If $N = N_1 \supset N_2 \supset \ldots \supset N_n = \{0\}$ is an invariant series of N, we will indicate $N_i/N_{i+1}, N_i/N_{i+2}, \ldots, N_i/N_{i+k}$ respectively with $N_i', N_i'', \ldots, N_k^k$ and with $f_i', f_i'', \ldots f_k^k$ the corresponding canonical epimorphisms.

1. 3J-near-rings

Definition 1. Let N be a finite near-ring. An *a-series* is an invariant series $N = N_1 \supset N_2 \supset \ldots \supset N_n = \{0\}$ whose factors belong to S_3 and such that, for every zero-symmetric N_i' and constant N_j' with $i < j$, $|N_j'|$ does not divide $|N_i'|$. The term *a-near-ring* describes a finite near-ring with an a-series.

Lemma 1. *A mixed a-near-ring N with an a-series*

$$(\alpha) \qquad N = N_1 \supset N_2 \supset \ldots \supset N_n = \{0\}$$

such that N_1' is zero-symmetric and N_i' is constant for every $i \in \{2, \ldots, n-1\}$ is isomorphic to $N_0 \oplus N_c$. Moreover there exists in N another series $N = M_1 \supset M_2 \supset \ldots \supset M_n = \{0\}$ such that $M_{n-1} \simeq N_1'$, hence zero-symmetric, and $M_{i-1}' \simeq N_i'$, therefore constant, for every $i \in \{2, 3, \ldots, n-1\}$.

Proof. Since N'_1 is zero-symmetric, $N_2 \supseteq N_c$. Hence $N_2 = (N_2)_0 + N_c$. Because the following factors are constant, for every $i \in \{3, 4, \dots, n\}$, $N_i \supseteq (N_2)_0$. Because $N_n = \{0\}$, we even get $(N_2)_0 = \{0\}$. Thus N_c is an ideal and N_0 is without proper subnear-rings, in fact it is isomorphic to N'_i. In [7] it was shown that a finite near-ring is without proper subnear-rings iff it is a simple p-singular near-ring and, in this case, its order is divisible by at most a prime number. Hence $|N_0| = p^b$ (p prime) and by Def. 1, $|N'_i| \neq p$ for every $i \in \{2, \dots, n-1\}$.

Now we prove that $|\overset{+}{N}/\overset{+}{N_0}|$ is prime with p and consequently N_0 is an ideal of N. In fact, if $|\overset{+}{N}/\overset{+}{N_0}| = kp$, then $|\overset{+}{N_c}| = kp$ and, because $|N_2/N_3| \neq p$, we have $|N_3| = k_1 p$. In this way we get $|N_n| = = k_{n-2}\, p = 0$ and this is absurd. Thus $|\overset{+}{N}/\overset{+}{N_0}|$ is relatively prime to p, $|N_0| = p^b$ and $\overset{+}{N_0}$, which is normal, is the unique Sylow p-subgroup of $\overset{+}{N}$. Since the homomorphic image of a Sylow p-subgroup is contained in a Sylow p-subgroup, $\overset{+}{N_0}$ is fully invariant in $\overset{+}{N}$. Therefore, since the left translations are endomorphisms of $\overset{+}{N}$, $nN_0 \subseteq N_0$ for every $n \in N$ and N_0 is an ideal of N. From this we can conclude that $N \simeq N_0 \oplus N_c$.

Finally, if we denote $M_i = N_{i+1} + N_0$, we have $M_{n-1} = N_0$ and the series $N = M_1 \supset M_2 \supset \dots \supset M_{n-1} \supset M_n = \{0\}$ is the series required. $\Diamond$

The following theorem gives a complete characterization of all a-near-rings.

Theorem 1. *A finite near-ring with an invariant series whose factors belong to S_3 is an a-near-ring iff its zero-symmetric part is an ideal.*

Proof. Suppose N a finite near-ring with an invariant series whose factors belong to S_3, and $N_0 \triangleleft N$. This series is a refinement of $N \supset \supset N_0 \supset \{0\}$. Therefore in such a series there are no zero-symmetric factors that precede the constant ones. Thus this series is an a-series.

Conversely, let

$$(\alpha) \qquad N = N_1 \supset N_2 \supset \dots \supset N_n = \{0\}$$

be an a-series of a finite near-ring. First we show that in N there is a series $N = M_1 \supset M_2 \supset \dots \supset M_n = \{0\}$ such that the constant factors precede the zero-symmetric ones. Let j be the highest index of the series (α) such that N'_j is zero-symmetric, $N'_{j+1}, \dots, N'_{j+h}$ are constant and N'_{j+k} is zero-symmetric for every $k > h$. Consider now the subseries of (α) $N_j \supset N_{j+1} \supset \dots \supset N_{j+h}$ and its image

$$(\beta) \qquad G = G_j \supset G_{j+1} \supset \ldots \supset G_{j+h} \supset G_{j+h+1} = \{0\}$$

under the homomorphism f_j^{h+1}, where $G_t = f_j^{h+1}(N_t)$ for $t \in \{j, j+1, \ldots, j+H\}$. Since $G'_j = G_j/G_{j+1}$ is isomorphic to N'_j, G'_j is zero-symmetric and G'_t is constant for every $t \in \{j+1, \ldots, j+h\}$. Therefore the series (β) is an a-series of G satisfying the hypotheses of Lemma 1. So applying Lemma 1, there is in G a series

$$(\gamma) \qquad G = F_j \supset F_{j+1} \supset \ldots \supset F_{j+h} = \{0\}$$

such that F_{j+h} is isomorphic to G'_j and is the zero-symmetric part of G, that is G_0, whereas F_t is isomorphic to G'_{t+1} for every $t \in \{j, j+1, \ldots, j+h-1\}$ and therefore constant. Using Lemma 1 again, we are able to say that F_t is fully invariant in G for every $t \in \{j, j+1, \ldots, j+h\}$.

Let now $M_t = (f_j^{h+1})^{\circ}(F_t)$, that is $F_t = M_t/N_{j+h+1}$. We prove that M_t is an ideal of N for every $t \in \{j, j+1, \ldots, j+h+1\}$. In fact M_{j+h} is a left ideal of N because every endomorphism of N_j which fixes N_{j+h+1}, fixes also M_{j+h}. Let $\varepsilon\colon N_j \to N_j$ be an endomorphism such that $\varepsilon(N_{j+h+1}) \subseteq N_{j+h+1}$. This endomorphism induces an endomorphism ε' in N_j/N_{j+h+1}, put $\varepsilon'(n_j + N_{j+h+1}) = \varepsilon(n_j) + N_{j+h+1}$. Let m now be an element of M_{j+h}. Obviously $m + N_{j+h+1} \in F_{j+h}$, therefore $\varepsilon'(m + N_{j+h+1}) = \varepsilon(m) + N_{j+h+1} \in F_{j+h}$ because, by Lemma 1 F_{j+h} is fully invariant in G. Hence there is an element $m' \in M_{j+h}$ and an element $n \in N_{j+h+1}$ such that $\varepsilon(m) = m' + n$, thus $\varepsilon(m) \in M_{j+h}$. Since every left translation γ_n, restricted to N_j is an endomorphism of N_j which fixes N_{j+h+1}, we have $nM_{j+h} \subseteq M_{j+h}$ for every $n \in N$. Thus M_{j+h} is a left ideal of N. Now we prove that $M_{j+h} = N_0 \cap N_j$ and, from this, that M_{j+h} is a right ideal. If $m \in N_0 \cap N_j$, then $f_j^{h+1}(m) \in G_0 = F_{j+1}$, therefore $m \in M_{j+h}$. On the other hand, M_{j+h} is obviously contained in N_j and it is zero-symmetric because one of its ideals and the respective factors are zero-symmetric. Thus we can conclude that M_{j+h} is an ideal of N.

Finally we show that $M_t = M_{j+h} + N_{t+1}$ for every $t \in \{j, j+1, \ldots, j+h\}$ and thus, M_t, as a sum of two ideals, is an ideal.

The series (γ) is obtained using Lemma 1, therefore $F_t = F_{t+h} + G_{t+1}$, where $F_{j+h} = G_0$. Hence $M_t/N_{j+h+1} = M_{j+h}/N_{j+h+1} + N_{t+1}/N_{j+h+1}$ and thus $M_t = M_{j+h} + N_{t+1}$. Finally we observe that M_{j+h} is exactly the zero-symmetric part of N_j and consequently $(N_j)_0$ is an ideal. ◊

It should be noted that not all the finite $3J$-near-rings are a-near-rings. For example $N = M_a(\mathbb{Z}p)$,[1] p prime is a $3J$-near-ring ($N \supset N_c \supset \{0\}$ is a series whose factors are in S_3) but N_0 is not an ideal.

2. 4J-near-rings of length 2

In this paragraph we study near-rings with an invariant series $N \supset \supset I \supset \{0\}$ whose factors are near-rings of prime order. Consequently they are near-rings of the order pq, where p, q are prime numbers. It is well known that the additive group of such a near-ring is a direct or semidirect sum of cyclic groups of prime order or is itself cyclic of the order p^2. We can also establish the following:

Proposition 1. *A $4J$-near-ring of length 2 has only one proper ideal or it is isomorphic to the direct sum of two of its ideals.*

Proof. Let $N \supset I \supset \{0\}$ be the invariant series of a $4J$-near-ring N with N/I and I of prime order: that means I is a maximal ideal. If J is another ideal of N, $I+J$ is also an ideal, thus $I = J$ or $I \oplus J = N$. $\Diamond$

From Th. 1 of [4] we know that a near-ring of prime order is constant or zero-symmetric. In the latter case, it is either an A-simple and strongly monogenic near-ring or a zero-ring. Thus, we will denote by

η'_c the class of constant near-rings;
η'_0 the class of zero-symmetric near-rings;
$\mathcal{A}$ the class of A-simple and strongly monogenic near-rings;
$\mathcal{O}$ the class of zero-rings.

Moreover we will denote by $[\mathcal{S}, \mathcal{T}]$ the class of near-rings with an invariant series $N \supset I \supset \{0\}$ such that $N/I \in \mathcal{S}$ and $I \in \mathcal{T}$, where $\mathcal{S}, \mathcal{T} \in \{\eta'_c, \eta'_0, \mathcal{A}, \mathcal{O}\}$. In this way, $4J$-near-rings of length 2 are the union of $[\eta'_c, \eta'_0]$, $[\eta'_0, \eta'_c]$, $[\eta'_0, \eta'_0]$, $[\eta'_c, \eta'_c]$, where $\eta'_0 = \mathcal{A} \cup \mathcal{O}$. Moreover we observe that the structure of the near-rings of $[\eta'_c, \eta'_c]$ is that of groups of order pq and therefore well-known.

Let $N \equiv A +_\varphi B$ be a semidirect sum of additive groups A and B. By Clay method, every near-ring on N can be constructed but, generally, from a multiplicative view-point, isomorphic images of semidirect

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[1] $M_a(\mathbb{Z}p) = \mathrm{Hom}(\mathbb{Z}p, \mathbb{Z}p) + M_c(\mathbb{Z}p)$, where $M_c(\mathbb{Z}p) = \{f\colon \mathbb{Z}p \to \mathbb{Z}p / f$ is constant$\}$.

summands are not even substructures. In [1] we study those functions introduced by Clay that provide the semidirect summands with a well defined multiplicative structure. We call *Φ-sum* of near-rings A and B, a near-ring N obtained by semidirect sum of the additive groups of A and B with a suitable Clay function preserving the multiplicative structure on semidirect summands.

CLASS $[\eta'_c, \eta'_0]$

The following theorem characterizes the near-rings belonging to $[\eta'_c, \eta'_0]$.

Theorem 2. *A near-ring N belongs to $[\eta''_c, \eta'_0]$ iff $N = A +_\Phi B$ with $f(\langle 0,0\rangle) = O_A$ and $\bar{f}(N) = \mathrm{id}$, where $A \in \eta'_0$ and $B \in \eta'_c$.*

Proof. If $N \in [\eta'_c, \eta'_0]$, then N_0 is an ideal. Thus N is isomorphic to $N_0 +_\Phi N_c$, where $f(\langle 0,0\rangle) = O_{N_0}$ and $\bar{f}(N) = \mathrm{id}$ (see [1], Th. 1, Cor. 1). Moreover, both N_0 and $N/N_0 \simeq N_c$ are of prime order. Conversely, suppose $N = A +_\Phi B$ with $f(\langle 0,0\rangle) = O_A$ and $\bar{f}(N) = \mathrm{id}$, where $A \in$ $\in \eta'_0$ and $B \in \eta'_c$. Then A° is an ideal of N; $A^\circ = N_0$; ${}^\circ B = N_c$ and $N/A^\circ \simeq {}^\circ B$ (see [1] Prop. 2). Whence $N \in [\eta'_c, \eta'_0]$. $\Diamond$

By construction described in Th. 2 we obtain each element of $[\eta'_c, \eta'_0]$. We observe also that, because $[\eta'_c, \eta'_0] = [\eta'_c \mathcal{A}] \cup [\eta'_c, \mathcal{O}]$, an element of $[\eta'_c, \eta'_0]$ belongs to $\eta'_c, \mathcal{A}]$ or to $[\eta'_c, \mathcal{O}]$ according to the choice of A in $\mathcal{A}$ or in $\mathcal{O}$ respectively.

CLASS $[\eta'_0, \eta'_c]$

The following theorem characterizes the near-rings belonging to $[\eta'_0, \eta'_c]$.

Theorem 3. *A near-ring N belongs to $[\eta'_0, \eta'_c]$ iff either $N = A \oplus B$, where $A \in \eta'_0$ and $B \in \eta'_c$, or N is an abstract affine near-ring of order p^2.*

Proof. Now the order of N is pq, where p, q are prime numbers.

(i) Suppose $p \neq q$. In this case $N \supset N_c \supset \{0\}$ is an a-series, thus $N = N_0 \oplus N_c$, by Lemma 1.

(ii) Let $p = q$. In this case the order of N is p^2 and $N^+ = N_0^+ \oplus$ $\oplus N_c^+$. If N_0 is an ideal of N, we still have $N = N_0 \oplus N_c$. If N_0 is not an ideal of N, we have $N_c N_0 = N_c$, because the order of N_c is a prime number. Moreover $r(N_c) = \{0\}$, due to $r(N_c) \subset N_0$. Thus N_c is a base of equality and N is an abstract affine near-ring (see [10], 9.85).

Conversely, if $N = A \oplus B$, where $A \in \eta'_0$ and $B \in \eta'_c$, it is clear that the theorem holds. Let N now be an abstract affine near-ring of order p^2. Obviously both N_0 and N_c are of the order p and, since N_c is an ideal, $N \in [\eta'_0, \eta'_c]$. ◊

Corollary 1. *A near-ring N belongs to $[\mathcal{O}, \eta'_c]$ iff $N \simeq A \oplus B$, where $A \in \mathcal{O}$ and $B \in \eta'_c$.*

Proof. Let $N \in [\mathcal{O}, \eta'_c]$. If N_0 is not an ideal of N, then $N_c N_0 = N_c$, thus $N_c = N_c N_0 = (N_c N_0) N_0 = N_c (N_0)^2 = \{0\}$, because N_0 is now a zero-ring. From this it follows that N_0 is an ideal of N, thus $N = N_0 \oplus N_c$ where $N_0 \in \mathcal{O}$ and $N_c \in \eta'_c$. The converse is trivial. ◊

Corollary 2. *A near-ring N belongs to $[\mathcal{A}, \eta'_c]$ iff either $N \simeq A \oplus B$, where $A \in \mathcal{A}$ and $B \in \eta'_c$, or N is an abstract affine near-ring of order p^2.*

In [2] we have shown a method for constructing abstract affine near-rings with a given zero-symmetric part and a given constant part (they are suitable Λ-sums). By using Th. 3 of [2], an abstract affine near-ring of order p^2 can be characterized as a Λ-sum of a field isomorphic to $\mathbb{Z}_p$ and the constant near-ring on $\mathbb{Z}_p$. We can also note that near-rings belonging to $[\eta'_0, \eta'_c] \cap [\eta'_c, \eta'_0]$ are direct sums of their zero-symmetric and constant parts. Those belonging to $[\eta'_0, \eta'_c] \setminus [\eta'_c, \eta'_0]$ are abstract affine near-rings of order p^2. Those belonging to $[\eta'_c, \eta'_0] \setminus [\eta'_0, \eta'_c]$ are Φ-sums (not direct sums) of two near-rings of prime order satisfying the conditions of Th. 2.

CLASS $[\eta'_0, \eta'_0]$

We now study the subclasses of $[\eta'_0, \eta'_0]$. Near-rings belonging to $[\mathcal{O}, \mathcal{O}]$ are characterized by the following

Theorem 4. *A near-ring N belongs to $[\mathcal{O}, \mathcal{O}]$ iff $|N| = pq$ and there is an ideal I of N such that $N^2 \subseteq I \subseteq A(N)$.*

Proof. If $N \in [\mathcal{O}, \mathcal{O}]$, obviously $|N| = pq$ and N has a proper ideal I, where I is a zero-ring of prime order including N^2. Moreover, suppose now that $NI \neq \{0\}$. We have $nI = I$ for some $n \in N$ and, because of $N^2 \subseteq I$, $I = nI = n^2 I = \{0\}$, a contradiction, due to I being a proper ideal. Thus $NI = \{0\}$ and $I \subseteq A_d(N)$. Now let $N^2 \neq \{0\}$, that means $nN = I$ for some $n \in N$. Thus $N^2 = I$ and $IN = N^2 N = NN^2 = NI = \{0\}$. From this it follows $I \subseteq A(N)$.

Conversely, let N be a near-ring of order pq with a proper ideal I such that $N^2 \subseteq I \subseteq A(N)$. Obviously the orders of I and N/I are prime numbers. Moreover, by $N^2 \subseteq I \subseteq A(N)$, both I and N/I are zero-rings, thus $N \in [\mathcal{O}, \mathcal{O}]$. $\Diamond$

Corollary 3. *A near-ring N belonging to $[\mathcal{O}, \mathcal{O}]$ is a zero-near-ring iff $I \subset A(N)$, where I is as in* Th. 4.

Proof. Let N be in $[\mathcal{O}, \mathcal{O}]$ and, by Th. 4, suppose the ideal I to be strictly contained in $A(N)$. Then $A(N)/I = N/I$ implies $A(N) = N$. The converse is trivial. $\Diamond$

Between near-rings of $[\mathcal{O}, \mathcal{O}]$ we can characterize the non-zero near-rings by the following

Theorem 5. *A near-ring N with $N^2 \neq \{0\}$, belongs to $[\mathcal{O}, \mathcal{O}]$ iff $|N| = = p^2$ and $N^3 = \{0\}$.*

Proof. Let $N \in [\mathcal{O}, \mathcal{O}]$ and $N^2 \neq \{0\}$. From Th. 4 and Cor. 3 it turns out that $I = A(N)$. Thus $N^3 = NN^2 \subseteq NI = \{0\}$. Moreover, due to an element $n \in N$ so that $nN = I$. we have $\operatorname{im} \gamma_n = nN = I = \ker \gamma_n$. Thus $|N| = |I|^2 = p^2$. The converse is trivial. $\Diamond$

Corollary 4. *A near-ring N, where $N^2 \neq \{0\}$ and N^+ is a cyclic group, belongs to $[\mathcal{O}, \mathcal{O}]$ iff $|N| = p^2$ and $A(N) \neq \{0\}$.*

Proof. If $N \in [\mathcal{O}, \mathcal{O}]$, the corollary holds trivially. Conversely, let N be a near-ring with $N^2 \neq \{0\}$, $|N| = p^2$, $A(N) \neq \{0\}$ and suppose that N^+ is a cyclic group. It should be noted that N has a proper ideal $A(N)$, which is a zero-ring of order p. Moreover, due to $|N| = p^2$, $N/A(N)$ is also of order p. It remains to show that $N/A(N)$ is a zero-ring. Because of $N^2 \neq \{0\}$, we get $\gamma_n \neq O_n$ for some $n \in N$. Since $\gamma_n(N)$ is an additive subgroup of N^+, if $\gamma_n \neq O_N$ then $\gamma_n(N) = N$ or $\gamma_n(N) = A(N)$. Now, $\gamma_n(N) = N$ implies $\ker \gamma_n = \{0\}$ and the last condition is absurd, because of $A(N) \subseteq \ker \gamma_n$. Thus $nN = A(N)$ or $nN = \{0\}$. In each case $N^2 \subseteq A(N)$. $\Diamond$

Theorem 6. *A near-ring N belongs to $[\mathcal{O}, \mathcal{O}]$ iff it arises by defining a Clay function $F\colon N \to \operatorname{End}(N)$ on an additive group N of order pq, where $F(N) \subseteq \operatorname{End}_I(N)^{(2)}$ and $F(I) = \{O_N\}$, I being a normal subgroup of N.*

Proof. Let $N \in [\mathcal{O}, \mathcal{O}]$; obviously $|N| = pq$. Let I be an ideal of N. From Th. 3 we have $I \subseteq A(N)$, that is $IN = \{0\}$. Thus the Clay function coupled with the product of N, satisfies the required conditions.

(2) $\operatorname{End}_I(N) = \{f \in \operatorname{End}(N) / f(N) \subseteq I \subseteq \ker f\}$.

Conversely, let N be a group of order pq, let I be one of its normal subgroups and $F\colon N \to \mathrm{End}(N)$ a Clay function such that $F(N) \subseteq \subseteq \mathrm{End}_I(N)$ and $F(I) = \{O_N\}$. Obviously $I \subseteq \ker f$, $\forall f \in F(N)$, thus $NI = \{0\}$, that means I is a left ideal of N and a zero-ring. Analogously $f(N) \subseteq I$, $\forall f \in F(N)$, thus $N^2 \subseteq I$, that means I is a right ideal of N and N/I is a zero-ring. Whence $N \in [\mathcal{O}, \mathcal{O}]$. $\Diamond$

Near-rings belonging to $[\mathcal{A}, \mathcal{A}]$ or to $[\mathcal{O}, \mathcal{A}]$ are characterized by the following theorems.

Theorem 7. *A near-ring N belongs to $[\mathcal{A}, \mathcal{A}]$ iff $N \simeq A \oplus B$, where $A, B \in \mathcal{A}$.*

Proof. Let $N \in [\mathcal{A}, \mathcal{A}]$ and let $N \supset I \supset \{0\}$ be the invariant series such that N/I and I belong to $\mathcal{A}$. From Th. 6 of [4], we know that the radical $J_2(N)$ is nilpotent and $N/J_2(N)$ is a direct sum of A-simple and strongly monogenic near-rings. So $J_2(N) \notin \{N, I\}$. If it is also $J_2(N) \neq \{0\}$, then $J_2(N) + I$ is an ideal of N and, recalling that I is a maximal ideal, $J_2(N) \oplus I = N$. Thus $N/I \simeq J_2(N)$ is nilpotent, and this is absurd. So $J_2(N) = \{0\}$ and the theorem holds. $\Diamond$

Theorem 8. *A near-ring N belongs to $[\mathcal{O}, \mathcal{A}]$ iff $N \simeq A \oplus B$, where $A \in \mathcal{O}$ and $B \in \mathcal{A}$.*

Proof. It is trivial, by Th. 4 of [4]. $\Diamond$

It should be noted that $[\mathcal{O}, \mathcal{A}]$ is included in $[\mathcal{A}, \mathcal{O}]$. The following example shows that the inclusion is strict.

Example 1. As additive group we consider the symmetric group of degree 3 and we define the following product

$*$	0	a	b	c	x	y
0	0	0	0	0	0	0
a	0	a	a	a	0	0
b	0	a	c	b	y	x
c	0	a	b	c	x	y
x	0	0	0	0	0	0
y	0	0	0	0	0	0

In this way N is a near-ring, $I = \{0, x, y\}$ is the only ideal of N, $I \in \mathcal{O}$ and $N/I \in \mathcal{A}$. Thus $N \in [\mathcal{A}, \mathcal{O}]$, but $N \notin [\mathcal{O}, \mathcal{A}]$. We can note that the near-ring under definition is isomorphic to a Φ-sum $I +_\Phi A$, where $A = \{0, a\}$, and this Φ-sum is not a direct sum. In general, proper Φ-sums (not direct sums) of $[\mathcal{A}, \mathcal{O}]$ are characterized by the following theorems.

Theorem 9. *Let N be a proper Φ-sum, then N belongs to $[\mathcal{A},\mathcal{O}]$ iff $N \simeq \simeq A+_\Phi B$, where $B \in \mathcal{A}$, $A \in \mathcal{O}$ and it is the only ideal of N, $f(\langle 0,0\rangle) = = O_A$, $\overline{f}(\langle 0,0\rangle) = O_B$, $\overline{f}_{a,b} = \overline{f}_{0,b}$ $\forall a \in A$, $\forall b \in B$.*

Proof. Let N be a proper Φ-sum of $[\mathcal{A},\mathcal{O}]$. Then N has a left invariant subgroup B and only one proper ideal A belonging to $\mathcal{O}$. Thus we can represent N as $A +_\Phi B$ and the theorem holds by Prop. 4 of [2]. Conversely, let $N = A +_\Phi B$, where $A \in \mathcal{O}$ and $B \in \mathcal{A}$, and let $f(\langle 0,0\rangle) = O_A$, $\overline{f}(\langle 0,0\rangle) = O_B$, $\overline{f}_{a,b} = \overline{f}_{0,b}$ ($\forall a \in A$, $\forall b \in B$). From Prop. 4 of [2] we see that N is zero-symmetric, A° is an ideal of N isomorphic to A, that means $A^\circ \in \mathcal{O}$, and also $N/A^\circ \simeq {}^\circ B \simeq B \in \mathcal{A}$. Thus $N \in [\mathcal{A},\mathcal{O}]$. $\Diamond$

Theorem 10. *A near-ring N belonging to $[\mathcal{A},\mathcal{O}]$ such that $\overset{+}{N} = = \overset{+}{A} \oplus \overset{+}{B}$ and where $|A|$ and $|B|$ are prime numbers p and q, with $p \neq q$, can be represented as a Φ-sum.*

Proof. Let $N \in [\mathcal{A},\mathcal{O}]$, with $\overset{+}{N} = \overset{+}{A} \oplus \overset{+}{B}$ and let $|A| = p$, $|B| = q$, $p \neq q$. Because N has a proper ideal, this one must be equal to A° or ${}^\circ B$. Suppose A° be the ideal of N. That means ${}^\circ B \in \mathcal{A}$. If $nb = 0$, $\forall n \in N$, $\forall b \in B$, then $n(a+b) = na \in {}^\circ A$, $\forall a \in A$, whence $N^2 \subseteq A^\circ$. This implies ${}^\circ B \simeq N/A^\circ \in \mathcal{O}$, which is a contradiction. Thus, there is an $n \in N$ for which $n{}^\circ B \neq \{0\}$. Because $n{}^\circ B$ is a proper subgroup of N of order q we have $n{}^\circ B = {}^\circ B$. Thus ${}^\circ B$ is a left invariant subgroup and, by Th. 1 of [2], $N = A +_\Phi B$. $\Diamond$

Finally, among Φ-sums of $[\mathcal{A},\mathcal{O}]$ such that their additive group is a direct sum of two groups, we can characterize the non-monogenic case.

Theorem 11. *Let $N = A +_\Phi B$ belonging to $[\mathcal{A},\mathcal{O}]$ with $\overset{+}{N} = \overset{+}{A} \oplus \oplus \overset{+}{B}$. Then N is non-monogenic iff $N \in [\mathcal{O},\mathcal{A}]$.*

Proof. If $N \in [\mathcal{O},\mathcal{A}]$, then $N \simeq A \oplus B$, by Th. 8, and, obviously, it is not monogenic.

Conversely, let N be a non-monogenic near-ring with $\overset{+}{N} = \overset{+}{A} \oplus \oplus \overset{+}{B}$, where $N = A +_\Phi B$ belongs to $[\mathcal{O},\mathcal{A}]$. To show that $N = A \oplus \oplus B$, it is sufficient to prove that ${}^\circ B$ is a right ideal of N. Firstly, we prove that $f_{a,b} = O_A$, $\forall a \in A$, $\forall b \in B$. Suppose $b^2 = 0$. We have $(\langle a,b\rangle)^2\langle a,0\rangle = \langle f_{a,b}(a), b^2\rangle\langle a,0\rangle = \langle 0,0\rangle$, but also $\langle a,b\rangle(\langle a,b\rangle\langle a,0\rangle) = = \langle a,b\rangle\langle f_{a,b}(a),0\rangle = \langle f^2_{a,b}(a),0\rangle$. Thus $f^2_{a,b}(a) = 0$ and $f_{a,b}$ is not an automorphism. Whence $f_{a,b} = O_A$. Suppose now $b^2 \neq 0$. Then $b{}^\circ B = = {}^\circ B$ and $f_{a,b} \neq O_A$ imply $f_{a,b}(A) = A^\circ$. Thus $\langle a,b\rangle N = N$, but

N is not monogenic, whence $f_{a,b} = O_A$. For this reason $(\langle a,b\rangle + +\langle 0,\bar{b}\rangle)\langle a',b'\rangle - \langle a,b\rangle\langle a',b'\rangle = \langle a,b+\bar{b}\rangle\langle a',b'\rangle - \langle a,b\rangle\langle a',b'\rangle = \langle f_{a,b+\bar{b}}(a'), (b+\bar{b})b'\rangle - \langle f_{a,b}(a'), bb'\rangle = \langle 0, (b+\bar{b})b' - bb'\rangle \in {}^\circ B$. Then ${}^\circ B$ is a right ideal. $\Diamond$

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BEWEGUNGSVORGÄNGE DES FLAGGENRAUMES MIT SPHÄRISCHEN BAHNEN II

DIE DREIPARAMETRIGEN BEWEGUNGSVORGÄNGE

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Abstract: In part I of this paper the four parameter motions of flag space, which move some of the points on spherical trajectories, were described. Part II. will now treat with three parameter motions. A complete classification of these motions is given. The two and one parameter cases turn out to be Bricard motions or motions, which can be imbedded into more parameter cases. This is why the posed problem is entirely solved for the case of flag space.

In J. Lang [1] wurde ein Weg beschrieben, der es erlaubt, Bewegungsvorgänge des Flaggenraumes, welche sphärische Bahnen besitzen, systematisch zu untersuchen. Im ersten Teil [2] dieser Arbeit[1] wurden die vierparametrigen Bewegungsvorgänge mit dieser Eigenschaft beschrieben, indem unter Verwendung eines Übertragungsprinzipes die Geraden eines sechsdimensionalen projektiven Raumes bezüglich einer Gruppe (10) klassifiziert wurden. Die fünfparametrigen Bewegungsvorgänge, welche Punkte mit sphärischen Bahnen besitzen, sind nicht von

[1] Die Verweise auf die Formeln (1) bis (74) beziehen sich auf diesen ersten Teil der Arbeit.

Interesse, da ihnen Punkte (0-dimensionale Unterräume) des Raumes $P(V)$ entsprechen. Sie sind entweder trivial oder führen nur die Punkte einer vollisotropen Geraden des Gangraumes auf Sphären (siehe J. Lang [1], Satz 2).

Die Klassifikation der Ebenen des „*Bedingungsraumes* $P(V)$" im Abschnitt 2 von [2] liefert die dreiparametrigen Bewegungsvorgänge des Flaggenraumes mit sphärischen Bahnen dreidimensionale Teilräume von $P(V)$ (Abschnitt 3) ergeben zweiparametrige Bewegungsvorgänge mit sphärischen Bahnen. In beiden Abschnitten werden wir auch auf Bricard-Bewegungsvorgänge stoßen, welche (unter Voraussetzung 4-maliger stetiger Differenzierbarkeit) schon von O. Röschel [3] betrachtet wurden.

Wie wir in Abschnitt 4 zeigen können, ergeben sich bei Betrachtung der vier- und höherdimensionalen Unterräume des Raumes $P(V)$ Teilzwangläufe mehrparametriger Bewegungsvorgänge, die schon in den vorangehenden Abschnitten beschrieben worden sind (siehe auch Fußnote 5). Somit ist *die betrachtete Fragestellung mit den hier behandelten drei- und zweiparametrigen Bewegungsvorgängen vollständig gelöst* (siehe dazu Fußnote 6).

2. Klassifikation der zweidimensionalen Bedingungsräume

Wir wollen zunächst alle jene Bewegungsvorgänge untersuchen, welche den Ebenen des Bedingungsraumes entsprechen. Sei T eine Ebene im Bedingungsraum $P(V)$. Der Raum W_{06} ist definiert durch $\omega_0 = \omega_6 = 0$.

2.1. Der Schnitt von T mit dem Raum W_{06} ist ein Punkt

Wir setzen also:

Fall 1: $T \cap W_{06} =: K$. Die Ebene T sei die Verbindungsebene dreier Punkte $T = [GKH]$, wobei wir neben $K \in W_{06}$ noch $G \in H_0 \setminus W_{06}$ und $H \in H_6 \setminus W_{06}$ wählen. Wir setzen an[2]:

[2] Wir haben zu berücksichtigen, daß der Punkt K ein ausgezeichneter Punkt der Ebene T ist, wohingegen die Punkte $G \in H_0$ und $H \in H_6$ willkürlich in der entsprechenden Spur $T \cap H_0$ bzw. $T \cap H_6$ gewählt wurden. Da wir aber Aussagen erwarten, die bezüglich der Gruppe (10) invariant sind, haben wir diese Punkte bei

$$
(75)\quad \begin{aligned}
&G\ldots(0\colon\gamma_1+\sigma\kappa_1\colon\gamma_2+\sigma\kappa_2\colon\gamma_3+\sigma\kappa_3\colon\gamma_4+\sigma\kappa_4\colon\gamma_5+\sigma\kappa_5\colon 1)\\
&K\ldots(0\colon\kappa_1\colon\kappa_2\colon\kappa_3\colon\kappa_4\colon\kappa_5\colon 0)\\
&H\ldots(1\colon\chi_1+\tau\kappa_1\colon\chi_2+\tau\kappa_2\colon\chi_3+\tau\kappa_3\colon\chi_4+\tau\kappa_4\colon\chi_5+\tau\kappa_5\colon 0).
\end{aligned}
$$

Man beachte, daß die Bedingung $\kappa_3 \neq 0$ oder auch $\kappa_4 \neq 0$ bezüglich (10) invariant ist. Wir können also unterscheiden:

Fall 1a: $\kappa_3\kappa_4 \neq 0$. Wir setzen $\kappa_3 = 1$ und erhalten[3]:

$$
(76)\quad \begin{aligned}
&G\ldots(0\colon 0\colon\gamma_2\colon 0\colon 0\colon 0\colon 1)\\
&K\ldots(0\colon 0\colon 0\colon 1\colon\kappa_4\colon\kappa_5\colon 0)\\
&H\ldots(1\colon\chi_1\colon\chi_2\colon 0\colon\chi_4\colon\chi_5\colon 0).
\end{aligned}
$$

Je nachdem[4], ob

1. $\chi_4 \neq 0$,
2. $\chi_4 = 0, \quad \chi_2 \neq 0$, oder
3. $\chi_2 = \chi_4 = 0, \quad \chi_5 \neq 0$

ist, löst man das aus den Bedingungen G, K, H

$$
(77)\quad \begin{aligned}
&G\ldots\gamma_2 Y_2 + Y_6 &&= 0\\
&K\ldots Y_3 + \kappa_4 Y_4 + \kappa_5 Y_5 &&= 0\\
&H\ldots Y_0 + \chi_1 Y_1 + \chi_2 Y_2 + \chi_4 Y_4 + \chi_5 Y_4 &&= 0
\end{aligned}
$$

und der Bedingung (13) bestehende System geeignet auf und erhält (78), (79) bzw. (80) also Normalformen der entsprechenden dreiparametrigen Bewegungsvorgänge:

$$
(78)\quad \begin{aligned}
&\bar{x} = x + t_1\\
&\bar{y} = y + t_2 x + \frac{\chi_1\kappa_4 t_1 + \chi_2\kappa_4 t_2 + \chi_5\kappa_4 t_3 + \kappa_4 t_1^2}{\chi_4} - \kappa_5 t_3 \qquad \chi_4 \neq 0.\\
&\bar{z} = z - \frac{\chi_1 t_1 + \chi_2 t_2 + \chi_5 t_3}{\chi_4} x + t_3 y + \frac{\chi_1\gamma_4 t_1 + \chi_2\gamma_4 t_2 + \chi_5\gamma_4 t_3 + \gamma_4 t_1^2}{\chi_4} - \gamma_2 t_2
\end{aligned}
$$

---

der Suche nach Normalformen zu spezialisieren: Für die im folgenden verwendeten Parameter σ, τ sind geeignete Werte zu wählen.

[3]Mit Hilfe von (10) erhalten wir bei

$$
t_a = \frac{-\gamma_3\kappa_4 + \kappa_3\gamma_4 + \kappa_2}{2},\ t_b = -\kappa_1,\ t_c = \frac{-\gamma_3\kappa_4\kappa_5 - \gamma_4\kappa_5 + 2\kappa_4\gamma_5 + \kappa_2\kappa_5}{2\kappa_4},
$$

$$
t_d = -\gamma_1 + \kappa_1\gamma_3,\ t_e = \frac{-\gamma_3\kappa_4 + \gamma_4 - \kappa_2}{2\kappa_4},\ \sigma = \frac{-\gamma_3\kappa_4 - \gamma_4 + \kappa_2}{2\kappa_3\kappa_4},\ \tau = -\chi_3
$$

die angegebene Normalform (76).

[4]Der Wert χ_4 ändert sich beim Übergang auf die Normalform (76) nicht mehr; ebenso ändert sich im Fall $\chi_4 = 0$ der Wert χ_5 beim Übergang zur Normalform nicht. Die entsprechenden Fallunterscheidungen sind also, nachdem $\tau = -\chi_3$ gesetzt wurde, einfach zu treffen.

$$(79)\qquad \begin{aligned} \bar{x} &= x + t_1 \\ \bar{y} &= y - \tfrac{\chi_1 t_1 + \chi_5 t_2 + t_1^2}{\chi_2} x - \kappa_4 t_3 - \kappa_5 t_2 \qquad \chi_2 \neq 0. \\ \bar{z} &= z + t_3 x + t_2 y + \tfrac{\chi_2 \gamma_2 t_1 + \chi_5 \gamma_2 t_2 + \gamma_2 t_1^2}{\chi_2} - \gamma_4 t_3 \end{aligned}$$

$$(80)\qquad \begin{aligned} \bar{x} &= x + t_1 \\ \bar{y} &= y + t_2 x + \tfrac{\kappa_5(\chi_1 t_1 + t_1^2)}{\chi_5} - \kappa_4 t_3 \qquad \chi_5 \neq 0. \\ \bar{z} &= z + t_3 x - \left(\tfrac{\chi_1}{\chi_5} t_1 + t_1^2\right) y - \gamma_2 t_2 - \gamma_4 t_3 \end{aligned}$$

Bei $\chi_2 = \chi_4 = \chi_5 = 0$ erhält man einen Bewegungsvorgang, der ein Teilzwanglauf eines vierparametrigen Bewegungsvorganges (siehe Abschnitt 1) ist[5].

In jedem dieser unter Fall 1a zusammengefaßten Typen betrachten wir den Schnitt der Ebene $T = [GKH] \subset P(V)$ mit der Kegelfläche Θ. Aus der Parameterdarstellung der Ebene T

$$(81)\qquad \begin{aligned} &(\omega_0 : \ldots : \omega_6) = \\ &= (\mu_2 : \chi_1 \mu_2 : \gamma_2 \mu_0 + \chi_2 \mu_2 : \mu_1 : \kappa_4 \mu_1 + \chi_4 \mu_2 : \kappa_5 \mu_1 + \chi_5 \mu_2 : \mu_0) \end{aligned}$$

und (8) erhalten wir die Gleichung

$$(82)\qquad T \cap \Theta \ldots \gamma_2 \mu_0^2 + \chi_2 \mu_0 \mu_2 - \kappa_4 \mu_1^2 - \chi_4 \mu_1 \mu_2 = 0$$

des Schnittes $T \cap \Theta$. Genau bei

$$(83)\qquad \gamma_2 = \kappa_4 = \chi_2 = \chi_4 = 0$$

ist $T \subset \Theta$. Eine Parameterdarstellung jener Punktmenge des Gangraumes, deren Elemente bei einem Bewegungsvorgang des betrachteten Falles Sphären durchlaufen, kann für jeden der angeführten Unterfälle von Fall 1a durch

$$(84)\qquad x_0(\mu_0, \mu_1, \mu_2) = \frac{\mu_1 \kappa_4 + \mu_2 \chi_4}{\mu_0}, \quad y_0(\mu_0, \mu_1, \mu_2) = \frac{\mu_1 \kappa_5 + \mu_2 \chi_5}{\mu_0}$$

angegeben werden, wobei (82) zu berücksichtigen ist. (84) ist genau dann die Darstellung einer isotropen Ebene, wenn

[5]Immer dann, wenn die Schnittkurve zwischen dem Raum T und der Hyperfläche Θ zerfällt, ist der zugehörige Bewegungsvorgang Teil eines vierparametrigen Bewegunsgsvorganges, der schon in Abschnitt 1 behandelt wurde. Die Untersuchung solcher Bewegungen ist nur dann interessant, wenn die Menge jener Punkte, die auf sphärischen Bahnen laufen, sich von jener des vierparametrigen Bewegungsvorganges unterscheidet. Dies ist hier nicht der Fall.

$$\kappa_4\chi_5 - \chi_4\kappa_5 = 0 \tag{85}$$

ist. Die Punktmenge ist also im allgemeinen eine vollisotrope Zylinderfläche 2. Ordnung[6]. Bei (85) degeneriert die Punktmenge (84) in eine isotrope oder vollisotrope Ebene und es ergibt sich ein Bewegungsvorgang, den wir hier nicht gesondert betrachten müssen (siehe Fußnote 5).

Ist (85) nicht erfüllt, so zerfällt die vollisotrope Zylinderfläche 2. Ordnung, deren Punkte sphärische Bahnen durchlaufen, genau dann, wenn die Schnittkurve $T \cap \Theta$ im Bedingungsraum selbst singulär ist, also bei

$$\kappa_4\chi_2^2 - \gamma_2\chi_4^2 = 0. \tag{86}$$

Die Typen (78), (79) umfassen auch Fälle singulärer Punktmengen; diese bilden jeweils ein Paar isotroper oder vollisotroper Ebenen, die im Sonderfall auch in die absolute Ebene fallen können. Beim Typ (80) ist die Punktmenge stets singulär.

Die Bahnsphären sind aus (9) zu erhalten und besitzen die Darstellung

$$\begin{aligned}\mu_0\mu_2x^2 + (\mu_0\mu_1\kappa_1 + \mu_0\mu_2\chi_1 - 2\mu_1\mu_2\kappa_4 - 2\mu_2^2\chi_4)x+\\ +\mu_0\mu_1y + \mu_0^2z + E = 0.\end{aligned} \tag{87}$$

Ihr Radius ist gegeben durch

$$R(\mu_0, \mu_1, \mu_2) = -\frac{\mu_0}{2\mu_2}. \tag{88}$$

Der Fernscheitel der Bahnsphäre hat die Koordinaten

$$U \ldots (0{:}\,0{:}\, -\mu_0{:}\, \mu_1). \tag{89}$$

Wir haben damit den

Satz 1. *Im Fall* 1a *ergeben sich Normalformen der Gestalt* (78), (79), (80). *Die Punkte mit sphärischen Bahnen erfüllen eine vollisotrope Zylinderfläche zweiter Ordnung, welche durch* (82) *und* (84) *gegeben ist. Die zugehörigen Bahnsphären – ihre Radien und ihre Fernscheitel sind durch* (88) *und* (89) *gegeben – sind durch die Gleichung* (87) *beschrieben. Ist* (83) *erfüllt, so handelt es sich um einen*

[6] Die Flächen zweiter Ordnung des Flaggenraumes wurden systematisch klassifiziert in [4]. Die hier auftretenden Zylinderflächen gehören im allgemeinen zum Typ 32 der dort angegebenen Klassifikation, in speziellen Fällen zum Typ 33, 43 oder 55.

Bricard-Bewegungsvorgang. Alle Punkte des Gangraumes werden auf Sphären geführt. Der Bewegungsvorgang gehört dann zu Typ (80). *Zwei Zwangläufe, die dieselben Invarianten*[7] $\gamma_2, \kappa_4, \kappa_5, \chi_2, \chi_4, \chi_5$ *besitzen, sind konjugiert bezüglich der Gruppe* $G_6^{(2)}$.

Der Bricard-Bewegungsvorgang aus (80) findet sich auch in O. Röschel [16].

Man beachte, daß auch χ_3 in (75) eine Invariante sowohl bezüglich der Gruppe (10) als auch bezüglich des Ansatzes (75) ist. Wir unterscheiden also:

Fall 1b: $\kappa_3 = 0$, $\kappa_4 \neq 0$. Sei vorerst $\chi_3 \neq 0$. Mit (75) und $\kappa_4 = 1$ erhalten wir[8]

$$
\begin{aligned}
&G \ldots (0{:}\, 0{:}\, \gamma_2{:}\, 0{:}\, 0{:}\, 0{:}\, 1)\\
&K \ldots (0{:}\, \kappa_1{:}\, \kappa_2{:}\, 0{:}\, 1{:}\, 0{:}\, 0)\\
&H \ldots (1{:}\, \chi_1{:}\, 0{:}\, \chi_3{:}\, 0{:}\, \chi_5{:}\, 0).
\end{aligned}
\tag{90}
$$

Daraus gewinnen wir die Normalform

$$
\begin{aligned}
\bar{x} &= x + t_1\\
\bar{y} &= y + t_2 x - \tfrac{\chi_5 t_3 + \chi_1 t_1 + t_1^2}{\chi_3} \qquad\qquad \chi_3 \neq 0.\\
\bar{z} &= z - (\kappa_1 t_1 + \kappa_2 t_2)x + t_3 y - \gamma_2 t_2
\end{aligned}
\tag{91}
$$

Die Punkte, welchen sphärische Bahnen zugewiesen sind, ergeben sich aus

$$
x_0(\mu_0, \mu_1, \mu_2) = \frac{\mu_1}{\mu_0}, \qquad y_0(\mu_0, \mu_1, \mu_2) = \frac{\mu_2}{\mu_0}\chi_5. \tag{92}
$$

und der Gleichung

$$
\mu_0^2 \gamma_2 + \mu_0 \mu_1 \kappa_2 - \mu_1 \mu_2 \chi_3 = 0. \tag{93}
$$

Dieser vollisotrope Zylinder zweiter Ordnung zerfällt genau dann, wenn entweder $\chi_5 = 0$ ist (eine isotrope Ebene) oder wenn

$$
\gamma_2 = 0 \tag{94}
$$

gilt. Bei (94) zerfällt die Menge aller sphärisch geführten Punkte in eine vollisotrope Ebene $\nu \ldots \mu_1 = 0$ und eine isotrope Ebene, welche

---

[7]Eine geometrische Deutung der auftretenden Invarianten wäre eine reizvolle Aufgabe, welche aber vom Umfang den Rahmen dieser Arbeit sprengen würde.

[8]Mit $t_a = \frac{\chi_2 - \kappa_2 \chi_4}{\chi_3}$, $t_b = \kappa_5$, $t_c = \frac{\gamma_5 \chi_3 + \kappa_5 \chi_2 - \kappa_2 \kappa_5 \chi_4 - \gamma_4 \kappa_5 \chi_3}{\chi_3}$, $t_d = \frac{\kappa_1 \kappa_2 \chi_4 - \gamma_1 \chi_3 + \kappa_1 \gamma_4 \chi_3 - \kappa_1 \chi_2 - \gamma_3 \kappa_5 \chi_3}{\chi_3}$, $t_e = -\gamma_3, \sigma = \frac{\chi_2 - \kappa_2 \chi_4 - \gamma_4 \chi_3}{\chi_3}$, $\tau = -\chi_4$ erhalten wir aus (75) mit Hilfe von (10) die Gestalt (90).

durch $\mu_0 \colon \mu_2 = \chi_3 \colon \kappa_2$ und (92) gegeben ist. Die Bahnsphären werden durch

$$\mu_2\mu_0 x^2 + (\mu_0\mu_1\kappa_1 + \mu_0\mu_2\chi_1 - 2\mu_1\mu_2)x + \mu_0\mu_2\chi_3 y + \mu_0^2 z + E = 0, \tag{95}$$

ihre Radien durch (88) und ihre Fernscheitel durch

$$U \ldots (0 \colon 0 \colon -\mu_0 \colon \mu_2\chi_3) \tag{96}$$

festgelegt.

Ist hingegen $\chi_3 = 0$, so ergibt sich in Fall 1b ein Bewegungsvorgang, der schon in einem vierparametrigen Bewegungsvorgang (siehe Fußnote 5) enthalten und deshalb hier nicht von Interesse ist.

Fall 1c: $\kappa_4 = 0, \kappa_3 \neq 0$. Sei etwa $\kappa_3 = 1$. Wir erhalten nur bei $\chi_4 \neq 0$ ein interessantes Resultat: Die Normalform des Unterraumes T wird aufgespannt von den Punkten[9]

$$\begin{aligned} &G \ldots (0 \colon 0 \colon \gamma_2 \colon 0 \colon 0 \colon 0 \colon 1) \\ &K \ldots (0 \colon 0 \colon \kappa_2 \colon 1 \colon 0 \colon \kappa_5 \colon 0) \\ &H \ldots (1 \colon \chi_1 \colon 0 \colon 0 \colon \chi_4 \colon \chi_5 \colon 0). \end{aligned} \tag{97}$$

und wir errechnen daraus die Normalform des Bewegungsvorganges zu

$$\begin{aligned} \bar{x} &= x + t_1 \\ \bar{y} &= y + t_2 x - \kappa_2 t_2 - \kappa_5 t_3 \qquad\qquad \chi_4 \neq 0. \\ \bar{z} &= z - \tfrac{t_3\chi_5 + t_1\chi_1 + t_1^2}{\chi_4} x + t_3 y - \gamma_2 t_2 \end{aligned} \tag{98}$$

Die Punkte mit sphärischen Bahnen sind gegeben durch

$$x_0(\mu_0, \mu_1, \mu_2) = \frac{\chi_4\mu_2}{\mu_0}, \qquad y_0(\mu_0, \mu_1, \mu_2) = \frac{\kappa_5\mu_1 + \chi_5\mu_2}{\mu_0} \tag{99}$$

und die Gleichung

$$-\chi_4\mu_1\mu_2 + \gamma_2\mu_0^2 + \kappa_2\mu_0\mu_1 = 0. \tag{100}$$

Das ist die Darstellung einer vollisotropen Zylinderfläche zweiter Ordnung, welche genau für

$$\gamma_2\kappa_5 = 0 \tag{101}$$

in ein Ebenenpaar ($\gamma_2 = 0$) oder eine Doppelebene ($\kappa_5 = 0$) zerfällt. Der Fall $\kappa_5 = 0$ ist also nicht von Interesse (siehe Fußnote 5). Die Bahnträgersphären sind von der Gestalt

[9] Wir setzen in (75) und (10) $t_a = \gamma_4$, $t_b = -\kappa_1$, $t_c = -\gamma_3\kappa_5 + \gamma_5 + \frac{\kappa_5\chi_2 - \kappa_2\kappa_5\chi_3}{\chi_4}$, $t_d = \kappa_1\gamma_3 - \gamma_1$, $t_e = \frac{\kappa_2\chi_3 - \chi_2}{\chi_4}$, $\sigma = \frac{-\gamma_3\chi_4 + \chi_2 - \kappa_2\chi_3}{\chi_4}$, $\tau = -\chi_3$.

$$\mu_2\mu_0 x^2 + (\mu_0\mu_2\chi_1 - \mu_2^2\chi_4)x + \mu_0\mu_1 y + \mu_0\mu_2\chi_4 z + E = 0, \tag{102}$$

ihre Radien sind durch (88), ihre Fernscheitel durch (89) gegeben.

Fall 1d: $\kappa_3 = \kappa_4 = 0$. Dieser Fall ist nur für $\kappa_2\kappa_5\chi_3\chi_4 \neq 0$ interessant[10]. Wir setzen also $\kappa_2 = 1$. Fall 1d liefert[11]

$$\begin{aligned} &G \ldots (0{:}\,0{:}\,0{:}\,0{:}\,0{:}\,0{:}\,1) \\ &K \ldots (0{:}\,\kappa_1{:}\,1{:}\,0{:}\,0{:}\,\kappa_5{:}\,0) \\ &H \ldots (1{:}\,\chi_1{:}\,0{:}\,\chi_3{:}\,\chi_4{:}\,0{:}\,0). \end{aligned} \tag{103}$$

Wir erhalten die Normalform des Bewegungsvorganges als

$$\begin{aligned} \bar{x} &= x + t_1 \\ \bar{y} &= y - (t_1\kappa_1 + t_3\kappa_5)x + t_2 \qquad\qquad \chi_4 \neq 0. \\ \bar{z} &= z - \tfrac{t_2\chi_3 + t_1\chi_1 + t_1^2}{\chi_4}x + t_3 y - \gamma_2 t_2 \end{aligned} \tag{104}$$

Die Punkte mit sphärischen Bahnen sind gegeben durch

$$x_0(\mu_0, \mu_1, \mu_2) = \frac{\chi_4\mu_2}{\mu_0}, \qquad y_0(\mu_0, \mu_1, \mu_2) = \frac{\kappa_5\mu_1}{\mu_0}, \tag{105}$$

wobei die Werte $\mu_0{:}\,\mu_1{:}\,\mu_2$ der Gleichung

$$\kappa_2\mu_0\mu_1 - \chi_3\chi_4\mu_2^2 = 0 \tag{106}$$

genügen müssen. Das ist die Darstellung einer vollisotropen Zylinderfläche zweiter Ordnung, welche genau für

$$\kappa_2\kappa_5\chi_3\chi_4 = 0 \tag{107}$$

in eine isotrope Ebene degeneriert. Die Bahnträgersphären sind von der Gestalt

$$\mu_2\mu_0 x^2 + (\mu_0\mu_2\kappa_1 - \mu_0\mu_2\chi_1 - 2\mu_2^2\chi_4)x + \mu_0\mu_2\chi_3 y + \mu_0^2 z + E = 0, \tag{108}$$

ihre Radien sind durch (88), ihre Fernscheitel durch (96) gegeben.

Wir fassen zusammen:

---

[10] Man beachte, daß die Werte $\kappa_2, \kappa_5, \chi_3, \chi_4$ wegen $\kappa_3 = \kappa_4 = \kappa_6 = \chi_6 = 0$ invariant sowohl bezüglich (10) als auch bezüglich des Ansatzes (75) sind; diese Fallunterscheidung ist also auch vor dem Übergang zu einer Normalform sinnvoll.

[11] Wir wählen hier $t_a = \gamma_4$, $t_b = \frac{\gamma_3\kappa_5\chi_4 + \chi_5 + \gamma_4\kappa_5\chi_3 - \kappa_5\chi_2}{\chi_4}$, $t_c = \gamma_3\gamma_4\kappa_5 - \gamma_2\kappa_5 + \gamma_5$, $t_d = \frac{-\gamma_1\chi_4 - \kappa_1\gamma_3\gamma_4\chi_4 + \kappa_1\gamma_2\chi_4 - \gamma_3^2\kappa_5\chi_4 - \gamma_3\chi_5 - \gamma_3\gamma_4\kappa_5\chi_3 - \gamma_3\kappa_5\chi_2}{\chi_4}$, $t_e = -\gamma_3$, $\sigma = \gamma_3\gamma_4 - \gamma_2$, $\tau = \gamma_2\chi_4 + \gamma_4\chi_3 - \chi_2$ und erhalten aus (75) die Gestalt (103).

Satz 2. *In den Fällen* 1b, 1c, 1d *ergeben sich Bewegungsvorgänge mit Normalformen* (91), (98), (104). *Die Menge der sphärisch geführten Punkte ist jeweils durch* (92) *und* (93), (99) *und* (100), (105) *und* (106) *gegeben. Sie ist im allgemeinen eine vollisotrope Zylinderfläche zweiter Ordnung, in speziellen Fällen auch ein Paar vollisotroper Ebenen oder eine einzige vollisotrope Ebene.* (91) *wird bei* $\gamma_2 = \kappa_2 = \chi_3 = 0 \neq \chi_5$ *zu einem Bricard-Bewegungsvorgang. Unter* (98) *treten keine solchen Spezialfälle auf. Bei* (104) *werden genau dann* alle *Punkte des Gangraumes auf sphärischen Bahnen geführt, wenn neben* $\kappa_2 = \chi_3 = 0$ *noch* $\kappa_5, \chi_4 \neq 0$ *erfüllt ist. Die in* (91), (98), (104) *angegebenen Koeffizienten* $\gamma_i, \kappa_i, \chi_i$ $(i \in \{1, \ldots, 5\})$ *sind Invarianten.*

2.2. Der Unterraum T schneidet W_{06} nach einer Geraden k.

Fall 2: $T \cap W_{06} = k$. Sei also $T = [Gk]$ mit $G \notin W_{06}$. Wir schreiben die Gerade k als Verbindungsgerade zweier Punkte $k = [KL]$ an, wobei wir $K \ldots (0{:}\, \kappa_1{:}\, \kappa_2{:}\, \kappa_3{:}\, \kappa_4{:}\, \kappa_5{:}\, 0)$ und $L \ldots (0{:}\, \lambda_1{:}\, \lambda_2{:}\, \lambda_3{:}\, \lambda_4{:}\, \lambda_5{:}\, 0)$ ansetzen. Die Unterräume $W_{036} \ldots \omega_0 = \omega_3 = \omega_6 = 0$ und $W_{046} \ldots \omega_0 = \omega_4 = \omega_6 = 0$ sind invariant gegenüber (10). Wir nehmen vorerst an, daß die Gerade k jeden dieser beiden Teilräume in genau einem Punkt trifft und daß diese beiden Punkte verschieden sind.

Fall 2a: $k \cap W_{036} = K \neq L = k \cap W_{046}$, $T \not\subset H_6$. Wir setzen an[12]:

$$(109)\quad \begin{aligned} &G \ldots (\gamma_0{:}\, \gamma_1 + \sigma\kappa_1 + \tau\lambda_1{:}\, \gamma_2 + \sigma\kappa_2 + \tau\lambda_2{:}\, \gamma_3 + \sigma\kappa_3 + \tau\lambda_3{:}\, \gamma_4 + \\ &\qquad + \sigma\kappa_4 + \tau\lambda_4{:}\, \gamma_5 + \sigma\kappa_5 + \tau\lambda_5{:}\, 1) \\ &K \ldots (0{:}\, \kappa_1{:}\, \kappa_2{:}\, 0{:}\, 1{:}\, \kappa_5{:}\, 0) \\ &L \ldots (0{:}\, \lambda_1{:}\, \lambda_2{:}\, 1{:}\, 0{:}\, \lambda_5{:}\, 0). \end{aligned}$$

Wir erhalten mit Hilfe von (10) aus diesem Ansatz bei geeigneter Wahl von $t_a, t_b, t_c, t_d, t_e, \sigma, \tau$ die Gestalt

[12] Die Punkt K, L sind in diesem Fall ausgezeichnete Punkte der Ebene T. Der Punkt $G \notin W_{06}$ hingegen wird willkürlich in $T \setminus W_{06}$ angenommen. Wie im Fall 1 haben wir bei der Suche nach Normalformen zu spezialisieren: Für die verwendeten Parameter σ, τ sind geeignete Werte zu wählen.

$$(110)\qquad \begin{aligned} &G\ldots(\gamma_0\colon 0\colon \gamma_2\colon 0\colon 0\colon 0\colon 1)\\ &K\ldots(0\colon \kappa_1\colon 0\colon 0\colon 1\colon 0\colon 0)\\ &L\ldots(0\colon \lambda_1\colon 0\colon 1\colon 0\colon \lambda_5\colon 0). \end{aligned}$$

Die zugehörige Normalform des Bewegungsvorganges ist:

$$(111)\qquad \begin{aligned} \bar{x} &= x + t_1\\ \bar{y} &= y + t_2 x - t_1\lambda_1 - t_3\lambda_5\\ \bar{z} &= z - \kappa_1 t_1 x + t_3 y - \gamma_2 t_2 - \gamma_0 t_1^2. \end{aligned}$$

Die Punkte

$$(112)\qquad x_0(\mu_0,\mu_1,\mu_2) = \frac{\mu_1}{\mu_0},\qquad y_0(\mu_0,\mu_1,\mu_2) = \frac{\lambda_5\mu_2}{\mu_0},$$

für die die Bedingung

$$(113)\qquad \gamma_2\mu_0^2 - \mu_1\mu_2 = 0$$

erfüllt ist, werden auf sphärischen Bahnen geführt. Diese vollisotropen Zylinderfläche zweiter Ordnung ist genau bei

$$(114)\qquad \lambda_5\gamma_2 = 0$$

singulär. Sie degeneriert bei $\lambda_5 = 0$ in eine isotrope Ebene, bei $\gamma_2 = 0$ in ein Paar isotroper Ebenen, von denen die eine vollisotrop ist. Die Bahnträgersphären sind von der Gestalt

$$(115)\qquad \mu_0\gamma_0 x^2 + (\mu_1\kappa_1 - \mu_2\lambda_1 - 2\mu_1\gamma_0)x + \mu_2 y + \mu_0 z + E = 0,$$

ihre Radien sind gegeben durch $R = -\frac{1}{2\gamma_0}$, ihre Fernscheitel durch $U\ldots$ $\ldots(0\colon 0\colon -\mu_0\colon \mu_2)$. Man beachte, daß in diesem Fall auch Bewegungsvorgänge inkludiert sind, bei denen alle angegebenen Punkte speziell auf ebenen Bahnen geführt werden ($\gamma_0 = 0$).

Fall 2b: $k \subset W_{036}$, $k \not\subset W_{046}$, $T \not\subset H_6$. Man zeigt durch Rechnung, daß dieser Fall im Sinne von Fußnote 5 nicht von Interesse ist. Dasselbe gilt für:

Fall 2c: $k \subset W_{046}$, $k \not\subset W_{036}$, $T \not\subset H_6$.

Fall 2d: k trifft den Schnittraum $W_{0346} = W_{036} \cap W_{046}$ in genau einem Punkt K. $T \not\subset H_6$. Wir setzen vorerst

$$(116)\quad \begin{aligned} &(\gamma_0 \colon \gamma_1 + \sigma\kappa_1 + \tau\lambda_1 \colon \gamma_2 + \sigma\kappa_2 + \tau\lambda_2 \colon \gamma_3 + \sigma\kappa_3 + \tau\lambda_3 \colon \\ &\qquad \colon \gamma_4 + \sigma\kappa_4 + \tau\lambda_4 \colon \gamma_5 + \sigma\kappa_5 + \tau\lambda_5 \colon 1) \\ &K \ldots (0 \colon \kappa_1 \colon \kappa_2 \colon 0 \colon 0 \colon \kappa_5 \colon 0) \\ &L \ldots (0 \colon \lambda_1 + \rho\kappa_1 \colon \lambda_2 + \rho\kappa_2 \colon 1 \colon \lambda_4 \colon \lambda_5 + \rho\kappa_5 \colon 0). \end{aligned}$$

Bei $\kappa_2 \neq 0$ ergibt sich (wir setzen $\kappa_2 = 1$) nach geeigneter Wahl von $t_a, t_b, t_c, t_d, \sigma, \tau, \rho$

$$(117)\quad \begin{aligned} &G \ldots (\gamma_0 \colon 0 \colon 0 \colon 0 \colon 0 \colon 0 \colon 1) \\ &K \ldots (0 \colon \kappa_1 \colon 1 \colon 0 \colon 0 \colon \kappa_5 \colon 0) \\ &L \ldots (0 \colon 0 \colon 0 \colon 1 \colon \lambda_4 \neq 0 \colon \lambda_5 \colon 0). \end{aligned}$$

Auch hier ist (bei $\gamma_0 = 0$) der Fall, daß alle betrachteten Punkte **ebene** Bahnen besitzen, miteingeschlossen (Darboux-Bewegungsvorgänge). Nur im Fall $\kappa_5 \neq 0$ ist ein (im Sinn von Fußnote 5) interessantes Resultat zu erwarten. Die Normalform lautet dann:

$$(118)\quad \begin{aligned} &\bar{x} = x + t_1 \\ &\bar{y} = y + t_2 x + \tfrac{\kappa_1\lambda_5 t_1 + \kappa_2\lambda_5 t_2}{\kappa_5} - \lambda_4 t_3 \qquad \lambda_4\kappa_5 \neq 0. \\ &\bar{z} = z + t_3 x - \tfrac{\kappa_1 t_1 + \kappa_2 t_2}{\kappa_5} y - \gamma_0 t_1^2 \end{aligned}$$

Die Punkte jenes vollisotropen Zylinders Φ zweiter Ordnung, der durch

$$(119)\quad x_0(\mu_0, \mu_1, \mu_2) = \lambda_4 \frac{\mu_2}{\mu_0}, \quad y_0(\mu_0, \mu_1, \mu_2) = \frac{\kappa_5\mu_1 + \lambda_5\mu_2}{\mu_0},$$

und

$$(120)\quad \mu_0\mu_1 - \lambda_4\mu_2^2 = 0.$$

bestimmt ist, durchlaufen sphärische Bahnen. Φ ist unter den obigen Voraussetzungen jedenfalls regulär. Die Bahnsphären sind gegeben durch

$$(121)\quad \mu_0\gamma_0 x^2 + (\mu_1\kappa_1 - 2\mu_2\lambda_4\gamma_0)x + \mu_2 y + \mu_0 z + E = 0,$$

ihre Radien werden angegeben durch $R = -\frac{1}{2\gamma_0}$, ihre Fernscheitel durch $U \ldots \ldots (0 \colon 0 \colon -\mu_0 \colon \mu_2)$ festgelegt.

Ist $\kappa_2 = 0$, so ergibt sich für $\kappa_1\lambda_4 + \kappa_5 \neq 0$ ein Fall, der sich als nicht interessant erweist.

Ist $\kappa_2 = 0$ und ist $\kappa_1\lambda_4 + \kappa_5 = 0$, so können wir $\kappa_1 = 1$ setzen. Nach geeigneter Wahl von $t_a, t_b, t_c, t_d, \sigma, \tau, \rho$ erhalten wir aus (116) die Normalgestalt

$$(122)\qquad \begin{aligned} &G\ldots(\gamma_0\colon 0\colon \gamma_2\colon 0\colon 0\colon 0\colon 1)\\ &K\ldots(0\colon \kappa_1\colon 0\colon 0\colon 0\colon -\kappa_1\lambda_4\colon 0)\\ &L\ldots(0\colon 0\colon 0\colon 1\colon \lambda_4\neq 0\colon \lambda_5\colon 0). \end{aligned}$$

Die Normalform des Bewegungsvorganges lautet dann:

$$(123)\qquad \begin{aligned} \bar{x} &= x+t_1\\ \bar{y} &= y+t_2x-\tfrac{\lambda_5}{\lambda_4}t_1-\lambda_4t_3 \qquad \lambda_4\neq 0.\\ \bar{z} &= z+t_3x+\tfrac{1}{\lambda_4}t_1y-\gamma_2t_2-\gamma_0t_1^2 \end{aligned}$$

Die Punkte

$$(124)\qquad x_0(\mu_0,\mu_1,\mu_2)=\lambda_4\frac{\mu_2}{\mu_0},\qquad y_0(\mu_0,\mu_1,\mu_2)=\frac{-\kappa_1\lambda_4\mu_1+\lambda_5\mu_2}{\mu_0}$$

mit der Eigenschaft

$$(125)\qquad \lambda_4\mu_2^2-\gamma_2\mu_0^2=0$$

erfüllen ein Paar vollisotroper Ebenen des Gangraumes, die für $\gamma_4\lambda_4 >$ > 0 reell und verschieden sind. Sie werden auf Sphären, bei $\gamma_0 = 0$ in Ebenen, geführt, welche in der Gestalt (115) gegeben sind.

Fall 2e: $T\subset H_6$. Nach J. Lang [1], Satz 2 (b), kann man, wenn man triviale Fälle außer acht läßt, die Punkte G, K, L hier durch

$$(126)\qquad \begin{aligned} &G\ldots(\gamma_0\colon \gamma_1\colon 1\colon 0\colon 0\colon 0\colon 0)\\ &K\ldots(0\colon \kappa_1\colon 1\colon 0\colon 0\colon 0\colon 0)\\ &L\ldots(0\colon 1\colon 0\colon 0\colon 0\colon 0\colon 0). \end{aligned}$$

ansetzen. Eine einfache Rechnung zeigt, daß alle durch solche Unterräume $T=[GKL]$ bestimmten Bewegungsvorgänge das Bündel vollisotroper Geraden elementweise festlassen. Die Punktbahnen sind durchwegs vollisotrope Geraden; dieser Fall ist trivial.

Satz 3. *Hat der Raum T mit dem Unterraum W_{06} eine Gerade k gemeinsam, so ergeben sich, wenn $T\not\subset H_6$ gilt, folgende nichttrivialen Fälle:*

Liegt die Schnittgerade k zu den Teilräumen W_{036} und W_{046} allgemein, so ergibt sich die Normalform (111), *welche die Punkte einer vollisotropen Zylinderfläche Φ zweiter Ordnung auf Sphären, im Spezialfall auch auf Ebenen führt. Φ kann auch in ein Paar isotroper Ebenen zerfallen, von denen eine vollisotrop ist.*

Trifft k die Räume W_{036} und W_{046} in einem gemeinsamen Punkt $K \ldots (0{:}\,\kappa_1{:}\,\kappa_2{:}\,0{:}\,0{:}\,\kappa_5{:}\,0)$, so ergibt sich bei $\kappa_2 \neq 0$ ein Bewegungsvorgang mit der Normalform (118). *Unter den obigen Voraussetzungen ergibt sich eine reguläre vollisotrope Zylinderlfäche zweiter Ordnung, bestehend aus sphärisch geführten Punkten. Im Fall $\kappa_2 = 0$ ergibt sich nur für $\kappa_1\lambda_4 + \kappa_5 = 0$ (siehe* (116)*) ein nichttrivialer Bewegungsvorgang* (123). *Er führt die Punkte zweier vollisotroper Ebene auf sphärischen Bahnen.*

Ist jedoch $T \subset H_6$, so ergibt sich nur ein trivialer Fall eines dreiparametrigen Bewegungsvorganges, bei dem alle Punktbahnen vollisotrope Geraden sind.

2.3. Der Unterraum T liegt in W_{06}.

Dieser Fall liefert, wie man nach kurzer Rechnung erkennt, nur triviale Bewegungsvorgänge.

3. Die höherdimensionalen Teilräume von $P(V)$.

Ist dim $T = 3$, so haben wir zweiparametrige Bewegungsvorgänge zu erwarten. Gemäß J. Lang [1] liefert die Projektion $(\omega_0{:}\ldots{:}\,\omega_6) \longrightarrow (\omega_4{:}\,\omega_5{:}\,\omega_6) \in W_{0123}$ als Bild von $T \cap \Theta$ 1. eine Gerade, 2. eine Kurve zweiter Ordnung oder 3. die gesamte Bildebene W_{0123}.

Im ersten und im zweiten Fall wählen wir drei Bildpunkte allgemeiner Lage und bestimmen zugehörige Urbildpunkte. Durch diese drei Punkte von $T \cap \Theta$ legen wir eine Ebene $T_2 \subset T$. Ihr Schnitt mit Θ liefert genau dieselbe Bildmenge wie $T \cap \Theta$ selbst.

Im dritten Fall schließen wir so: Existiert eine Ebene $T_2 \subset T$, welche ganz in Θ liegt, so ist der zugehörige dreiparametrige Bewegungsvorgang eine Bricard-Bewegung, die wir schon in Abschnitt 2 betrachtet haben; der zu T gehörige Bewegungsvorgang ist ein Teilzwanglauf davon. Gibt es eine solche Ebene $T_2 \subset \Theta \cap T$ nicht, so erhalten wir einen Fall eines zweiparametrigen Bricard-Bewegungsvorganges. Er ist im Sinne von Fußnote 5 nicht Teilzwanglauf eines schon behandelten Typs. Solche zweiparametrige Bricard-Bewegungsvorgänge wurden von O. Röschel [3] als Typ III (13) gefunden.

Eine Klassifikation der vierdimensionalen Unterräume von $P(V)$

würde eine Klassifikation der einparametrigen Bewegungsvorgänge mit sphärischen Bahnen und deren Invarianten liefern. Wir haben aber vorausgesetzt, zwei Bewegungsvorgänge ξ und ξ', bei denen ξ' ein Teilzwanglauf von ξ ist, nur dann gesondert zu betrachten, wenn sich die Menge der sphärisch geführten Punkte bei beiden Bewegungsvorgängen unterscheidet. Wie in Abschnitt 3 erkennt man: Ergibt sich gemäß J. Lang [1] bei der Projektion (24) als Bild von $T \cap \Theta$ eine Kurve zweiter Ordnung, so kann man dasselbe Bild auch durch Projektion eines geeigneten *ebenen* Schnittes von Θ erhalten, wobei die Schnittebene Teilmenge des 4-Raumes T ist. Der zu T gehörige Bewegungsvorgang wäre dann ein Teilzwanglauf eines Typs, wie er schon in Abschnitt 2 behandelt worden ist. Ist die Bildmenge von $T \cap \Theta$ jedoch die gesamte Bildebene W_{0123}, so ist der Schnitt $T \cap \Theta$ selbst entweder ein dreidimensionaler Teilraum von Θ oder eine Hyperfläche zweiter Ordnung in T, zu der aber die Gerade $Z \cap T =: z$ gehört. Im ersten Fall liefert jede nichtprojizierende Ebene $\varepsilon \subset T \cap \Theta$ dasselbe Bild. Im zweiten Fall findet man einen dreidimensionalen Teilraum $T_3 \subset T$, der mit z nur einen Punkt Z_0 gemeinsam hat und T nach einer Fläche zweiter Ordnung schneidet. Die Projektion von $T \cap \Theta$ liefert ebenfalls die gesamte Ebene W_{0123} als Bild. Der zu T_3 gehörige zweiparametrige Bewegungsvorgang ist aber schon behandelt worden: Der zu T gehörige Zwanglauf ist ein Teilzwanglauf eines Bricard-Bewegungsvorganges, wie er schon in Abschnitt 3 gefunden wurde.

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NON-REGULARITY OF SOME TOPOLOGIES ON $\mathbb{R}^n$ STRONGER THAN THE STANDARD ONE[1]

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Abstract: The finest topology on $\mathbb{R}^n$ ($n \geq 2$) which induces the Euclidean topology on each line is not regular and have big character and extent. The same holds for the finest topology which induces the Euclidean topology on each line parallel to a coordinate axis; this latter topology is symmetrizable.

0. Introduction

Investigating Minkowski space M (the "real" 4-dimensional space-time continuum) Zeeman suggested some alternative topologies for M, [12, 13]. The *fine topology* on M induces the 3-dimensional Euclidean topology on every space-axis and the 1-dimensional Euclidean topology on every time-axis, and it is the finest topology satisfying this property, [13]. The reader could consult also [1], [4], [5], [7, 8, 9, 10] for a more detailed view of this topic. In this paper we investigate some of the properties of the finest topology on $\mathbb{R}^n$ which induces the Euclidean topology on each line in $\mathbb{R}^n$ (resp. each line parallel to a coordinate axis).

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1. Preliminaries

Let $\mathcal{A}_*^n = \{\ell\colon \ell$ is a line in $\mathbb{R}^n\}$ and $\mathcal{A}_+^n = \{\ell \in \mathcal{A}_*^n\colon \ell$ is parallel to a coordinate axis$\}$. Let $\mathbb{R}_*^n$ (resp. $\mathbb{R}_+^n$) be the set $\mathbb{R}^n$ with the following topology: a set $U \subseteq \mathbb{R}^n$ is open in $\mathbb{R}_*^n$ (resp. $\mathbb{R}_+^n$) if and only if $U \cap \ell$ is open with respect to the standard topology on ℓ, for each line $\ell \in \mathcal{A}_*^n$ (resp. $\ell \in \mathcal{A}_+^n$). The topology of $\mathbb{R}_*^n$ is stronger than the standard topology of $\mathbb{R}^n$, and the topology of $\mathbb{R}_+^n$ is stronger than the topology of $\mathbb{R}_*^n$. For instance, any circle without a point is closed in $\mathbb{R}_*^n$ although it is not closed in the standard topology of $\mathbb{R}^n$; any non-parallel to a coordinate axis line without a point is closed in $\mathbb{R}_+^n$ although it is not closed in $\mathbb{R}_*^n$. The space $\mathbb{R}_+^n$ is symmetrizable by the symmetric $\mathrm{s}(x, y)$ defined as follows:

$$\mathrm{s}(x,y) = \begin{cases} \|x-y\| & \text{if } x \text{ and } y \text{ lie on a line that is} \\ & \text{parallel to a coordinate axis;} \\ 1 & \text{otherwise} \end{cases}$$

($\|\cdot\|$ stands for the usual norm in $\mathbb{R}^n$). We recall that a *symmetric* s on a topological space X is a function from $X \times X$ into $\mathbb{R}$ such that: a) $\mathrm{s}(x,y) = \mathrm{s}(y,x) \geq 0$ for each $x, y \in X$; b) $\mathrm{s}(x,y) = 0$ iff $x = y$; c) a set $U \subseteq X$ is open iff for each $x \in U$ there exists $r > 0$ such that the "ball" $K(x,r) = (y : s(x,y) < r\}$ is contained in U ([2], [3], see also [11]).

In Section 2 we prove that, for $n \geq 2$, both $\mathbb{R}_*^n$ and $\mathbb{R}_+^n$ are not regular. The first, for $n = 3$, answers a question of Prof. Otto Laback (of Technical University Graz, Austria). The second, for $n = 2$, answers a question of Prof. Stoyan Nedev (Institute of Mathematics, BAN, Sofia, Bulgaria). In Section 3 we investigate some cardinal functions of $\mathbb{R}_*^n$ and $\mathbb{R}_+^n$ (density, weight, character, extent and spread) and the results once more show that $\mathbb{R}_*^n$ and $\mathbb{R}_+^n$ are not regular. Throughout the paper $\mathbb{Q}$ denotes the set of all rational numbers. The i-th coordinate of an $x \in \mathbb{R}^n$ is denoted by x_i, $x = (x_1, \ldots, x_n)$.

2. Non-regularity of $\mathbb{R}_*^n$ and $\mathbb{R}_+^n$

Lemma 2.1. *Let $n \geq 2$ and U is a subset of $\mathbb{R}^n$ which is open with respect to the standard topolgy of $\mathbb{R}^n$. Then there exist a set $F \subseteq U$ such that:*

(a) $\mathrm{cl}\, F = \mathrm{cl}\, U$, where "cl" denotes the standard closure in $\mathbb{R}^n$.

(b) Each line in $\mathbb{R}^n$ passes through at most two points of F (and hence, F is closed in both $\mathbb{R}^n_*$ and $\mathbb{R}^n_+$).

Proof. The lemma is trivial if $U = \emptyset$, so let $U \neq \emptyset$. Let $B(x,r)$ denotes the ball $\{y \in \mathbb{R}^n : \|x-y\| < r\}$. Then the family $\mathcal{B}(U) = \{B(x,\frac{1}{k}) : x \in \mathbb{Q}^n, k \in \mathbb{N}, B(x,\frac{1}{k}) \subseteq U\}$ is countable and we can write it as $\mathcal{B}(U) = \{B_i : i \in \mathbb{N}\}$. By induction we will pick points $x^i \in B_i$ and define sets $F_i = \{x^j : j \leq i\}$ such that:

(∗) each line in $\mathbb{R}^n$ passes through at most two points of F_i.

Finally we will set $F = \cup\{F_i : i \in \mathbb{N}\}$.

In order to do this let us pick an $x^1 \in B_1$ and let $F_1 = \{x^1\}$. Let us suppose that, for some $i \in \mathbb{N}$ and for each $j \leq i$ we have picked points $x^j \in B_j$ such that the condition (∗) holds. Let $\mathcal{A}(F_i)$ be the family of these lines in $\mathbb{R}^n$ that passes through exactly two points of F_i. Since $\mathcal{A}(F_i)$ is finite, the set $B_{i+1} \setminus (\cup\mathcal{A}(F_i))$ is not empty. We can pick a point x^{i+1} from it and define $F_{i+1} = \{x^j : j \leq i+1\}$. The condition (∗) holds for F_{i+1}, because $x^{i+1} \notin \cup\mathcal{A}(F_i)$. The induction step is completed.

Now, let $F = \cup\{F_i : i \in \mathbb{N}\}$. It is clear that (b) holds. In order to prove (a) let $y \in \operatorname{cl} U$. For any standard neighbourhood V of y, $U \cap V \neq \emptyset$. Because $\mathbb{Q}^n$ is dense in $\mathbb{R}^n$, we can pick an $x \in \mathbb{Q}^n \cap U \cap V$. There is some $k \in \mathbb{N}$ such that $B(x,\frac{1}{k}) \subseteq U \cap V$. There is some $i \in \mathbb{N}$ for which $B(x,\frac{1}{k}) = B_i$ and hence $x^i \in U \cap V$. Hence $F \cap V \neq \emptyset$ and $y \in \operatorname{cl} F$; $\operatorname{cl} U \subseteq \operatorname{cl} F$. ◊

Proposition 2.2. *For $n \geq 2$, the spaces $\mathbb{R}^n_*$ and $\mathbb{R}^n_+$ are not regular.*

Proof. Because $\mathbb{R}^2_*$ (resp. $\mathbb{R}^2_+$) is a closed subspace of $\mathbb{R}^n_*$ (resp. $\mathbb{R}^n_+$) it suffices to show that $\mathbb{R}^2_*$ and $\mathbb{R}^2_+$ are not regular.

By Lemma 2.1, there is a set $F \subseteq \mathbb{R}^2 \setminus \{O\}$ (where $O = (0,0)$) such that $\operatorname{cl} F = \mathbb{R}^2$ and F is closed in both $\mathbb{R}^n_*$ and $\mathbb{R}^n_+$. We will show that the O and F have no disjoint neighbourhoods in $\mathbb{R}^2_+$ (and hence, also in $\mathbb{R}^2_*$).

Let $O \in U$ and $F \subseteq V$ where U and V are open subsets of $\mathbb{R}^2_+$. We will show that $U \cap V \neq \emptyset$. There is an $\varepsilon > 0$ such that the horizontal interval $J = \{x : |x_1| < \varepsilon,\ x_2 = 0\}$ is included in U (x_i denotes the i-th coordinate of a given point x). For each $x \in J$ and $n \in \mathbb{N}$ let $U(x,n)$ is the vertical interval with base x and highness $\frac{1}{n}$, i.e. $U(x,n) = \{(x_1,\delta) : 0 \leq \delta \leq \frac{1}{n}\}$. For each $n \in \mathbb{N}$ let $A_n = \{x \in J : U(x,n) \subseteq U\}$. Since $J \subseteq U$ and U is open in $\mathbb{R}^2_+$ we have that

$\bigcup\{A_n : n \in \mathbb{N}\} = J$. Since J is of second category there are an $n \in \mathbb{N}$ and a nonempty open interval $J' \subseteq J$ such that the standard closure of A_n contains J'. The set $P = \{x\colon x_1 \in J', 0 < x_2 < \frac{1}{n}\}$ is nonempty and open (in $\mathbb{R}^2$) and hence we can pick a point $y \in P \cap F$. Since $y \in \in V$ there exists $\mu > 0$ such that the horizontal interval $Y = \{z \in \mathbb{R}^2_+ : z_2 = y_2, |z_1 - y_1| < \mu\}$ is contained in V. Let x be a point from $A_n \cap J'$ such that $|x_1 - y_1| < \mu$. Then $U(x, n) \cap Y \neq \emptyset$ and hence $U \cap V \neq \emptyset$. $\Diamond$

Corollary 2.3. *The fine topology of Minkowski space M is not regular.*

Proof. We have that $\mathbb{R}^2_+$ is a closed subspace of M. In fact, $\mathbb{R}^2_+ \simeq \simeq T \times \mathbb{R} \hookrightarrow T \times \mathbb{R}^3 = M$, where $T = \mathbb{R}$ is the time, $\simeq$ denotes homeomorphism, $\hookrightarrow$ denotes homeomorphic embedding, and $\times$ denotes product of sets (not topological product). $\Diamond$

3. Some cardinal functions of $\mathbb{R}^n_*$ and $\mathbb{R}^n_+$

Let us recall the definitions of the cardinal functions *weight, character* and *density*, denoted (for a given topological space X) by $\mathrm{w}(X)$, $\chi(X)$ and $\mathrm{d}(X)$ respectively:

$$\mathrm{w}(X) = \min\{|\mathcal{B}| : \mathcal{B} \text{ is a base for the topology of } X\};$$
$$\chi(X) = \sup\{\chi(x, X) : x \in X\}, \text{ where}$$
$$\chi(x, X) = \min\{|\mathcal{B}_x| : \mathcal{B}_x \text{ is a base for the topology of } X \text{ at } x\};$$
$$\mathrm{d}(X) = \min\{|A| : A \text{ is dense in } X\}.$$

A space X is called *separable* if $\mathrm{d}(X) = \aleph_0$. For a detailed survey on cardinal functions see [6]. Another approach for showing that $\mathbb{R}^n_*$ and $\mathbb{R}^n_+$ are not regular is to use that for a regular space X, $\mathrm{w}(X) \leq 2^{\mathrm{d}(X)}$. In fact, $\mathbb{R}^n_*$ and $\mathbb{R}^n_+$ are separable but they have weight and character strongly greater than $\mathfrak{c} = 2^{\aleph_0}$.

Proposition 3.1. *For $n \geq 2$, $\chi(\mathbb{R}^n_*) > \mathfrak{c}$ and $\chi(\mathbb{R}^n_+) > \mathfrak{c}$.*

Proof. We consider the case $n = 2$. Suppose that $x = (x_1, x_2) \in \mathbb{R}^2$ and $\mathcal{U} = \{U_\alpha : \alpha < \mathfrak{c}\}$ is a family of neighbourhoods of x (either in $\mathbb{R}^2_*$ or in $\mathbb{R}^2_+$); we shall show that $\mathcal{U}$ cannot be a base at x. By induction, for each $\alpha < \mathfrak{c}$ we shall pick a point $x^\alpha \in U_\alpha \setminus \{x\}$ such that the following condition holds:

(C_α) each line in $\mathbb{R}^2$ contains at most two points of the set $C_\alpha = = \{x^\beta : \beta \leq \alpha\}$.

Then the set $C = \{x^\alpha : \alpha < \mathfrak{c}\}$ will be closed in $\mathbb{R}^2_*$ and $\mathbb{R}^2_+$. The set $V = \mathbb{R}^n \setminus C$ will be a neighbourhood of x (in $\mathbb{R}^n_*$ and $\mathbb{R}^n_+$) that does not contain any element of $\mathcal{U}$ (because V misses each x^α, $\alpha < \mathfrak{c}$).

Let $x^0 \in U_0 \setminus \{x\}$ and $C_0 = \{x^0\}$. Let $1 \leq \alpha < \mathfrak{c}$ and suppose that for each $\gamma < \alpha$ the points x^γ have already been picked so that the conditions (C_γ) hold. There is an $\varepsilon > 0$ such that the vertical interval $J = \{y : \mathrm{y}_1 = \mathrm{x}_1, |\mathrm{y}_2 - \mathrm{x}_2| < \varepsilon\}$ is contained in U_α. Since $\alpha < \mathfrak{c}$ there is an h, $\mathrm{x}_2 - \varepsilon < h < \mathrm{x}_2 + \varepsilon$, such that the horizontal line $\hbar = \{y : \mathrm{y}_2 = h\}$ misses x^γ, for each $\gamma < \alpha$. There is a $\delta > 0$ such that the horizontal interval $H = \{y : |\mathrm{y}_1 - \mathrm{x}_1| < \delta, \mathrm{y}_2 = h\}$ is included in $U_\alpha \cap \hbar$. Let $\mathcal{A} = \{\ell : \ell \text{ is a line in } \mathbb{R}^2 \text{ passing through two points of } \{x^\gamma : \gamma < \alpha\}\}$. Since $|\mathcal{A}| \leq \alpha^2 < \mathfrak{c}$ and $\hbar \notin \mathcal{A}$, we have that $|H \setminus \cup\mathcal{A}| = \mathfrak{c}$. We can pick an $x^\alpha \in H \setminus \cup\mathcal{A}$, $x^\alpha \neq x$, and define $C_\alpha = \{x^\gamma : \gamma \leq \alpha\}$. Because $x^\alpha \notin \cup\mathcal{A}$, (C_α) holds. ◊

Corollary 3.2. *For* $n \geq 2$, $\mathrm{w}(\mathbb{R}^n_*) = \chi(\mathbb{R}^n_*) > \mathfrak{c}$ *and* $\mathrm{w}(\mathbb{R}^n_+) = \chi(\mathbb{R}^n_+) > \mathfrak{c}$.

The author conjectures that $\mathrm{w}(\mathbb{R}^n_*) = \chi(\mathbb{R}^n_*) = \mathrm{w}(\mathbb{R}^n_+) = \chi(\mathbb{R}^n_+) = 2^\mathfrak{c}$ (the assumption $\mathfrak{c}^+ = 2^\mathfrak{c}$ implies these equations).

Proposition 3.3. *For* $n \geq 2$, *both* $\mathbb{R}^n_*$ *and* $\mathbb{R}^n_+$ *are separable.*

Proof. We shall show that the set $\mathbb{Q}^2$ is dense in $\mathbb{R}^2_+$ (and hence in $\mathbb{R}^2_*$). The cases $n \geq 3$ are similar to this one. Let U be a nonempty open subset of $\mathbb{R}^2_+$. Let us pick an $h \in \mathbb{R}$ such that the horizontal line $\hbar = \{x : \mathrm{x}_2 = h\}$ intersects U. Since $U \cap \hbar$ is open in $\hbar$ there is a $p \in \mathbb{Q}$ for which $(p, h) \in U \cap \hbar$. Let $\nu = \{x : \mathrm{x}_1 = p\}$. There is a $q \in \mathbb{Q}$ for which the point $(p, q) \in U \cap \nu$ and hence $\mathbb{Q}^2 \cap U \neq \emptyset$. ◊

Finally, let us recall that the *extent* $\mathrm{e}(X)$ of a space X is defined as

$$\mathrm{e}(X) = \sup\{|C| : C \text{ is a closed discrete subspace of } X\},$$

and the *spread* $\mathrm{s}(X)$ is defined as

$$\mathrm{s}(X) = \sup\{|C| : C \text{ is a discrete subspace of } X\}.$$

In the proof of Prop. 3.1, the set C (looked at as a subspace of either $\mathbb{R}^n_*$ or $\mathbb{R}^n_+$) is closed discrete, and $|C| = \mathfrak{c}$. So, we have proved:

Proposition 3.4. *For* $n \geq 2$, $\mathrm{e}(\mathbb{R}^n_*) = \mathrm{e}(\mathbb{R}^n_+) = \mathrm{s}(\mathbb{R}^n_*) = \mathrm{s}(\mathbb{R}^n_+) = \mathfrak{c}$.

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DEPTH OF DENDROIDS

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Abstract: It is shown that for every countable ordinal α there exists a fan whose depth is α and that depth of uniformly arcwise connected dendroids (in particular smooth fans) is less than or equal to two.

1. Introduction

S. D. Iliadis has constructed in [4] an uncountabe family of hereditarily decomposable and hereditarily unicoherent continua $X(\alpha)$ numbered with ordinals $\alpha < \omega_1$ having the property that

(1.1) for each $\alpha < \omega_1$ the depth of the α-th member of the family is just α (see also [7], Th. 24, p. 24).

The continua $X(\alpha)$ are arclike (they are called snake-like in [1] and [4]), and thus $X(1)$ only, being an arc, is arcwise connected. A question can be asked if it is possible to construct such a family (having property (1.1)) composed exclusively of arcwise connected continua. In this paper we construct a family of continua satisfying (1.1) and moreover such that each member of the family is hereditarily arcwise connected, hereditarily unicoherent, and has exactly one ramification point. In other words the family consists of fans. All the continua $X(\alpha)$ were planable (as arclike ones, since each arclike continuum is planable, see [1], Th. 4, p. 654). Our family also keeps this property.

It is shown in the final part of the paper that if we replace the assumption of arcwise connectedness by a stronger one, uniform arcwise connectedness, then no such a family does exist. Namely depth of uniformly arcwise connected continua is either 1 if they are locally connected, or 2 otherwise. Since smoothness of dendroids implies uniform arcwise connectedness, depth of a smooth dendroid is at most 2.

2. Preliminaries

A *continuum* means a compact connected metric space. A continuum is said to be *hereditarily unicoherent* provided that the intersection of any two its subcontinua is connected. A continuum that is hereditarily unicoherent and arcwise connected is called a *dendroid.* A locally connected dendroid is called a *dendrite.* A point p in a dendroid X is called an *end point* of X if p is an end point of every arc contained in X. The set of all end points of a dendroid X is denoted by $E(X)$. A point p in a dendroid X is called a *ramification point* of X if p is the vertex of a simple triod contained in X (i.e. if there are three points a, b and c in X such that any two of the three arcs pa, pb and pc have the point p in common only). The set of all ramification points of a dendroid X is denoted by $R(X)$. A dendroid having exactly one ramification point is called a *fan*, and the point is called the *vertex* of the fan. A fan is said to be *countable* provided that the set of all its end points is countable.

A continuum X is said to be *hereditarily decomposable* provided that every subcontinuum of X is the union of two its proper subcontinua. A hereditarily decomposable and hereditarily unicoherent continuum is called a *λ-dendroid.* Given a λ-dendroid X we denote by $\mathcal{P}(X)$ the family of all subcontinua S of X such that for each finite cover of X the elements of which are subcontinua of X, the continuum S is contained in a member of the cover. A (transfinite) well-ordered sequence (numbered with ordinals α) of nondegenerate subcontinua X_α of a λ-dendroid X is said to be *normal* provided that the following conditions are satisfied:

(2.1) $$X_1 = X;$$

(2.2) $$X_{\alpha+1} \in \mathcal{P}(X_\alpha);$$

(2.3) $$X_\beta = \bigcap\{X_\alpha : \alpha < \beta\} \text{ for each limit ordinal } \beta.$$

The *depth* $k(X)$ of a λ-dendroid X is defined as the minimum ordinal number γ such that the order type of each normal sequence of subcontinua of X is not greater than γ. The reader is referred to [4], [6] and [8] for an additional information about this concept. The following three important facts concerning the depth will be needed in the present paper (see [4], Ths. 1, 2 and 3, p. 94 and 95).

Fact 2.4. *For every two λ-dendroids X and Y if $Y \subset X$ then $k(Y) \leq \leq k(X)$.*

Fact 2.5. *A λ-dendroid X is locally connected (i.e., it is a dendrite) if and only if $k(X) = 1$.*

Fact 2.6. *If a λ-dendroid Y is a continuous image of a λ-dendroid X, then $k(Y) \leq k(X)$.*

Let a continuum X with a metric d be given, and let A and B be two its closed subsets. The *Hausdorff distance* dist from A to B is defined by

$$\operatorname{dist}(A, B) = \max\{\sup\{d(a, B)\colon a \in A\}, \sup\{d(b, A)\colon b \in B\}\}.$$

As usual, the symbol $\mathbb{N}$ stands for the set of all positive integers.

3. Fans

The main result of the paper is the following theorem.

Theorem 3.1. *For every ordinal number $\alpha < \omega_1$ there exists a countable plane fan $F[\alpha]$ having its depth $k(F[\alpha])$ equal to α.*

Proof. We proceed by transfinite induction. Let $\alpha = 1$. In the rectangular Cartesian coordinate system in the plane let $v = (0,0)$ be the origin. For each $n \in \mathbb{N}$ put $e(1,n) = (1/n, 1/n^2)$ and denote by $I(1,n)$ the straight line segment joining v and $e(1,n)$. Then the union

(3.2) $$F[1] = \bigcup\{I(1,n)\colon n \in \mathbb{N}\}$$

is a plane fan having the origin v as its vertex. The set $E(F[1]) = = \{e(1,n)\colon n \in \mathbb{N}\}$ of end points of $F[1]$ is countable, so the constructed fan is countable by its definition. Furthermore, since $F[1]$ is locally connected, we infer from Fact 2.5 that

$$k(F[1]) = 1. \tag{3.3}$$

Let $\beta \geq 1$ be an ordinal number. Assume that countable plane fans $F[\alpha]$ are defined for all ordinals $\alpha < \beta$ in such a way that

$$v \text{ is the vertex of } F[\alpha]; \tag{3.4}$$

(3.5) $F[\alpha]$ contains the union of a sequence of arcs $\{I(1,n)\colon n \in \mathbb{N}\}$;

$$k(F[\alpha]) = \alpha. \tag{3.6}$$

If $\beta = \alpha + 1$, we perform the following construction. For each $n \in \mathbb{N}$ we define an arc $I(\beta, n)$ such that

$$v \text{ is an end point of } I(\beta, n); \tag{3.7}$$

$$I(\beta, n) \cap F[\alpha] = \{v\}; \tag{3.8}$$

$$I(\beta, m) \cap \big(I(\beta, 1) \cup \ldots \cup I(\beta, m-1)\big) = \{v\} \text{ for each } m \in \mathbb{N} \text{ and } m \geq 2; \tag{3.9}$$

$$\operatorname{dist}\big(I(\beta, n), \bigcup\{I(\alpha, m)\colon m \in \{1, \ldots, n\}\}\big) < 1/2^n. \tag{3.10}$$

Put

$$F[\beta] = F[\alpha] \cup \bigcup\{I(\beta, n)\colon n \in \mathbb{N}\}. \tag{3.11}$$

Condition (3.10) implies that the arcs $I(\beta, n)$ better and better approximate the unions $\bigcup\{I(\alpha, m)\colon m \in \{1, \ldots, n\}\}$ and therefore it guarantees that the resulting space $F[\beta]$ is compact. Further, it can be observed by (3.10) and (3.11) that

(3.12) the union $\bigcup\{I(\beta, n)\colon n \in \mathbb{N}\}$ is a dense (and thus $F[\alpha]$ is a nowhere dense) subset of $F[\beta]$.

Connectedness of $F[\beta]$ follows from (3.8), and thus (3.11) assures that $F[\beta]$ is arcwise connected. Further, conditions (3.7)–(3.9) imply hereditary unicoherence of $F[\beta]$, so that $F[\beta]$ is a dendroid, and also they lead to the equality $R(F[\beta]) = R(F[\alpha]) = \{v\}$, so $F[\beta]$ is a fan having v as its vertex.

Denote by $e(\beta, n)$ this end point of $I(\beta, n)$ which is distinct from v. It can easily be seen from the construction that

$$(3.13) \qquad E(F[\beta]) = E(F[\alpha]) \cup \{e(\beta, n) \colon n \in \mathbb{N}\}.$$

Since $E(F[\alpha])$ is countable by assumption, (3.13) implies that $E(F[\beta])$ is countable, too, and thus the fan $F[\beta]$ is countable.

If a covering of $F[\beta]$ by finitely many subcontinua is considered, then condition (3.12) implies that $F[\alpha]$ is contained in a member of the covering. Thus $F[\alpha] \in \mathcal{P}(F[\beta])$, whence we infer that if a normal sequence $X_1, X_2, \ldots$ of subcontinua of $X = F[\beta]$ is considered, then the second term of it, X_2, is $F[\alpha]$, so we conclude from (3.6) that

$$(3.14) \qquad k(F[\alpha + 1]) = \alpha + 1.$$

If β is a limit ordinal, we consider a sequence

$$(3.15) \qquad \{\alpha_n \colon n \in \mathbb{N}\} \text{ with } \alpha_n < \beta \text{ and } \beta = \lim \alpha_n.$$

For each $n \in \mathbb{N}$ we take a copy of the fan $F[\alpha_n]$. Roughly speaking, we locate these copies in the plane in such a manner that their union, $F[\beta]$, is obtained from $F[\alpha_n]$'s in the same way as $F[1]$ is obtained from the segments $I(1, n)$. More rigorously, we assume that the considered copies of the fans $F[\alpha_n]$ satisfy the following conditions:

$$(3.16) \qquad \lim \operatorname{diam} F[\alpha_n] = 0,$$

$$(3.17) \qquad F[\alpha_n] \cap F[\alpha_m] = \{v\} \text{ if } n \neq m,$$

and we put

$$(3.18) \qquad F[\beta] = \bigcup \{F[\alpha_n] \colon n \in \mathbb{N}\}.$$

Condition (3.16) guarantees compactness, and (3.17) implies connectedness (thus arcwise connectedness) and hereditary unicoherence of the resulting space $F[\beta]$ which thereby is a countable plane fan. Finally Fact 2.4 implies by (3.6) and (3.18) that $\alpha_n = k(F[\alpha_n]) \leq k(F[\beta])$, whence by (3.15) we infer that

$$k(F[\beta]) = \beta.$$

Thus the fan $F[\alpha]$ is defined for each ordinal number $\alpha < \omega_1$, and it satisfies the needed equality (3.6). ◊

Remark 3.19. After constructing, for each ordinal $\alpha < \omega_1$, an arclike continuum $X(\alpha)$ with $k(X(\alpha)) = \alpha$ a question is asked in [4], Remark 3, p. 98, whether there exists 1° a λ-dendroid, or 2° an arclike λ-dendroid X having the depth ω_1. And it is shown in Section 3 of [6], p. 719, that an answer to 2° is negative, while an answer for the general case (i.e. for 1°) remains unknown. Thus, in connection with Th. 3.1, the following question seems to be natural.

Question 3.20. Does there exist a fan X such that $k(X) = \omega_1$?

Remark 3.21. If we consider, instead of arbitrary finite coverings of a λ-dendroid X by its subcontinua, finite coverings having a fixed number $n \geq 2$ of elements, we get a (similarly defined) concept of the n-depth (see [8], p. 587). Since for this concept the results that correspond to Facts 2.4 and 2.5 are also true (see [9], Theorems 4 and 5), and since only Facts 2.4 and 2.5 were used in the proof of Th. 3.1 above, one can repeat all the arguments of that proof replacing the depth $k(X)$ by the n-depth $k_n(X)$ for any continuum X considered in that proof. In this way we have the following corollary to Th. 3.1.

Corollary 3.22. *Let an ordinal number $\alpha < \omega_1$ be given, and let $F[\alpha]$ denote the countable plane fan of* Th. 3.1. *Then for each natural number $n \geq 2$ the n-depth $k_n(F[\alpha])$ of $F[\alpha]$ equals α.*

4. Uniformly arcwise connected dendroids

The cone F_c over the Cantor middle-thirds set is called the *Cantor fan.* A continuum X is said to be *uniformly pathwise connected* provided that it is a continuous image of the Cantor fan. The original definition of this concept, given in [5], p. 316, is more complicated, but it agrees with the above one by Th. 3.5 of [5], p. 322. A space X is said to be *uniformly arcwise connected* provided that it is arcwise connected and that for each $\varepsilon > 0$ there is a $j \in \mathbb{N}$ such that every arc in X contains j points that cut it into subarcs of diameters less than ε. By Th. 3.5 of [5], p. 322, each uniformly arcwise connected continuum is uniformly pathwise connected (but not conversely), and it is easy to see that for uniquely arcwise connected continua these two notions coincide (see [5], p. 316). In particular, the coincidence holds for dendroids.

Proposition 4.1. *The depth $k(F_c)$ of the Cantor fan F_c equals* 2.

Proof. Denote by v the vertex of F_c. It can easily be observed that a subcontinuum S of F_c is in $\mathcal{P}(F_c)$ if and only if it is an arc contained in the straight line segment vc for some end point c of F_c. Thus every normal sequence in F_c has the form $\{F_c, S\}$ for some $S \in \mathcal{P}(F_c)$, and the conclusion follows. ◊

Proposition 4.2. *If a dendroid X is uniformly arcwise connected, then $k(X) \leq 2$.*

Proof. Since each such a dendroid is a continuous image of the Cantor fan F_c by the definition, the conclusion is an immediate consequence of Prop. 4.1 and Fact 2.6. ◊

Remark 4.3. Note that the converse implication to that of Prop. 4.2 is not true. Namely there is a non-uniformly arcwise connected fan F_{P1} (see [2], (52), p. 201) for which we have $k(F_{P1}) = 2$.

As particular examples of uniformly arcwise connected dendroids one can consider smooth ones. A dendroid X is said to be *smooth* provided that there is a point $v \in X$ such that for each point $x \in X$ and for each sequence of points $\{x_n \in X : n \in \mathbb{N}\}$ tending to x we have $vx = \operatorname{Lim} vx_n$. It is known that every smooth dendroid is uniformly arcwise connected (see [3], Cor. 16, p. 318). Thus Prop. 4.2 leads to the following corollary.

Corollary 4.4. *If a dendroid X is smooth, then $k(X) \leq 2$.*

Remark 4.5. It is evident from the construction of the fans $F[\alpha]$ of Th. 3.1, especially by condition (3.12), that they are not smooth for $\alpha > 1$. And it follows from Cor. 4.3 that it is not possible to construct any family of smooth dendroids (therefore of smooth fans) $X[\alpha]$ (for every $\alpha < \omega_1$) having property (1.1), i.e. such that $k(X[\alpha]) = \alpha$.

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ON TOTALLY UMBILICAL SUBMANIFOLDS OF MANIFOLDS WITH CERTAIN RECURRENT CONDITION IMPOSED ON THE CURVATURE TENSOR

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Abstract: Totally umbilical submanifolds in manifolds which are generalisation of recurrent manifolds are investigated. At the end of the paper two examples are given.

1. Introduction

Investigating *conformally flat Riemannian manifolds of class one*, i.e. manifolds characterized by the property that at least $n-1$ principal normal curvatures are equal to one another, R. N. Sen and M. C. Chaki ([10]) found that if the remaining one is zero, then the curvature tensor satisfies

(1) $$R_{hijk,l} = 2a_l R_{hijk} + a_h R_{lijk} + a_i R_{hljk} + a_j R_{hilk} + a_k R_{hijl}\,,$$

where the "comma" denotes covariant derivative with respect to the metric. Hereafter, Riemannian manifolds with condition (1) imposed on the curvature tensor were examined ([1], [2], [3]). Some further generalisations of the condition (1) for various tensor fields were considered by L. Tamássy and T. Q. Binh ([11]). In [3] the present author proved

Proposition ([3]). *If the curvature tensor satisfies*

$$R_{hijk,l} = \sum_p \overset{p}{v}_{i_1} R_{i_2 i_3 i_4 i_5} ,$$

where the sum includes all permutation p of the indices (h,i,j,k,l) *and* $\left\{\overset{p}{v} = \left(\overset{p}{v}_1, \ldots, \overset{p}{v}_n\right)\right\}$ *is a set of some vectors, then there exists a vector* a_l *such that relation* (1) *holds.*

Hence it follows that on a *recurrent manifold*, i.e. on a manifold satisfying the condition

$$R_{hijk,l}R_{pqrs} - R_{hijk}R_{pqrs,l} = 0 ,$$

at each point where R_{hijk} does not vanish relation (1) is satisfied. Moreover, it was proved that on a neighbourhood of a generic point the vector a_l is a gradient ([3]).

In the paper we begin investigation of totally umbilical submanifolds of manifold satisfying the condition (1) for some vector field a_l. Throughout the paper all manifolds under consideration are assumed to be smooth connected Hausdorff manifolds and their metrics need not be definite.

2. Preliminaries

Let N be an n-dimensional Riemannian manifold with not necessarily definite metric g_{rs}, covered by a system of coordinate neighbourhoods $\{U; x^r\}$. We denote by Γ_{ij}^k, R_{hijk}, R_{hk}, R the Christoffel symbols, the curvature tensor, the Ricci tensor and the scalar curvature of N respectively. Here and in the sequel the indices h, i, j, k, l, r, s, t, u run over the range $1, 2, \ldots, n$. Let M be an m-dimensional manifold covered by a system of coordinate neighbourhoods $\{V; y^a\}$ immersed in manifold N and let $x^r = x^r(y^a)$ be its local expression in N. Then the local components g_{ab} of the induced metric tensor of M are related to g_{rs} by $g_{ab} = g_{rs}B_a^r B_b^s$, where $B_a^r = \frac{\partial x^r}{\partial y^a}$. In what follows we shall adopt the convention

$$B_{ab}^{rs} = B_a^r B_b^s , \quad B_{abc}^{rst} = B_a^r B_b^s B_c^t , \quad B_{abcd}^{rstu} = B_a^r B_b^s B_c^t B_d^u .$$

We denote by Γ_{ab}^c, K_{abcd}, K_{ad}, K the Christoffel symbols, the curvature tensor, the Ricci tensor and the scalar curvature of M with respect to g_{ab} respectively. Here and in the sequel the indices a, b, c, d, e, f run over the range $1, 2, \ldots, m$ $(m < n)$. The *van der Waerden-Bertolotti covariant derivative* ([12], [13]) of B_a^r is given by

(2) $$B^r_{a\,/\,b} = B^r_{a\cdot b} + \Gamma^{\,r}_{st} B^{st}_{ab} - B^r_c \Gamma^{\,c}_{ab} ,$$
where the "comma" and the dot denote covariant derivative with respect to g_{ab} and partial derivative.

The vector field H^r defined by $H^r = \frac{1}{m} g^{ab} B^r_{a\,/\,b}$ is called the *mean curvature vector* of M. Using (2) and the equation
$$\Gamma^{\,a}_{bc} = \left(B^r_{b\cdot c} + \Gamma^{\,r}_{st} B^{st}_{bc} \right) B^u_d g^{da} g_{ru}$$
we obtain on M
$$g_{rs} H^r B^s_a = 0 .$$
The *Schouten curvature tensor* H^r_{ab} of M is defined by
$$H^r_{ab} = B^r_{a\,/\,b} .$$
If the tensor H^r_{ab} satisfies the condition
$$H^r_{ab} = g_{ab} H^r ,$$
then M is said to be a *totally umbilical submanifold* of N.

Let N^r_x $(x, y, z = m+1, \dots, n)$ be pairwise orthogonal unit vectors normal to M. Then

(3) $$g_{rs} N^r_x N^s_x = e_x , \quad g_{rs} N^r_x N^s_y = 0 \quad (x \neq y), \quad g_{rs} N^r_x B^s_a = 0$$
and

(4) $$g^{rs} = B^{rs}_{ab} g^{ab} + \sum_x e_x N^r_x N^s_x ,$$
where e_x is the indicator of the vector N^r_x. On a totally umbilical submanifold M of a manifold N the Gauss and Codazzi equations take the form ([7])

(5) $$K_{abcd} = R_{rstu} B^{rstu}_{abcd} + H (g_{bc} g_{ad} - g_{bd} g_{ac})$$
and

(6) $$R_{rstu} B^{rst}_{abc} N^u_x = A_{ax} g_{ac} - A_{bx} g_{ac}$$
respectively, where
$$H = g_{rs} H^r H^s , \quad A_{ax} = H_{x\cdot a} + \sum_y e_y L_{ayx} H_y , \quad H_y = H^r N^s_y g_{rs}$$
and
$$L_{azy} = g_{rs} N^r_y N^s_{z\,/\,a} .$$
Moreover, we have ([6], [7])

(7) $$R_{rstu} H^r B^{stu}_{bcd} = \frac{1}{2} \left(g_{bc} H_d - g_{bd} H_c \right) , \quad H_c = H_{/\,c} ,$$

(8) $$K_{abcd/e} = R_{hijk;l}B^{hijkl}_{abcde} + H_e\left(g_{bc}g_{ad} - g_{bd}g_{ac}\right) + \frac{1}{2}\Big[H_a\left(g_{bc}g_{ed} - g_{bd}g_{ec}\right) + H_b\left(g_{ec}g_{ad} - g_{ed}g_{ac}\right) + H_c\left(g_{be}g_{ad} - g_{bd}g_{ae}\right) + H_d\left(g_{bc}g_{ae} - g_{be}g_{ac}\right)\Big],$$

where the semicolon denotes covariant derivative with respect to the metric of the ambient space,

(9) $$H^r{}_{/a} = -HB^r_a + \sum_z e_z A_{az} N^r_z .$$

We shall also use

Lemma 1 ([8]). (I) *Let (A_i), (B_i) be two sequences of numbers which are linearly independent as elements of the space $\mathbb{R}^n$. If T_{ij}, S_{ij} are numbers satisfying conditions*

$$T_{ij}A_k + T_{jk}A_i + T_{ki}A_j + S_{ij}B_k + S_{jk}B_i + S_{ki}B_j = 0,$$

$$T_{ij} = T_{ji}, \quad S_{ij} = S_{ji}$$

then there exist numbers D_i such that

$$T_{ij} = -B_iD_j - B_jD_i, \quad S_{ij} = A_iD_j + A_jD_i .$$

(II) *Let T_{ij}, A_k be numbers satisfying conditions*

$$T_{ij}A_k + T_{jk}A_i + T_{ki}A_j = 0, \quad T_{ij} = T_{ji} .$$

Then either each T_{ij} is zero or each A_i is zero.

Lemma 2 ([4], Lemma 1). *Let M be a Riemannian manifold of dimension $n \geq 3$. If B_{hijk} is a tensor field on M such that*

(10) $$B_{hijk} = -B_{ihjk} = B_{jkhi}, \quad B_{hijk} + B_{hjki} + B_{hkij} = 0, \qquad B_{hijk/[lm]} = 0,$$

and a_l, A_l are vectors fields on M satisfying

$$a_r R^r{}_{ijk} = g_{ij}A_k - g_{ik}A_j ,$$

then

$$A_l\left[B_{hijk} - \frac{S}{n(n-1)}(g_{ij}g_{hk} - g_{ik}g_{hj})\right] = 0,$$

where $S = B_{pqrs}g^{ps}g^{qr}$.

Lemma 3 ([9], Lemma 3). *If c_j, p_j, B_{hijk} are numbers satisfying* (10) *and*

$$c_lB_{hijk} + p_hB_{lijk} + p_iB_{hljk} + p_jB_{hilk} + p_kB_{hijl} = 0,$$

then either each $b_j = c_j + 2p_j$ is zero or each B_{hijk} is zero.

3. Main results

Theorem 1. *Let M ($\dim M > 2$) be a totally umbilical submanifold of the manifold N satisfying the condition* (1) *for some vector field a_i. Then the relation*

(11)
$$(g_{rs}H^rH^s - a_rH^r)C_{abcd} = 0$$

holds on M, where C_{abcd} are components of the Weyl conformal curvature tensor of the submanifold M.

Proof. Transvecting (1) with $H^hB^{ijkl}_{bcde}$ and applying (5) and (7) we obtain

(12)
$$R_{hijk,l}H^hB^{ijkl}_{bcde} =$$
$$= a_e(g_{bc}H_d - g_{bd}H_c) + VK_{ebcd} - VH(g_{bc}g_{ed} - g_{bd}g_{ec}) +$$
$$+\frac{1}{2}a_b(g_{ec}H_d - g_{ed}H_c) + \frac{1}{2}a_c(g_{be}H_d - g_{bd}H_e) + \frac{1}{2}a_d(g_{bc}H_e - g_{be}H_c),$$

where $a_e = a_rB^r_e$, $V = a_rH^r$. On the other hand, differentiating covariantly the left hand side of (7), in virtue of (9), (5) and (6), we get

(13)
$$\left[R_{hijk}H^hB^{ijk}_{bcd}\right]_{,e} = R_{hijk,l}H^hB^{ijkl}_{bcde} - HK_{ebcd} +$$
$$+H^2(g_{bc}g_{ed} - g_{bd}g_{ec}) + g_{bc}E_{de} - g_{bd}E_{ce} - g_{ce}S_{bd} + g_{de}S_{bc},$$

where $E_{de} = \sum_x e_xA_{dx}A_{ex} = E_{ed}$ and $S_{bc} = R_{hijk}H^hB^{ij}_{bc}H^k = S_{cb}$. Then, substituting (12) into (13) and taking into account relation (7), we find

(14)
$$(H - V)K_{ebcd} =$$
$$= H(H - V)(g_{bc}g_{ed} - g_{bd}g_{ec}) + g_{bc}E_{de} - g_{bd}E_{ce} - g_{ce}S_{bd} + g_{de}S_{bc} +$$
$$+a_e(g_{bc}H_d - g_{bd}H_c) - \frac{1}{2}(g_{bc}H_{,de} - g_{bd}H_{,ce}) +$$
$$+\frac{1}{2}(g_{ec}a_bH_d - g_{ed}a_bH_c + g_{be}a_cH_d - g_{bd}a_cH_e + g_{bc}a_dH_e - g_{be}a_dH_c).$$

Hereafter, contracting (14) with g^{ed} and alternating the resulting equation in (b, c), we obtain

(15)
$$a_bH_c = a_cH_b.$$

Therefore, alternating (14) in (e, b) and using (15), we get

$$2(H-V)K_{ebcd}= \tag{16}$$
$$=2H(H-V)(g_{bc}g_{ed}-g_{bd}g_{ec})+$$
$$+g_{bc}(E_{de}+S_{de})-g_{bd}(E_{ce}+S_{ce})+g_{de}(E_{bc}+S_{bc})-g_{ce}(E_{bd}+S_{bd})+$$
$$+g_{bc}a_eH_d-g_{bd}a_cH_e+g_{ed}a_bH_c-g_{ec}a_bH_d-$$
$$-\frac{1}{2}(g_{bc}H_{,ed}-g_{bd}H_{,ce}+g_{ed}H_{,bc}-g_{ce}H_{,bd}),$$

whence we obtain

$$2(H-V)K_{bc}=2(m-1)H(H-V)g_{bc}+(m-2)(E_{bc}+S_{bc})+ \tag{17}$$
$$+g_{bc}(E+S+P-\frac{1}{2}Q)+(m-2)a_bH_c-\frac{m-2}{2}H_{,bc}$$

and

$$2(H-V)K=(m-1)[2mH(H-V)+2(E+S)+2P-Q], \tag{18}$$

where

$$E=E_{bc}g^{bc},\quad S=S_{bc}g^{bc},\quad P=a_bH_cg^{bc},\quad Q=H_{,bc}g^{bc}.$$

Finally, using equations (16)–(18), by an immediate calculations, we check that (11) holds good. ◊

Transvecting (1) with B^{hijkl}_{abcde} and making use of (5) and (8) we find

$$K_{abcd,e}=2a_eK_{abcd}+a_aK_{ebcd}+a_bK_{aecd}+a_cK_{abed}+a_dK_{abce}+ \tag{19}$$
$$+2Z_e(g_{bc}g_{ad}-g_{bd}g_{ac})+Z_a(g_{bc}g_{ed}-g_{bd}g_{ec})+$$
$$+Z_b(g_{ec}g_{ad}-g_{ed}g_{ac})+Z_c(g_{be}g_{ad}-g_{bd}g_{ae})+Z_d(g_{bc}g_{ae}-g_{be}g_{ac}),$$

where $Z_e=\frac{1}{2}H_{,e}-a_eH$, whence we obtain

$$K_{bc,e}=2a_eK_{bc}+a_bK_{ec}+a_cK_{be}+a_fK^f{}_{bce}+a_fK^f{}_{cbe}+ \tag{20}$$
$$+2mg_{bc}Z_e+(m-2)(g_{ec}Z_b+g_{be}Z_c),$$

$$K_{,e}=2a_eK+4a_fK^f{}_e+2(m-1)(m+2)Z_e. \tag{21}$$

Suppose, that at a point $x\in M$ the relation

$$K_{abcd,e}=2b_eK_{abcd}+b_aK_{ebcd}+b_bK_{aecd}+b_cK_{abed}+b_dK_{abce} \tag{22}$$

is satisfied for a certain vector b_e. Then we have

$$K_{bc,e}=2b_eK_{bc}+b_bK_{ec}+b_cK_{be}+b_fK^f{}_{bce}+b_fK^f{}_{cbe}. \tag{23}$$

Subtracting (23) from (20), permuting cyclically the such obtained equality in (b,c,e) and adding the resulting equations, we get

(24) $$K_{bc}(a_e - b_e) + K_{ce}(a_b - b_b) + K_{eb}(a_c - b_c) +$$
$$+(m-1)(g_{bc}Z_e + g_{ce}Z_b + g_{eb}Z_c) = 0\,.$$

If $a_e - b_e$ and Z_e are linearly independent, then by Lemma 1 (I) we have $\operatorname{rank} g_{ab} \leq 2$. Thus, for $m > 2$, either $Z_b = 0$ or $Z_b \neq 0$ and $Z_e = f(a_e - b_e)$, for $f \in \mathbb{R} - \{0\}$. Subtracting (22) from (19), then substituting $Z_e = f(a_e - b_e)$ and applying Lemma 3 we get $(a_e - - b_e)[K_{abcd} + f(g_{bc}g_{ad} - g_{bd}g_{ac})] = 0$ at x. Thus, if $Z_e(x)$ does not vanish and $\dim M > 2$, then on a neighbourhood of x we have $f_{,e} = 0$. Moreover, we have

(25) $$(a_f - b_f)K^f{}_{bce} + g_{bc}Z_e - g_{be}Z_c = 0$$

at a point $x \in M$ where Z_e does not vanish.

From the above made considerations we are in a position to obtain

Theorem 2 (cf [5], Th. 3.3). *Let M be a totally umbilical submanifold of a manifold N satisfying the condition* (1) *for some vector field a_l and suppose that a_l is not orthogonal to M. If condition* (22) *is satisfied on M for some vector field b_b which does not vanish on a dense subset of M and* $\dim M > 2$*, then $Z_b = 0$ on M. Conversely, if $Z_b = 0$, then condition* (22) *holds on M with $b_b = a_b$.*

Theorem 3 (cf. [5], Ths. 3.6 and 3.7). *Let M* ($\dim M > 2$) *be a totally umbilical submanifold of a manifold N satisfying the condition* (1) *for some vector field a_l and suppose that a_l is not orthogonal to M. If Z_b does not vanish on a dense subset of M, then M is a space of constant curvature.*

Theorem 4. *Let M* ($\dim M > 2$) *be a totally umbilical submanifold of a manifold N satisfying the condition* (1) *for some vector field a_l. If M is semi-symmetric (i.e. $K_{abcd,[ef]} = 0$) and Z_b does not vanish on a dense subset of M, then M is a space of constant curvature and* $Z_b + \frac{K}{m(m-1)}a_b = 0$.

Proof. Follows from (25) and Lemma 2. ◊

Theorem 5 (cf. [5], Th. 4.1). *Let M* ($\dim M > 2$) *be a totally umbilical submanifold of the manifold N satisfying the condition* (1). *If the vector a_l is orthogonal to M, then M is a conformally symmetric manifold.*

Proof. If a_l is orthogonal to M, then $a_e = a_r B^r_e$ vanishes. Using the formulas (19)–(21), by an immediate calculations, we check that $C_{abcd,e} = 0$ holds on M. ◊

Theorem 6. *Let M* ($\dim M > 2$) *be a totally umbilical submanifold of a manifold N satisfying the condition* (1) *for some vector field a_l and suppose that a_l is not orthogonal to M. Then the relation*

(26) $$K_{abcd\,/\,e} = c_e K_{abcd}$$
holds on M for some vector field c_e which does not vanish on a dense subset of M, if and only if

(27) $$Z_e = 0$$
and

(28) $$a_e K_{abcd} + a_c K_{abde} + a_d K_{abec} = 0$$
on M.

Proof. Suppose that relation (26) holds on M, i.e. at each point there exists a vector c_e satisfying (26). Consequently, we have on M

(29) $$c_e K_{abcd} + c_c K_{abde} + c_d K_{abec} = 0$$

and relation of the form (22) is also satisfied ([3], Prop. 1). According to the Th. 2, the last condition is equivalent to $Z_e = 0$. Hence, we have (19) with $Z_e = 0$. Substituting (26) and (27) into (19) we obtain

$$(-c_e + 2a_e)K_{abcd} + a_a K_{ebcd} + a_b K_{aecd} + a_c K_{abed} + a_d K_{abce} = 0\,,$$

whence, in virtue of Lemma 3, $c_e = 4a_e$. Therefore, using (29), we get (28) on M. Conversely, if $Z_e = 0$ and (28) holds on M, then (19) yields $K_{abcd\,/\,e} = 4a_e K_{abcd}$. ◊

Suppose now that $K_{abcd\,/\,e}(x) = 0$, $x \in M$. If a_e and Z_e are not linearly dependent, then (20) and Lemma 1(I) yield $\operatorname{rank} g_{ab} \leq 2$. Thus, for $m > 2$, we have either

$$Z_e = 0 \quad \text{and} \quad a_e = 0 \quad \text{or}$$

$$Z_e = 0 \quad \text{and} \quad a_e \neq 0 \quad \text{or}$$

$$Z_e \neq 0\,, \quad a_e \neq 0 \quad \text{and} \quad Z_e = f a_e\,, \quad f \in \mathbb{R} - \{0\}\,.$$

Therefore relation (19) and Lemma 3 result in

Theorem 7. *Let M $(\dim M > 2)$ be a totally umbilical submanifold of a manifold N satisfying the condition (1) for some vector field a_l. If a_l is orthogonal to M, then $Z_e = 0$ if and only if $K_{abcd\,/\,e} = 0$.*

Theorem 8. *Let M $(\dim M > 2)$ be a totally umbilical submanifold of a manifold N satisfying the condition (1) for some vector field a_l and suppose that M is locally symmetric. If $a_e(x) \neq 0$ and $Z_e = 0$, then M is flat. If a_l is not orthogonal to M and Z_e does not vanish at any point of M, then M is a non-flat space of constant curvature.*

4. Some examples

Let N be an open subset of $\mathbb{R}^n$, $(n > 2)$, endowed with the metric

$$\tilde{g}_{ij}dx^i dx^j = (dx^1)^2 + p^2 f_{\alpha\beta}dx^\alpha dx^\beta ,$$

$\alpha, \beta, \gamma, \cdots = 2, \ldots, n$, where $f_{\alpha\beta}dx^\alpha dx^\beta$ is a flat metric and p is a function in x^1 variable satisfying the equation

$$pp'p''' + 3(p')^2 p'' - 4p(p'')^2 = 0 .$$

For suitable choosen of N there exist solutions such that the condition (1) holds on N and N is not recurrent ([3], Th. 6, Props. 5 and 6).

Let V be a flat manifold of dimension m endowed with the metric $h_{PQ}dx^P dx^Q$, $P, Q = n+1, \ldots, n+m$. On the manifold $N \times V$ define the metric

$$g_{rs}dx^r dx^s = \tilde{g}_{ij}dx^i dx^j + h_{PQ}dx^P dx^Q .$$

Then on $(N \times V, g)$, for suitable function p, the condition (1) is fulfilled while $R_{hijk\,,\,l} = c_l R_{hijk}$ is not satisfied.

Example 1. Let M be an n-dimensional manifold covered by a system of coordinate neighbourhoods $\{W; y^a\}$, $a, b, \cdots = 1, \ldots, n$, immersed in $N \times V$ and let $x^1 = Q(y^a)$, $x^\alpha = y^\alpha$, $x^P = C_P$, $C_P =$ const be its local expression in $N \times V$. Then $B^1_d = Q_d$, $B^\alpha_\beta = \delta^\alpha_\beta$, $B^P_d = 0$, where $Q_d = Q_{\,,\,d}$. The covariant and contravariant components of the induced metric tensor of M are respectively

$$g_{11} = (Q_1)^2 , \quad g_{1\alpha} = Q_1 Q_\alpha , \quad g_{\alpha\beta} = Q_\alpha Q_\beta + \tilde{g}_{\alpha\beta} ,$$

$$g^{11} = (Q_\alpha Q_\beta \tilde{g}^{\alpha\beta} + 1)(Q_1)^{-2} , \quad g^{1\alpha} = -Q_\beta \tilde{g}^{\beta\alpha}(Q_1)^{-1} , g^{\alpha\beta} = \tilde{g}^{\alpha\beta} .$$

The only components of the Christoffel symbols which may not vanish are

$$\Gamma^1_{11} = \frac{Q_{\cdot 11}}{Q_1} , \quad \Gamma^1_{1\alpha} = \frac{Q_{\cdot 1\alpha}}{Q_1} - \frac{p'}{p}Q_\alpha ,$$

$$\Gamma^1_{\alpha\beta} = (Q_{\cdot\alpha\beta} - \frac{p'}{p}\tilde{g}_{\alpha\beta})(Q_1)^{-1} - 2\frac{p'}{p}Q_\alpha Q_\beta (Q_1)^{-1} - \overline{\Gamma}^\nu_{\alpha\beta} Q_\nu (Q_1)^{-1} ,$$

$$\Gamma^\alpha_{\beta\gamma} = \frac{p'}{p}(Q_\beta \delta^\alpha_\gamma + Q_\gamma \gamma^\alpha_\beta) + \overline{\Gamma}^\alpha_{\beta\gamma} , \quad \Gamma^\alpha_{1\beta} = \frac{p'}{p}Q_1 \delta^\alpha_\beta ,$$

where $\overline{\Gamma}^\alpha_{\beta\gamma}$ are Christoffel symbols of $f_{\alpha\beta}dx^\alpha dx^\beta$. Then, using (2), we check that $B^r_{a\,,\,b} = 0$, so the submanifold M is a totally geodesic one. Consequently, M is a totally umbilical submanifold in $N \times V$ and the vector field $Z_e = \frac{1}{2}H_{\,,\,e} - a_e H$ vanishes. Moreover, the components of the projection of the vector $\tilde{a}_l$ of $N \times V$ onto the submanifold M are $a_d = \tilde{a}_1 Q_d$. If $\tilde{a}_1 = \frac{p''}{p'} - \frac{p'}{p} \neq 0$ ([3]) and $Q_d \neq 0$, then, according to the Th. 2, condition (22) holds on M with $b_d = a_d$.

The components of the curvature tensor and the Ricci tensor of M are

$$K_{\alpha\beta\gamma\delta} = \frac{p''}{p}\left[Q_\beta Q_\gamma \tilde{g}_{\alpha\delta} - Q_\beta Q_\delta \tilde{g}_{\alpha\gamma} + Q_\alpha Q_\delta \tilde{g}_{\beta\gamma} - Q_\alpha Q_\gamma \tilde{g}_{\beta\delta}\right] +$$

$$+(p')^2(\tilde{g}_{\beta\gamma}\tilde{g}_{\alpha\delta} - \tilde{g}_{\beta\delta}\tilde{g}_{\alpha\gamma}),$$

$$K_{1\beta\gamma\delta} = \frac{p''}{p} Q_1(\tilde{g}_{\beta\gamma}Q_\delta - \tilde{g}_{\beta\delta}Q_\gamma),$$

$$K_{1\beta\gamma 1} = \frac{p''}{p}(Q_1)^2\tilde{g}_{\beta\gamma},$$

$$K_{1d} = (n-1)\frac{p''}{p}Q_1 Q_d,$$

$$K_{\alpha\beta} = (n-1)\frac{p''}{p}Q_\alpha Q_\beta + \left[\frac{p''}{p} + (n-2)(p')^2\right]\tilde{g}_{\alpha\beta}.$$

Moreover, the scalar curvature of M is given by

$$K = (n-1)\left[2\frac{p''}{p} + (n-2)(p')^2\right].$$

One can check, that M is a conformally flat submanifold in $N \times V$. Setting in (28) $a = \alpha$, $b = \beta$, $c = \gamma$, $d = \delta$, $e = \nu$ we easily obtain $(n-3)(n-1)Q_\nu = 0$. Thus we have

Proposition 1. *For each $n > 3$ and $t > n$ there exists t-dimensional manifold satisfying* (1) *admitting n-dimensional totally umbilical and conformally flat submanifold M such that the condition* (1) *holds on M whereas* (26) *is not satisfied.*

Example 2. Let M be an $(n-1)$-dimensional manifold covered by a system of coordinate neighbourhoods $\{W; z^\alpha\}$, $\alpha, \beta, \gamma = 2, \dots n$, immersed in $N \times V$ and let $x^1 = C_1$, $x^P = C_P$, $C_P = \text{const}$, $x^\alpha = X^\alpha(z^2, \dots, z^n)$ be its local expression in $N \times V$. Then we have $B^1_\alpha = B^P_\alpha = 0$, $B^\alpha_\beta = \frac{\partial X^\alpha}{\partial z^\beta}$, $g_{\alpha\beta} = \tilde{g}_{\mu\nu}B^{\mu\nu}_{\alpha\beta}$, whence $B^1_{\alpha\,/\,\beta} = -\frac{p'}{p}g_{\alpha\beta}$, $B^P_{\alpha\,/\,\beta} = 0$. Moreover, in virtue of (3) and (4), we get

$$B^\nu_{\alpha\,/\,\beta} = \left(B^\rho_{\alpha\cdot\beta} + \overline{\Gamma}^\rho_{\mu\eta}B^{\mu\eta}_{\alpha\beta}\right)\left(\sum_x e_x N^\tau_x N^\nu_x \tilde{g}_{\tau\rho}\right).$$

Setting $f_{\alpha\beta} = \delta_{\alpha\beta}$, $x^\alpha = z^2 + \dots + z^n$ we get

$$a_\alpha = \tilde{a}_r B^r_\alpha = 0\,, \quad H^1 = -\frac{p'}{p}\,, \quad H^\alpha = H^P = 0\,,$$

$$g_{rs} H^r H^s - \tilde{a}_r H^r = \frac{p''}{p}\,, \quad Z_e = 0\,,$$

$$K_{\alpha\beta\gamma\delta} = 2(p')^2 (p)^{-2} (g_{\beta\gamma} g_{\alpha\delta} - g_{\beta\delta} g_{\alpha\gamma})\,.$$

Hence we obtain

Proposition 2. *For each $n > 3$ and $t > n$ there exists t-dimensional manifold satisfying* (1) *admitting $(n-1)$-dimensional totally umbilical submanifold M such that the recurrence vector is orthogonal to M* (cf. Th. 5).

Proposition 3. *For each $n > 3$ and $t > n$ there exists t-dimensional manifold satisfying* (1) *admitting $(n-1)$-dimensional totally umbilical submanifold M such that $g_{rs} H^r H^s - a_r H^r$ does not vanish identically on M* (cf. Th. 1).

Proposition 4. *For each $n > 3$ and $t > n$ there exists t-dimensional manifold satisfying* (1) *admitting $(n-1)$-dimensional totally umbilical locally symmetric submanifold* (cf. Th. 7).

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L-MULTIFUNCTIONS AND THEIR PROPERTIES

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Abstract: In this paper a concept of L-multifunction is introduced and other related objects are defined. Next their properties are presented.

1. Introduction

The notion of Cartesian product plays an important role in the usual theory of functions and multifunctions. The Cartesian product of two fuzzy subsets $A \in I^X$ and $B \in I^Y$ may be defined as the subset $A \times B$ of $X \times Y$ characterized by $(A \times B)(x, y) = \min(A(x), B(y))$. This definition has the inconvenience that when $A \times B$ is known and $A \times B \neq \neq \emptyset$, it is impossible to retrieve again the subsets A and B. The notion of fuzzy Cartesian product which is introduced in paper [1] is free from this inconvenience. The L-multifunctions which are introduced in this paper are the subsets of a special case of Cartesian product and also free from this inconvenience. We will introduce and develop basic ideas of the L-multifunction theory, necessary for our further considerations on economical systems.

2. Introduction and general properties of L-multifunctions

Let X, Y and Z denote arbitrary but for further considerations fixed reference spaces. Next $\mathcal{P}(X)$, $\mathcal{P}(Y)$ and $\mathcal{P}(Z)$ denote respectively the families of all non-void subsets of X, Y and Z.

Definition 2.1. An *L-multifunction*, $F : X \to \mathcal{P}(Y)$ say, is a subset of the Cartesian product $X \times \mathcal{P}(Y) \times I \times I^Y$ satisfying the following conditions:

(i) if $(x, B, r, f) \in F$, then $\operatorname{supp} f = B$,

(ii) if $(x, B, r, f) \in F$, then $r = 0$ implies $B = \emptyset$ and $r = 1$ implies $f(y) = 1$ for any $y \in B$,

(iii) if $(x, B, r_1, f_1) \in F$ and $(x, B, r_2, f_2) \in F$, then $r_1 > r_2$ implies $f_1 \geq f_2$.

Let $\{x, r\}$ denote a fuzzy singleton in X with support x and value r. This fuzzy singleton is now transformed by F to a family of the fuzzy subsets in Y.

Definition 2.2. A *converse L-multifunction, F^{-1}* say, to an L-multifunction $F : X \to \mathcal{P}(Y)$ is a subset of the Cartesian product $Y \times \mathcal{P}(X) \times$ $\times I \times I^X$ satisfying the following condition:

— $(y, A, t, h) \in F^{-1}$ if there exists $(x, B, r, f) \in F$ such that $x \in A$, $y \in B$, $f(y) = t$, $h(x) = r$.

Definition 2.3. A *composite, $G \circ F : X \to \mathcal{P}(Z)$* say, of two L-multifunctions $F : X \to \mathcal{P}(Y)$ and $G : Y \to \mathcal{P}(Z)$ is an L-multifunction such that

— $(x, C, r, h) \in G \circ F$ iff there exist $(x, B, r, f) \in F$ and $(y, C, t, h) \in$ $\in G$ such that $Y \in B$, $f(y) = t$.

Let X, Y and Z denote the linear spaces.

Definition 2.4. An L-multifunction, $F : X \to \mathcal{P}(Y)$ say, is called *conical* iff for any $(x, B, r, f) \in F$ and for any $\alpha > 0$ $(\alpha x, \alpha B, r, \alpha f) \in$ $\in F$, where $(\alpha f)(y) = f\left(\frac{1}{\alpha} y\right)$ for any $y \in Y$.

Theorem 2.1. *If an L-multifunction is conical, then its converse L-multifunction is conical too.*

Proof. As a matter of fact, let F be a conical L-multifunction. Let $(y, A, t, h) \in F^{-1}$. So, taking into account Def. 2 there exists $(x, B, r, f) \in F$ such that $x \in A$, $y \in B$, $f(y) = t$, $h(x) = r$. So, with respect to Def. 4 for any $\alpha > 0$ we have $(\alpha x, \alpha B, r, \alpha f) \in F$.

Moreover $\alpha x \in \alpha A$, $\alpha y \in \alpha B$, $\alpha f(\alpha y) = f(y) = t$, $\alpha h(\alpha x) = h(x) = r$. This means that $(\alpha y, \alpha A, t, \alpha h) \in F^{-1}$. So, F^{-1} is a conical L-multifunction. ◊

Theorem 2.2. *If F and G are conical L-multifunctions then $G \circ F$ is a conical L-multifunction too.*

Proof. Let $(x, C, r, h) \in G \circ F$. So, taking into account Def. 3 there exist $(x, B, r, f) \in F$ and $(y, C, t, h) \in G$ such that $y \in B$, $f(y) = t$. F and G are conical L-multifunctions, so for any $\alpha > 0$ we have $(\alpha x, \alpha B, r, \alpha f) \in F$ and $(\alpha y, \alpha C, t, \alpha h) \in G$. Because $y \in B$, $f(y) = t$, so $\alpha y \in \alpha B$, $\alpha f(\alpha y) = f(y) = t$. This means that $(\alpha x, \alpha C, r, \alpha h) \in G \circ F$. ◊

Definition 2.5. An L-multifunction, $F : X \to \mathcal{P}(Y)$ say, is called *superadditive* iff for any $(x_1, B_1, r_1, f_1) \in F$ and $(x_2, B_2, r_2, f_2) \in F$, we have $(x_1 + x_2, B_1 + B_2, \min(r_1, r_2), f_1 + f_2) \in F$, where $(f_1 + f_2)(y) = \sup_{y_1+y_2=y} \min(f_1(y_1), f_2(y_2))$ for any $y \in Y$.

Theorem 2.3. *If an L-multifunction is superadditive then its converse L-multifunction is superadditive as well.*

Proof. In point of fact, let an L-multifunction, $F : X \to \mathcal{P}(Y)$ say, satisfy the assumption of the theorem. Let

$$(y_1, A_1, t_1, h_1) \in F^{-1} \quad \text{and} \quad (y_2, A_2, t_2, h_2) \in F^{-1}.$$

Then from the Def. 2 it follows that there exist

$$(x_1, B_1, r_1, f_1) \in F \quad \text{and} \quad (x_2, B_2, r_2, f_2) \in F$$

such that

$$x_1 \in A_1, \quad y_1 \in B_1, \quad x_2 \in A_2, \quad y_2 \in B_2,$$
$$h_1(x_1) = r_1, \quad h_2(x_2) = r_2, \quad f_1(y_1) = t_1, \quad f_2(y_2) = t_2.$$

We can assume that x_1, x_2, f_1 and f_2 are such that

$$h_1(x_1) = \sup_{x \in A_1} h_1(x) = r_1, \quad h_2(x_2) = \sup_{x \in A_2} h_2(x) = r_2$$

and

$$f_1(y_1) = \sup_{y \in B_1} f_1(y) = t_1, \quad f_2(y_2) = \sup_{y \in B_2} f_2(y) = t_2.$$

Because F is superadditive L-multifunction, so

$$(x_1 + x_2, B_1 + B_2, \min(r_1, r_2), f_1 + f_2) \in F.$$

Moreover

$$x_1 + x_2 \in A_1 + A_2, (h_1 + h_2)(x_1 + x_2) = \min(r_1, r_2)$$

and

$$y_1 + y_2 \in B_1 + B_2, (f_1 + f_2)(y_1 + y_2) = \min(t_1, t_2).$$

This means that

$$(y_1 + y_2, A_1 + A_2, \min(t_1, t_2), h_1 + h_2) \in F^{-1}.$$

So, F^{-1} is a superadditive L-multifunction. ◊

Theorem 2.4. *If F and G are superadditive L-multifunctions then $G \circ F$ is a superadditive L-multifunction too.*

Proof. Let $(x_1, C_1, r_1, h_1) \in G \circ F$ and $(x_2, C_2, r_2, h_2) \in G \circ F$. Then, from the Def. 3 it follows that there exist $(x_1, B_1, r_1, f_1) \in F$, $(y_1, C_1, t_1, h_1) \in G$ such that $y_1 \in B_1$, $f_1(y_1) = t_1$ and there exist $(x_2, B_2, r_2, f_2) \in F$, $(y_2, C_2, t_2, h_2) \in G$ such that $y_2 \in B_2$, $f_2(y_2) = t_2$. We can assume that

$$f_1(y_1) = \sup_{y \in B_1} f_1(y) = t_1 \quad \text{and} \quad f_2(y_2) = \sup_{y \in B_2} f_2(y) = t_2.$$

Because F and G are superadditive L-multifunctions, so

$$(x_1 + x_2, B_1 + B_2, \min(r_1, r_2), f_1 + f_2) \in F$$

and

$$(y_1 + y_2, C_1 + C_2, \min(t_1, t_2), h_1 + h_2) \in G.$$

Moreover

$$y_1 + y_2 \in B_1 + B_2 \quad \text{and} \quad (f_1 + f_2)(y_1 + y_2) = \min(t_1, t_2).$$

This means that

$$(x_1 + x_2, C_1 + C_2, \min(r_1, r_2), h_1 + h_2) \in G \circ F,$$

i.e. $G \circ F$ is superadditive L-multifunction. ◊

Definition 2.6. By the *graph* of L-multifunction, $F : X \to \mathcal{P}(Y)$ say, it is understood a set W_F of the elements $(x, y, r, t) \in X \times Y \times I \times I$ such that there exist $B \in \mathcal{P}(Y)$ and $f \in I^Y$ satisfying the following conditions:

(i) $y \in B$,

(ii) $t = f(y)$,

(iii) $(x, B, r, f) \in F$.

Definition 2.7. Let $\alpha, \beta \in I$. An *α,β-cut* of W_F, $W_F^{\alpha,\beta}$ in symbol, is a set of the elements $(x, y) \in X \times Y$ such that for $r \geq \alpha$ and $t \geq \beta$ $(x, y, r, t) \in W_F$.

Theorem 2.5. *If $F : X \to \mathcal{P}(Y)$ is a conical L-multifunction then for any $\alpha, \beta \in I$, the α,β-cut of W_F is a cone.*

Proof. Let $(x, y) \in W_F^{\alpha,\beta}$. Then for $r \geq \alpha$ and $t \geq \beta$, $(x, y, r, t) \in$ $\in W_F$. This means that there exist $B \in \mathcal{P}(Y)$ and $f \in I^Y$ such that

$y \in B$, $t = f(y)$ and $(x, B, r, f) \in F$. Because F is a conical L-multifunction, so for any $\lambda > 0$, $(\lambda x, \lambda B, r, \lambda f) \in F$. Moreover $\lambda y \in \lambda B$, $t = \lambda f(\lambda y) = f(y)$. This means that $(\lambda x, \lambda y, r, t) \in W_F$ and finally $(\lambda x, \lambda y) \in W_F^{\alpha,\beta}$. $\Diamond$

Theorem 2.6. *If F is a conical and superadditive L-multifunction then for any $\alpha, \beta \in I$, $W_F^{\alpha,\beta}$ is a convex set.*

Proof. Let (x_1, y_1), $(x_2, y_2) \in W_F^{\alpha,\beta}$. Then for any $r_1, r_2 \geq \alpha$ and $t_1, t_2 \geq \beta$

$$(x_1, y_1, r_1, t_1) \in W_F\,, \quad (x_2, y_2, r_2, t_2) \in W_F\,.$$

This means that there exist $B_1, B_2 \in \mathcal{P}(Y)$ and $f_1, f_2 \in I^Y$ such that $y_1 \in B_1$, $t_1 = f_1(y_1)$, $y_2 \in B_2$, $f_2(y_2) = t_2$ and $(x_1, B_1, r_1, f_1) \in F$, $(x_2, B_2, r_2, f_2) \in F$. Because F is a conical and superadditive L-multifunction so for any $\lambda > 0$

$$(\lambda x_1, \lambda B_1, r_1, \lambda f_1) \in F\,, \quad ((1-\lambda)x_2(1-\lambda)B_2\,, r_2\,, (1-\lambda)f_2) \in F\,,$$

and

$$(\lambda x_1 + (1-\lambda)x_2\,, \lambda B_1 + (1-\lambda)B_2\,, \min(r_1, r_2)\,, \lambda f_1 + (1-\lambda)f_2) \in F\,.$$

This means that

$$(\lambda x_1 + (1-\lambda)x_2, \lambda y_1 + (1-\lambda)y_2, \min(r_1, r_2), \min(t_1, t_2)) \in W_F\,.$$

Because $\min(r_1, r_2) \geq \alpha$ and $\min(t_1, t_2) \geq \beta$ so

$$(\lambda x_1 + (1-\lambda)x_2, \lambda y_1 + (1-\lambda)y_2) \in W_F^{\alpha,\beta}\,,$$

i.e. a set $W_F^{\alpha,\beta}$ is convex. $\Diamond$

3. Some topological properties of L-multifunctions

Now, let us assume that the reference spaces X, Y and Z are finite dimensional Euclidean spaces.

Definition 3.1. An L-multifunction, $F : X \to \mathcal{P}(Y)$ say, is called *closed* iff its graph W_F is a closed set.

Corollary. *For any closed L-multifunction its converse L-multifunction is closed.*

Definition 3.2. An L-multifunction $F : X \to \mathcal{P}(Y)$ say, is *sequentially bounded* iff for any bounded sequence $S = \{x_n\}$ and any sequence $R = \{r_n\}$, $x_n \in X$, $r_n \in (0,1\rangle$, the set

$$\{(y,t) \in Y \times I : (x_n, y, r_n, t) \in W_F, x_n \in S, r_n \in R\}$$

is bounded.

Theorem 3.1. *If $F : X \to \mathcal{P}(Y)$ and $G : Y \to \mathcal{P}(Z)$ are closed L-multifunctions and F is sequentially bounded, then $G \circ F$ is a closed L-multifunction.*

Proof. Let $(x_n, z_n, r_n, p_n) \in W_{G\circ F}$ and let $(x_n, z_n, r_n, p_n) \to (x_0, z_0, r_0, p_0)$ as $n \to \infty$ (the convergence may be taken with respect to each coordinate separately), $x_n \in X$, $z_n \in Z$, $r_n, p_n \in (0,1\rangle$. We will prove that $(x_0, z_0, r_0, p_0) \in W_{G\circ F}$. In fact, for any n, (x_n, z_n, r_n, p_n) belongs to $W_{G\circ F}$ iff there exist $C_n \in Z$, $h_n \in I^Z$ such that $(x_n, C_n, r_n, h_n) \in \in G \circ F$ and $z_n \in C_n$, $h_n(z_n) = p_n$. An element $(x_n, C_n, r_n, h_n) \in G \circ F$ iff there exist $(x_n, B_n, r_n, f_n) \in F$ and $(y_n, C_n, t_n, h_n) \in G$ such that $y_n \in B_n$ and $f_n(y_n) = t_n$. From the above conditions it follows that

$$(x_n, y_n, r_n, t_n) \in W_F \quad \text{and} \quad (y_n, z_n, t_n, p_n) \in W_G .$$

Because F is a sequentially bounded and closed L-multifunction we observe that the sequences $\{y_n\}$ and $\{t_n\}$ are bounded and without losing generality we may assume that $y_n \to y_0$ and $t_n \to t_0$ as $n \to \infty$. Moreover

$$(x_0, y_0, r_0, t_0) \in W_F \quad \text{and} \quad (y_0, z_0, t_0, p_0) \in W_G .$$

This means that there exist $B_0 \in \mathcal{P}(Y)$, $f_0 \in I^Y$ such that $y_0 \in \in B_0$, $f_0(y_0) = t_0$, $(x_0, B_0, r_0, f_0) \in F$ and there exist $C_0 \in \mathcal{P}(Z)$, $h_0 \in I^Z$ such that $z_0 \in C_0$, $h_0(z_0) = p_0$, $(y_0, C_0, t_0, h_0) \in G$. This means that $(x_0, C_0, r_0, h_0) \in G \circ F$. Because $z_0 \in C_0$, $h_0(z_0) = p_0$, so $(x_0, z_0, r_0, p_0) \in W_{G\circ F}$. $\Diamond$

Theorem 3.2. *If an L-multifunction $F : X \to \mathcal{P}(Y)$ is closed and conical and for any $r, t \in I$, $(0, y, r, t) \notin W_F$ for $y \neq 0$, then F is a sequentially bounded L-multifunction.*

Proof. According to Def. 3.2 it suffices to show that for any bounded sequence $S = \{x_n\}$ and any sequence $R = \{r_n\}$, $x_n \in X$, $r_n \in (0,1\rangle$ the set $T = \{(y,t) \in Y \times I : (x_n, y, r_n, t) \in W_F,\ x_n \in S,\ r_n \in R\}$ is bounded. Suppose that the set T is unbounded for some S and some R. Then there exist the sequences $\{y_n\}$, $\{t_n\}$, $(y_n, t_n) \in T$ such that $||y_n|| \to \infty$ as $n \to \infty$. But $(x_n, y_n, r_n, t_n) \in W_F$ and F is a conical L-multifunction, so $(x_n/||y_n||, y_n/||y_n||, r_n, t_n) \in W_F$. Hence, there exist subsequences x_{n_k} y_{n_k}, r_{n_k}, t_{n_k} such that

$$(x_{n_k}/||y_{n_k}||, y_{n_k}/||y_{n_k}||, r_{n_k}, t_{n_k}) \to (0, y_0, r_0, t_0)$$

as $k \to \infty$, where $y_0 \neq 0$ because $\lim_n y_{n_k}/||y_{n_k}|| = 1 = y_0$. Because F is a closed L-multifunction, so $(0, y_0, r_0, t_0) \in W_F$ for $y_0 \neq 0$, a contradiction. $\Diamond$

Definition 3.3. A *fixed point* of the L-multifunction $F : X \to \mathcal{P}(X)$ is an element $\overline{x} \in X$ such that there exist $r, t \in I$ such that $(\overline{x}, \overline{x}, r, t) \in$ $\in W_F$.

Theorem 3.3 (Fixed point theorem). *Let C be a nonempty, convex and compact subset of X. If $F : C \to \mathcal{P}(C)$ is a closed, conical and superadditive L-multifunction, then F has a fixed point in C.*

Proof. Let us consider a point-to-set mapping $\hat{F} : C \to \mathcal{P}(C)$ such that for any $x \in C$

$$\hat{F}(x) = \{y \in C : \exists r, t, \in I, (x, y, r, t) \in W_F\}.$$

First we will prove that $\overline{x}$ is a fixed point of F iff $\overline{x}$ is a fixed point of $\hat{F}$. If $\overline{x}$ is a fixed point of $\hat{F}$, then $\overline{x} \in \hat{F}(\overline{x})$. This means that there exist $r, t \in I$ such that $(\overline{x}, \overline{x}, r, t) \in W_F$, i.e. $\overline{x}$ is a fixed point of F. Now, if $\overline{x}$ is a fixed point of F then from Def. 3.3 it follows that there exist $r, t \in I$ such that $(\overline{x}, \overline{x}, r, t) \in W_F$. This means that $\overline{x} \in \hat{F}(\overline{x})$, i.e. $\overline{x}$ is a fixed point for $\hat{F}$. $\Diamond$

Now, we will show that $\hat{F}$ satisfies the hypothesis of Kakutani fixed point theorem, i.e. that $\hat{F}$ is a closed mapping and for any $x \in C$ $\hat{F}(x)$ is a convex set.

Let y_1, y_2 be elements from $\hat{F}(x)$. From the definition of $\hat{F}$ it follows that there exist elements $r_1, r_2, t_1, t_2 \in I$ such that $(x, y_1, r_1, t_1) \in$ $\in W_F$ and $(x, y_2, r_2, t_2) \in W_F$. This means that there exist $B_1, B_2 \in$ $\in \mathcal{P}(C)$ and $f_1, f_2 \in I^C$ such that

$$(x, B_1, r_1, f_1) \in F, \quad (x, B_2, r_2, f_2) \in F$$

and

$$y_1 \in B_1, \quad y_2 \in B_2, \quad f_1(y_1) = t_1, \quad f_2(y_2) = t_2.$$

Because F is a conical and superadditive L-multifunction, so for any $\alpha \geq 0$

$$(x, \alpha B_1 + (1 - \alpha) B_2, \min(r_1, r_2), \alpha f_1 + (1 - \alpha) f_2) \in F.$$

This means that

$$(x, \alpha y_1 + (1 - \alpha) y_2, \min(r_1, r_2), t) \in W_F,$$

where

$$t = (\alpha f_1 + (1 - \alpha) f_2)(\alpha y_1 + (1 - \alpha) y_2),$$

i.e. $\alpha y_1 + (1 - \alpha) y_2 \in \hat{F}(x)$.

Now, let us consider a sequence $\{x_n\}$, $x_n \in C$ such that $x_n \to x_0$ as $n \to \infty$. Let $y_n \in \hat{F}(x_n)$ and $y_n \to y_0$ as $n \to \infty$. We will prove that $y_0 \in \hat{F}(x_0)$. If $y_n \in \hat{F}(x_n)$ then for any n there exist $r_n, t_n \in I$ such that $(x_n, y_n, r_n, t_n) \in W_F$. Without losing generality we may assume that $r_n \to r_0$, $t_n \to t_0$ as $n \to \infty$. Because F is a closed L-multifunction, so $(x_0, y_0, r_0, t_0) \in W_F$. This means that $y_0 \in \hat{F}(x_0)$, i.e. $\hat{F}$ is a closed mapping. So, according to the Kakutani theorem there exists $\overline{x} \in C$ such that $\overline{x} \in \hat{F}(\overline{x})$. This means that an L-multifunction F has a fixed point $\overline{x}$ in C. $\Diamond$

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A SANDWICH WITH CONVEXITY

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Abstract: We prove that real functions f and g, defined on a real interval I, satisfy

$$f(tx + (1-t)y) \leq tg(x) + (1-t)g(y)$$

for all $x, y \in I$ and $t \in [0,1]$ iff there exists a convex function $h : I \to \mathbb{R}$ such that $f \leq h \leq g$. Using this sandwich theorem we characterize solutions of two functional inequalities connected with convex functions and we obtain also the classical one-dimensional Hyers-Ulam Theorem on approximately convex functions.

Introduction

It is the aim of this note to characterize real functions which can be separated by a convex function. This leads us to functional inequality

$$(1) \qquad f(tx+(1-t)y) \le tg(x)+(1-t)g(y).$$

Using this characterization we describe also solutions of the inequalities

$$(2) \qquad f(tx+(T-t)y) \le tf(x)+(T-t)f(y)$$

and

$$(3) \quad f(tx+(T-t)y+(1-T)z_0) \le tf(x)+(T-t)f(y)+(1-T)f(z_0).$$

Functions fulfilling (2) appear in a connection with the converse of Minkowski's inequality in the case where the measure of the space considered is less than 1 (see [4; pp. 671–672] and [5; Remark 16]).

1. A sandwich theorem

Our main result reads as follows.

Theorem 1. *Real functions f and g, defined on a real interval I, satisfy (1) for all $x, y \in I$ and $t \in [0,1]$ iff there exists a convex function $h : I \to \mathbb{R}$ such that*

$$(4) \qquad f \le h \le g.$$

Proof. We argue as in [1; proof of Th. 2]. Assume that functions $f, g : I \to \mathbb{R}$ satisfy (1) and denote by E the convex hull of the epigraph of g:

$$E = \operatorname{conv}\{(x,y) \in I \times \mathbb{R} : g(x) \le y\}.$$

Let $(x,y) \in E$. It follows from the Carathéodory Theorem (see [3; Cor. 17.4.2] or [6; Th. 31E] or [7; the lemma on p. 88]) that (x,y) belongs to a two-dimensional simplex S with vertices in the epigraph of g. Denote

$$y_0 = \inf\{z \in \mathbb{R} : (x,z) \in S\}.$$

Then $y \ge y_0$ and (x, y_0) belongs to the boundary of S. Consequently $(x,y_0) = t(x_1,y_1)+(1-t)(x_2,y_2)$ with some $t \in [0,1]$ and (x_1,y_1), $(x_2,y_2) \in I \times \mathbb{R}$ such that $g(x_1) \le y_1$ and $g(x_2) \le y_2$. Hence, using also (1), we get

$$y \ge y_0 = ty_1+(1-t)y_2 \ge tg(x_1)+(1-t)g(x_2) \ge$$
$$\ge f(tx_1+(1-t)x_2) = f(x).$$

This allows us to define a function $h : I \to \mathbb{R}$ by the formula

$$h(x) = \inf\{y \in \mathbb{R} : (x,y) \in E\}$$

and gives $f \le h$. Moreover, since $(x, g(x)) \in E$ for every $x \in I$, we have also $h \le g$. It remains to show that h is convex. To this end fix arbitrarily $x_1, x_2 \in I$ and $t \in [0,1]$. Then, for any reals y_1, y_2 such that

$(x_1, y_1), (x_2, y_2) \in E$ we have $(tx_1 + (1-t)x_2, ty_1 + (1-t)y_2) \in E$, whence $h(tx_1 + (1-t)x_2) \le ty_1 + (1-t)y_2$. Passing to infimum we obtain the desired inequality: $h(tx_1+(1-t)x_2) \le th(x_1)+(1-t)h(x_2)$. This ends the proof (of the "only if" part but the "if" part is obvious). ◊

The following example shows that Th. 1 cannot be generalized for functions defined on a convex subset of the (complex) plane.

Example 1. Let $D \in \mathbb{C}$ be the open ball centered at zero and with the radius 2, and let z_1, z_2, z_3 be the (different) third roots of the unity. Define the functions f and g on D by the formulas

$$f(z) = \begin{cases} 0 & \text{if } z \neq 0 \\ 1 & \text{if } z = 0 \end{cases} \qquad g(z) = \begin{cases} 0 & \text{if } z \in \{z_1, z_2, z_3\} \\ 3 & \text{if } z \in D \setminus \{z_1, z_2, z_3\}. \end{cases}$$

It is easy to check that (1) holds for all $x, y \in D$ and $t \in [0,1]$. Suppose that there exists a convex function $h : D \to \mathbb{R}$ satisfying (4). Then

$$1 = f(0) = f\left(\frac{1}{3}(z_1 + z_2 + z_3)\right) \le h\left(\frac{1}{3}(z_1 + z_2 + z_3)\right) \le$$
$$\le \frac{1}{3}(h(z_1) + h(z_2) + h(z_3)) \le \frac{1}{3}(g(z_1) + g(z_2) + g(z_3)) = 0,$$

a contradiction.

Arguing as in the proof of Th. 1 we can get however the following results.

Theorem 1a. *Real functions f and g, defined on a convex subset D of an $(n-1)$-dimensional real vector space, satisfy*

$$f\left(\sum_{j=1}^{n} t_j x_j\right) \le \sum_{j=1}^{n} t_j g(x_j) \tag{5}$$

for all vectors $x_1, \ldots, x_n \in D$ and reals $t_1, \ldots, t_n \in [0,1]$ summing up to 1 iff there exists a convex function $h : D \to \mathbb{R}$ satisfying (4).

Theorem 1b. *Real functions f and g, defined on a convex subset D of a vector space, satisfy* (5) *for each positive integer n, vectors $x_1, \ldots, x_n \in D$ and reals $t_1, \ldots, t_n \in [0,1]$ summing up to 1 iff there exists a convex function $h : D \to \mathbb{R}$ satisfying* (4).

2. Applications

We start with an application of Th. 1 connected with approximately convex functions.

If ε is a positive real number and a real function f, defined on a real interval I, satisfies

$$f(tx+(1-t)y) \le tf(x)+(1-t)f(y)+\varepsilon$$

for all $x, y \in I$ and $t \in [0,1]$, then (1) holds with $g = f+\varepsilon$ and it follows from Th. 1 that there exists a convex function $h : I \to \mathbb{R}$ such that

$$f(x) \le h(x) \le f(x)+\varepsilon \quad \text{for} \quad x \in I.$$

Putting $\varphi(x) = h(x)-\varepsilon/2$ we obtain a convex function $\varphi : I \to \mathbb{R}$ such that

$$|\varphi(x)-f(x)| \le \varepsilon/2 \quad \text{for} \quad x \in I.$$

This is the classical one-dimensional Hyers-Ulam Stability Theorem (see [2; Th. 2]; cf. also [1; Th. 2] and [3; Th. 17.4.2]).

Further applications of our Th. 1 concern solutions of the inequalities (2) and (3). Denote by J either $[0,+\infty)$ or $(0,+\infty)$. Given $T > 0$ and $f : J \to \mathbb{R}$ we define the function $f_T : J \to \mathbb{R}$ by the formula

$$f_T(x) = T^{-1}f(Tx).$$

Theorem 2. *Let T be a positive real number. A function $f : J \to \mathbb{R}$ satisfies (2) for all $x, y \in J$ and $t \in [0,T]$ iff there exists a convex function $\varphi : J \to \mathbb{R}$ such that*

$$\varphi_T \le f \le \varphi. \tag{6}$$

Proof. Assume that $f : J \to \mathbb{R}$ satisfies (2). Putting $T \cdot t$ in place of t in (2) we have

$$f_T(tx+(1-t)y) \le tf(x)+(1-t)f(y) \tag{7}$$

for all $x, y \in J$ and $t \in [0,1]$. Applying Th. 1 we obtain a convex function $h : J \to \mathbb{R}$ such that

$$f_T \le h \le f. \tag{8}$$

Define now $\varphi : J \to \mathbb{R}$ by the formula

$$\varphi(x) = Th(T^{-1}x). \tag{9}$$

Then φ is convex and (6) holds.

Conversely, if (6) holds with a convex function $\varphi : J \to \mathbb{R}$ then (9) defines a convex function $h : J \to \mathbb{R}$ which satisfies (8) whence (7) follows for all $x, y \in J$ and $t \in [0,1]$. But this means that (2) holds for all $x, y \in J$ and $t \in [0,T]$. ◊

Example 2. If $T \in (0,1)$, then taking $\varphi(x) = x^2$ for $x \in [0,+\infty)$ we get by Th. 2 that every function $f : [0,+\infty) \to \mathbb{R}$ satisfying

$$Tx^2 \le f(x) \le x^2 \quad \text{for} \quad x \in [0,+\infty)$$

is a solution of (2). Similarly, if $T \in (1,+\infty)$, then taking $\varphi(x) = 1/x$ for $x \in (0,+\infty)$ we see that every function $f : (0,+\infty) \to \mathbb{R}$ such that

$$1/(T^2x) \le f(x) \le 1/x \quad \text{for} \quad x \in (0,+\infty)$$

satisfies (2).

Now we pass to inequality (3). Fix a real interval I and a point $z_0 \in I$. For $T \in (0,1)$ put

$$I_T^* = TI + (1-T)z_0 .$$

Given a real function φ with the domain containing I_T^*, we define $\varphi_T^* : I \to \mathbb{R}$ by the formula

$$\varphi_T^*(x) = T^{-1}(\varphi(Tx+(1-T)z_0) - (1-T)\varphi(z_0)).$$

Theorem 3. *Let $T \in (0,1)$. A function $f : I \to \mathbb{R}$ satisfies (3) for all $x, y \in I$ and $t \in [0,T]$ iff there exists a convex function $\varphi : I_T^* \to \mathbb{R}$ such that*

(10) $\varphi_T^*(x) \le f(x)$ *for* $x \in I$ *and* $f(x) \le \varphi(x)$ *for* $x \in I_T^*$.

Proof. Assume that f satisfies (3). Putting $T \cdot t$ in place of t in (3) we have

(11) $$f_T^*(tx+(1-t)y) \le tf(x) + (1-t)f(y)$$

for all $x, y \in I$ and $t \in [0,1]$. Applying Th. 1 we obtain a convex function $h : I \to \mathbb{R}$ such that

(12) $$f_T^* \le h \le f .$$

Since $f_T^*(z_0) = f(z_0)$, we have $h(z_0) = f(z_0)$. Define $\varphi : I_T^* \to \mathbb{R}$ by the formula

(13) $$\varphi(x) = Th(T^{-1}(x-(1-T)z_0)) + (1-T)f(z_0).$$

Then φ is a convex function, $\varphi(z_0) = f(z_0)$,

$$\varphi_T^*(x) = h(x) \le f(x) \quad \text{for} \quad x \in I$$

and

$$\varphi(x) \ge Tf_T^*(T^{-1}(x-(1-T)z_0)) + (1-T)f(z_0) = f(x) \quad \text{for} \quad x \in I_T^*.$$

Conversely, if (10) holds with a convex function $\varphi : I_T^* \to \mathbb{R}$ then $f(z_0) = \varphi(z_0)$ and (13) defines a convex function $h : I \to \mathbb{R}$ which satisfies (12). This implies (11) for all $x, y \in I$ and $t \in [0,1]$. Consequently f satisfies (3) for all $x, y \in I$ and $t \in [0,T]$. ◊

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