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A CONCRETE ANALYSIS OF THE RADICAL CONCEPT

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Abstract: The notion of isolators, representing well–defined sets of proper ideals in an arbitrary ring, is employed to define preradicals through intersection. The various basic properties that a preradical may have are analysed through this technique. Also, isolators are emphasized in their role as "tangible mediators" between preradicals and their semisimple classes. Finally, the three classical nil radicals are characterized as radicals generated by a very natural sequence of isolators.

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1. Introduction

In describing the structure of certain types of algebraic systems, also such "radicals" which are not Kurosh–Amitsur radicals, may play an important role, for instance this is the case in the theory of near–rings, groups and lattice–ordered groups. Thus it seems to be useful to revisit the connection and interrelationship among the various notions of (not necessarily Kurosh–Amitsur) radicals.

The purpose of this paper is to analyse the concept of a radical of an arbitrary universal class $\mathcal{A}$ of not necessarily associative rings or Ω–groups, in particular near–rings. For the sake of simplicity we shall refer to the objects of $\mathcal{A}$ as *rings*. Let us recall that a *universal class* $\mathcal{A}$ of rings is one which is closed under taking ideals and homomorphic images. At this point we may mention that our investigation could be carried out on an even higher level of abstraction by taking arbitrary universal algebras for rings and congruence relations for ideals – this basis was selected by Hoehnke for his work in [3].

Our analysis is carried out progressively along the sequence:

$$\text{preradical} \longrightarrow \text{quasi–radical} \longrightarrow \text{radical}$$

A *Kurosh–Amitsur radical* r of $\mathcal{A}$ is a function $\mathcal{A} \to \mathcal{A}$, $A \mapsto rA \triangleleft A$ which satisfies the following conditions:
(A) for every homomorphism $f : A \to fA$, $(A \in \mathcal{A})$, $frA \subseteq rfA$;
(B) $r(A/rA) = 0$ for all $A \in \mathcal{A}$;
(C) for all $A \in \mathcal{A}$: $(I \triangleleft A$ and $rI = I) \Rightarrow I \subseteq rA$;
(D) for all $A \in \mathcal{A}$: $rrA = rA$.
A function $r : A \mapsto rA \triangleleft A$ which satisfies (A) is called a *preradical* (of $\mathcal{A}$). A preradical r which satisfies (B) is called a *quasi–radical* (also a *Hoehnke radical* due to [3]). A preradical r which satisfies (C) resp. (D) is said to be *complete* resp. *idempotent.* Connections between preradicals and (hereditary) quasi–radicals were first investigated by Michler in [8]. (Cf. also [9].) We follow a different line of approach.

The basic technique in our analysis is that of *definition through intersection of ideals* with the focus on basic properties of the *sets* of ideals to be intersected. For this purpose we employ the notion of an *isolator* (cf. our Def. 1, and [1]), and we impose various conditions on isolators, these conditions (stable, transferring, 0–extending) being generalizations of well–known properties of the set of semiprime ideals in an arbitrary associative ring (cf. definitions 2–4). We attempt to emphasize isolators as "concrete preradical definers" on the one hand, and "tangible mediators" between quasi–radicals and their semisimple classes, on the other. Our analyses culminate in three sequences of one–to–one correspondences (cf. corollaries 1–3) of the form:

$$r \longleftrightarrow \Lambda \longleftrightarrow \mathbf{S}$$

where r, Λ and $\mathbf{S}$ represent specific types of quasi–radicals, maximal stable isolators and subdirectly closed classes respectively.

As an application (section 5) we recover the three well-known nil-based radicals within our framework by giving new characterizations of these radicals in terms of natural cardinality conditions.

2. Quasi–radicals

We start off our analysis of the concept of a radical by defining our basic instrument.

Definition 1. An *isolator* Δ is a function which assigns to every ring A (in $\mathcal{A}$) a set $\Delta(A)$ of proper ideals of A satisfying the condition:

(α) if $f : A \to fA$ is any homomorphism, then for every $K \in \Delta(fA)$ there exists an $I \in \Delta(A)$ such that $fI \subseteq K$.

It easily follows that the following sets of proper ideals in an arbitrary associative ring A define isolators:

$\pi(A)$: the prime ideals;
$\Pi(A)$: the prime maximal ideals;
$\mu(A)$: the maximal ideals;
$\sigma(A)$: the semiprime ideals;
$\kappa(A)$: the quasi–semiprime ideals, (cf. [2]).

Proposition 1. *If Δ is an isolator then the assignment $A \mapsto rA = \cap(I \in \Delta(A))$ is a preradical.*

Proof. Let $f : A \to fA$ be any homomorphism. Then $frA = f(\cap(I \in \Delta(A))) \subseteq \cap(fI : I \in \Delta(A))$. Using ($\alpha$) we get $frA \subseteq \cap(K \in \Delta(f(A))) = rfA$. $\Diamond$

The preradical defined by the assignment $A \mapsto rA = \cap(I \in \Delta(A))$ will be referred to as the preradical *generated* by Δ. As to a converse to Prop. 1 we note that every preradical is trivially generated by an isolator: Let r be a preradical and define the function Δ by $\Delta(A) := \{P \triangleleft A : rA \subseteq P\}$. Then $rA = \cap(P \in \Delta(A))$ for all A. Let $f : A \to fA$ be any homomorphism, and $K \in \Delta(fA)$. Then $frA \subseteq rfA \subseteq K$;

and we have that (α) is satisfied with $I = rA$. Hence Δ is an isolator which generates r.

Our first condition on isolators is the one in:

Definition 2. An isolator Δ may be called *stable* if it satisfies the condition:

(β) if $f : A \to fA$ is any homomorphism, then for every $I \in \Delta(A)$ with $\ker f \subseteq I$ there exists a $K \in \Delta(fA)$ such that $K \subseteq fI$.

There exist isolators which are *not* stable. We construct such an isolator: Let n be a positive integer ≥ 2. Then the assignment $\overline{n} : A \mapsto \overline{n}A := \{a \in A : na = 0\}$ is a preradical. Define the function Δ by $\Delta(A) := \{\overline{m}A : m \geq n \text{ and } n|m\}$. Then Δ is an isolator, and it generates $\overline{n}$. However, Δ is not stable. For let us consider $f : A \to A/\overline{n}A$, and $I = \overline{n}A = \ker f$. Now if $K \in \Delta(fA)$ and $K \subseteq fI$, we must have that $K = \overline{s}(A/\overline{n}A) = \overline{n}A$ for some $s \geq n$ with $n|s$. This implies that $\{a + \overline{n}A : a \in \overline{sn}A\} = \overline{n}A$, i.e. $\overline{n}A = \overline{sn}A$, which is in general not true. Thus we have that Δ is not stable.

We note here (cf. [1]) that a function Δ assigning to every ring A a set $\Delta(A)$ of proper ideals of A, and satisfying:

(χ) for all A and every homomorphism $f : A \to fA$, the assignment $P \mapsto fP$ defines a bijection $\{P \in \Delta(A) : \ker f \subseteq P\} \to \Delta(fA)$,

satisfies (α) and (β), and hence it defines a stable isolator. In particular, if $\mathcal{P}$ is any abstract property of rings and the function Δ is defined by

$$\Delta(A) := \{P \triangleleft A : A/P \quad \text{has property} \quad \mathcal{P}\}$$

then Δ satisfies condition (χ), because by the isomorphism $A/P \cong fA/fP$ there is a bijection between $\{P \in \Delta(A) : \ker f \subseteq P\}$ and $\Delta(fA)$. (It is easy to construct a stable isolator which does *not* satisfy (χ).) It now easily follows that the isolators listed above are stable. Moreover, every preradical gives rise to a stable isolator. This is:

Proposition 2. *If r is a preradical then the assignment $A \mapsto \Delta(A) = \{I \triangleleft A : r(A/I) = 0\}$ is a stable isolator.*

Proof. Let f be a homomorphism, and $fK \in \Delta(fA)$. Then $r(A/K) \cong r(fA/fK) = 0$ shows that $K \in \Delta(A)$, and (α) is satisfied. The validity of (β) follows in a similar way. $\Diamond$

Quasi–radicals and stable isolators stand in a very special relationship with one another. This is:

Theorem 1. *Let r be a preradical. Then r is a quasi–radical if and only if r is generated by a stable isolator.*

Proof. Suppose that r is a quasi–radical. Then $r(R/rR) = 0$ for all R. This shows that for an arbitrary ring A, $\cap(I \triangleleft A : r(A/I) = 0) \subseteq rA$. On the other hand, if $I \triangleleft A$ such that $r(A/I) = 0$ then the natural homomorphism $A \to A/I$ induces that $rA \to r(A/I) = 0$, so that $rA \subseteq I$. Thus we have that $rA = \cap(I \triangleleft A : r(A/I) = 0)$, and it remains to show that the function Λ defined by $\Lambda(A) = \{I \triangleleft A : r(A/I) = 0\}$ is a stable isolator. This follows by Prop. 2.

Conversely, suppose that r is generated by a stable isolator Δ. Let A be an arbitrary ring and consider $f : A \to A/rA$. Now $r(A/rA) = rfA := \cap(K \in \Delta(fA))$. Using ($\beta$) and ($\alpha$) we get $r(A/rA) \subseteq \cap(fI : I \in \Delta(A), rA \subseteq I) = \cap(fI : I \in \Delta(A)) = \cap(I/ \cap (I \in \Delta(A)) : I \in \Delta(A)) = 0.$ $\Diamond$

The preradical r in Prop. 2 and the quasi–radical implied there in view of Th. 1, say s, are comparable:

Proposition 3. *$rA \subseteq sA$ for all rings A.*

Proof. Let $I \triangleleft A$ such that $r(A/I) = 0$ and f the natural homomorphism $A \to A/I$. Then $frA \subseteq r(A/I) = 0$, and this together with $frA = (rA + I)/I$ shows that $rA \subseteq I$. Thus we have that $rA \subseteq \cap(I \triangleleft A : r(A/I) = 0) = sA$. $\Diamond$

Referring back to our list of well–known stable isolators we recall the fact that $\cap(I \in \pi(A)) = \cap(I \in \sigma(A))$. This implies that a given quasi–radical may be generated by *different* stable isolators. In terms of the partial order on isolators defined by "$\Delta \leq \Delta' \Leftrightarrow \Delta(A) \subseteq \Delta'(A)$ for all A" we have:

Proposition 4. *If r is a quasi–radical then the function Λ defined by $\Lambda(A) = \{I \triangleleft A : r(A/I) = 0\}$ is a stable isolator such that Λ satisfies condition (χ) and the quasi-radical generated by Λ is r. Moreover, if Δ is any stable isolator generating the same quasi–radical r, then $\Delta \leq \Lambda$.*

Proof. The first claim was already verified in the proof of Th. 1. Let us therefore consider any stable isolator generating r. Let A be an arbitrary ring, and $I \in \Delta(A)$. Applying (β) to $A \to A/I$ we find that $K := 0 \in \Delta(A/I)$. Since $r(A/I) \subseteq X/I$ for all $X/I \in \Delta(A/I)$ we have that $r(A/I) = 0$, i.e. $I \in \Lambda(A)$. Hence $\Delta \leq \Lambda$. $\Diamond$

The unique maximal stable isolator Λ corresponding to the quasi–radical r will be referred to as the *maximal generating isolator* for r, and denoted by $\Lambda[r]$, or just by Λ where no ambiguity can occur. For any given quasi–radical r, $\Lambda[r]$ has another unique feature, as is exhibited in the following characterization.

Proposition 5. *Let Δ be a stable isolator. Then $\Delta = \Lambda[r]$ for some* *quasi–radical r if and only if Δ satisfies the condition:*

$$(\phi) \qquad \forall A((\Gamma \subseteq \Delta(A)) \Rightarrow \cap(I \in \Gamma) \in \Delta(A))$$

Proof. Suppose $\Delta = \Lambda$ for a quasi–radical r. Let A be an arbitrary ring and $\Gamma \subseteq \Delta(A)$; and consider $K := \cap(I \in \Gamma)$. Using (β) for Λ we get

$$r(A/K) := \cap(M/K \in \Lambda(A/K)) =$$
$$= \cap(M/K : K \subseteq M \in \Lambda(A)) \subseteq \cap(I/K \in \Gamma) = 0\,.$$

Thus we have that $\cap(I \in \Gamma) = K \in \Lambda(A) = \Delta(A)$.

Conversely, let Δ be a stable isolator satisfying (ϕ), and r the quasi–radical generated by Δ. Then $\Delta \leq \Lambda$. Let $P \in \Lambda(A)$. Then $r(A/P) = 0$. This implies (since Δ generates r) that $\cap(I/P \in \Delta(A/P)) = 0$, and this in its turn implies that $\cap(I : I/P \in \Delta(A/P)) = P$. By (α), for each $I_\nu/P \in \Delta(A/P)$ there is an $M_\nu \in \Delta(A)$ such that $M_\nu/P \subseteq I_\nu/P$. Set $\Gamma := \{M_\nu\}$. Then we have $\cap(M_\nu \in \Gamma) \subseteq \subseteq \cap(I : I/P \in \Delta(A/P)) = P$. By (β), for each $M_\nu \in \Gamma$ there is an $L_\nu/P \in \Delta(A/P)$ such that $L_\nu/P \subseteq M_\nu/P$. Hence $P = = \cap(I : I/P \in \Delta(A/P)) \subseteq \cap L_\nu \subseteq \cap(M \in \Gamma)$. Thus we have shown that $P = \cap(M \in \Gamma)$; and condition (ϕ) yields $P \in \Delta(A)$. Hence $\Lambda \leq \Delta$; and the equality $\Delta = \Lambda[r]$ follows. $\Diamond$

A quasi–radical is, as mentioned in the introduction, just a Hoehnke radical. It is known from [3] that there is a one–to–one correspondence between quasi–radicals and subdirectly closed classes: if r is a quasi–radical then the class $\mathbf{S}_r := \{A \in \mathcal{A} : rA = 0\}$, (which is usually called the *semisimple class* of the quasi–radical r), is closed under taking subdirect sums; and if $\mathbf{S}$ is a subclass of $\mathcal{A}$ being closed under subdirect sums, then the assignment $r : A \mapsto rA$ defined by $rA = \cap(I \triangleleft A : A/I \in \mathbf{S})$ is a quasi–radical with semisimple class $\mathbf{S}$. In view of this and Th. 1 and Prop. 4 we have:

Corollary 1. *There exist one–to–one correspondences $r \longleftrightarrow \Lambda \longleftrightarrow \mathbf{S}$ between quasi–radicals r, maximal stable isolators Λ, and subdirectly closed classes $\mathbf{S}$.*

An example: $\beta : A \mapsto \beta A := \cap(P \in \pi(A)) = \cap(S \in \sigma(A))$ is a quasi–radical. (β is the well–known prime radical for associative rings.) Using Prop. 5 together with well–known properties of prime and semiprime ideals, we see that $\pi \neq \Lambda[\beta]$ while $\sigma = \Lambda[\beta]$.

3. Complete quasi–radicals

In this section we carry our analysis one step further : we consider those preradicals satisfying conditions (B) and (C) stated in the introduction, i.e. the complete quasi–radicals. For this purpose we shall need a further condition on isolators:

Definition 3. An isolator Δ may be called *transferring* if it satisfies the condition:

(γ) if $P \in \Delta(A)$ and $I \triangleleft A$ with $P \cap I \neq I$ then there exists a $K \triangleleft I$, $K \neq I$, such that $P \cap I \subseteq K$ and $K \in \Delta(I)$.

Of the five examples of isolators listed in section 2, the first four are transferring. In fact, in the case of any $\Delta \in \{\pi, \Pi, \mu, \sigma\}$, well–known properties of the ideals concerned show that $(P \in \Delta(A), I \triangleleft A, P \cap I \neq I)$ implies that $P \cap I \in \Delta(I)$, and (γ) is satisfied with $K = P \cap I$. The isolator κ, however, is not transferring, e.g. if A is a ring with identity and having a nilpotent ideal $N \neq 0$, the $\cap(P \in \kappa(A)) = 0$, so that $P \cap N \neq N$ for at least one $P \in \kappa(A)$, though $\kappa(N) = \emptyset$. (Cf. [2].)

A basic relationship between transferring isolators and complete quasi–radicals is exhibited in:

Theorem 2. *The following three conditions on a quasi–radical r are equivalent:*

(1) *r is complete;*

(2) *the maximal generating isolator $\Lambda = \Lambda\ [r]$ of r is transferring;*

(3) *the semisimple class $\mathbf{S}_r$ of r is regular, i.e.* $(0 \neq I \triangleleft A \in \mathbf{S}_r) \Rightarrow$ $\Rightarrow$ ($\exists\ K \triangleleft I$ *such that* $0 \neq I/K \in \mathbf{S}_r$).

Proof. (1) $\Rightarrow$ (2): Assume that Λ is not transferring. Then there is a $P \in \Lambda(A)$ and an $I \triangleleft A$ with $P \cap I \neq I$ such that $\Lambda(I/(P \cap I)) = \emptyset$. The latter implies that $r(I/(I \cap P)) = I/(I \cap P)$, and hence we get $r((I + P)/P) = (I + P)/P$. From the completeness of r it follows that $(I + P)/P \subseteq r(A/P) = 0$, giving the contradiction $I \subseteq P$. Hence Λ is transferring.

(2) $\Rightarrow$ (3): Let A be a ring with $rA = 0$, and $0 \neq I \triangleleft A$. From $rA = 0$ and the maximality of Λ it follows that $0 \in \Lambda(A)$. Since Λ is transferring, there exists a $K \triangleleft I$, $K \neq I$ such that $K \in \Lambda(I)$, i.e. $r(I/K) = 0$.

(3) $\Leftrightarrow$ (1): has been proved in Prop. 2.2 of [7]. $\Diamond$

In the structure theory of 0–symmetric near–rings the most important quasi–radical assignments are $\mathcal{J}_\nu : A \mapsto \mathcal{J}_\nu(A)$, $\nu = 0, 1, 2$. It is known that $\mathcal{J}_0 < \mathcal{J}_1 < \mathcal{J}_2$, that $\mathcal{J}_0$ and $\mathcal{J}_1$ are complete but not idempotent, (cf. [4]), and $\mathcal{J}_2$ is a Kurosh–Amitsur radical. For any *ring* A, $\mathcal{J}_0(A) = \mathcal{J}_1(A) = \mathcal{J}_2(A)$ is the usual Jacobson radical. For details we refer to [6]. By Th. 2 we have:

Corollary 2. *There exist one-to-one correspondences* $r \longleftrightarrow \Lambda \longleftrightarrow \mathbf{S}$ *between the complete quasi-radicals* r*, the maximal stable transferring isolators* Λ*, and the regular subdirectly closed classes* $\mathbf{S}$.

Remark: Applying our Th. 2 and Th. 1.2 of [7] to the property "the quasi-radical r is complete", and translating into the language of isolators, we get: If Δ is an isolator satisfying condition (χ) and generating the quasi-radical r, then Δ is transferring if and only if the maximal stable isolator $\Lambda[r]$ is transferring.

4. Idempotent complete quasi-radicals

We now proceed to isolating the (Kurosh–Amitsur) radicals from among the preradicals. First we shall briefly look at idempotent preradicals.

Proposition 6. *Let* Δ *be an isolator generating the preradical* r. r *is idempotent if and only if* $\Delta(rA) = \emptyset$ *for all* $A \in \mathcal{A}$. (The proof is straightforward.)

Proposition 7. *Let* r *be a quasi-radical and* Λ *the corresponding maximal stable isolator. The following conditions are equivalent:*

(1) r *is idempotent,*

(2) $\Lambda(rA) = \emptyset$ *for all* $A \in \mathcal{A}$,

(3) $r(rA/M) \neq 0$ *for all* $A \in \mathcal{A}$ *and all proper ideals* M *of* rA.

Proof. (1) $\Rightarrow$ (2): this follows from Prop. 6.

(2) $\Rightarrow$ (3): If $r(rA/M) = 0$ for a proper ideal M of A, then $M \in \Lambda(rA)$, contradicting (2).

(3) $\Rightarrow$ (1) If r is not idempotent, then $rrA \neq rA$ for some $A \in$ $\in \mathcal{A}$. Since r is a quasi-radical, for $M = rrA$ we have $r(A/M) = 0$, contradicting (3). $\Diamond$

Another criterion for the idempotence of a quasi-radical has been given in Prop. 2.4 of [7].

The quasi-radical r determined by κ is (as has already been indicated) not complete. It is, however, idempotent because: for any ring A and $I \triangleleft A$, $rI \supseteq I \cap rA$ (cf. [2], Lemma 4.6), and, setting $I := rA$, we have $rA \supseteq rrA \supseteq rA \cap rA = rA$.

For 0–symmetric near–rings the isolator π defines an idempotent quasi–radical, called the *prime* radical. As it has been shown in [5], this prime radical is not a Kurosh–Amitsur radical, so it is not complete, and π is not transferring.

The independence of being complete and being idempotent has been exhibited also in examples 1 and 2 of [7].

We shall need one more condition on isolators:

Definition 4. An isolator Δ may be called 0–*extending* if it satisfies the condition:

$$(\delta) \qquad \text{if} \quad I \in \Delta(A) \quad \text{and} \quad 0 \in \Delta(I) \quad \text{then} \quad 0 \in \Delta(A).$$

The best–known example of a 0–extending isolator is the isolator σ isolating the semiprime ideals in an arbitrary associative ring. As one easily sees π is not 0–extending, though it generates the same preradical as σ. Also the isolator μ is not 0–extending.

For the purposes of our next result we shall need a further construction. We consider an arbitrary fixed preradical r and define a function

$$\Psi : A \mapsto \Psi(A) := \{P \triangleleft A : (Q \triangleleft A \quad \text{and} \quad rQ = Q) \Rightarrow Q \subseteq P\};$$

and then a function $r' : A \mapsto r'A := \cap(P \in \Psi(A))$. It is easy to see that Ψ is an isolator, and hence r' is a preradical. Moreover, if Δ is a transferring isolator generating r, then $\Delta \leq \Psi$. In this notation we have:

Theorem 3. *The following three conditions on a complete quasi–radical r are equivalent:*

(1) *r is idempotent (and hence a Kurosh–Amitsur radical);*

(2) *for all A, $rrA \triangleleft A$; and $\Lambda = \Lambda[r]$ is 0–extending;*

(3) *for all A, $rrA \triangleleft A$ and $rA = r'A$.*

Proof. (1) $\Rightarrow$ (2): Since $rrA = rA$, $rrA \triangleleft A$. Let $I \in \Lambda(A)$ and $0 \in \Lambda(I)$, and assume that $0 \notin \Lambda(A)$, i.e. $rA \neq 0$. By the completeness of r it follows that $(rA + I)/I \subseteq r(A/I) = 0$. Hence $0 \neq rA \subseteq I$, contradicting $rI = 0$. It follows that Λ is 0–extending.

(2) $\Rightarrow$ (3): Assuming (2) we prove that $rA = r'A$. From $r((A/rrA)/(rA/rrA)) \cong r(A/rA) = 0$ we see that $rA/rrA \in \in \Lambda(A/rrA))$, and it is clear that $0 \in \Lambda(rA/rrA)$. Hence (2) implies that $0 \in \Lambda(rA/rrA)$. This gives the inclusion $rA \subseteq rrA$, and we have that $rrA = rA$. This equality shows that $rA \subseteq T$ for all $T \in \Psi(A)$, so that $rA \subseteq r'A$. On the other hand, if $P \in \Lambda(A)$ and $Q \triangleleft A$ with

$rQ = Q$, it follows by the completeness of r that $Q \subseteq rA \subseteq P$, so that $P \in \Psi(A)$. Hence $r'A \subseteq rA$; and now $rA = r'A$ follows.

(3) $\Rightarrow$ (1): Let $Q \triangleleft A$ such that $rQ = Q$. Since r is complete, $Q \subseteq rA$. Since now $Q \triangleleft rA$, it follows (again by the completeness of r) that $Q \subseteq rrA$. Hence it follows that $rrA \in \Psi(A)$. We now have that $r'A \subseteq rrA \subseteq rA = r'A$, showing that $rrA = rA$. $\Diamond$

Our final observation in this section comes in view of Th. 3 and known facts about the semisimple classes of Kurosh–Amitsur radicals (cf. [10]). This is:

Corollary 3. *There exist one–to–one correspondences* $r \longleftrightarrow \Lambda \longleftrightarrow \mathbf{S}$ *between Kurosh–Amitsur radicals* r*, those maximal stable transferring 0–extending isolators* Λ *for which* $r_\Lambda r_\Lambda A \triangleleft A$ *for all* A*, and those subdirectly closed, regular, extensionally closed classes* $\mathbf{S}$ *for which* $r_{\mathbf{S}} r_{\mathbf{S}} A \triangleleft A$ *for all* A*, where* r_Λ *and* $r_{\mathbf{S}}$ *represent the quasi–radicals associated with* Λ *and* $\mathbf{S}$ *respectively.*

5. An application — the nil radicals

In this final section — by way of an application — we use the isolator approach to construct a single formula function which yields the three classical nil radicals — the prime radical β, the locally nilpotent radical $\mathcal{L}$ and the nil radical $\mathcal{N}$. We confine our attention here to associative rings. We shall need the following:

Lemma. *A proper ideal* S *of a ring* A *is a semiprime ideal if and only if every nonzero ideal of* A/S *has a potent countably generated subring.*

Proof. Suppose S is a semiprime ideal of a ring A and let $0 \neq X/S \triangleleft \triangleleft A/S$. If every countably generated subring of X/S is nilpotent, then X/S itself is a nilpotent ring. (Suppose X/S is not nilpotent. Then for every natural number n we may select a sequence $r_{1n}, r_{2n}, \ldots, r_{nn}$ in X/S such that $r_{1n}r_{2n}\ldots r_{nn} \neq 0$. Thus we would have selected a countable subset of X/S which generates a countable *non*-nilpotent (i.e. potent) subring of X/S).** But then $X^m \subseteq S$ for some positive integer m, and this would imply that $X \subseteq S$.

Conversely, suppose that every nonzero ideal of A/S has a potent countably generated subring. Let $X \triangleleft A$ such that $X^2 \subseteq S$. If $X \not\subseteq S$,

**Professor Otto Kegel has on inquiry pointed out this subproof to the first author in a personal communication in 1981. We are indebted to professor Kegel.

then $(X+S)/S$ must have a potent countably generated subring T/S. But this is impossible since $(T/S)^2 \subseteq ((X+S)/S)^2 = 0$. $\Diamond$

We are now in a position to prove the main result of this section. This is:

Theorem 4. *Let Δ_i $(1 \le i \le 3)$ be the function defined by*
$\Delta_i(A) = \{P \triangleleft A, P \neq A : (P \subset X \triangleleft A) \Rightarrow X/P$ *has a potent* δ_i*–subring*$\}$

where the δ_i are defined as follows:

δ_1 : *"countably generated";*

δ_2 : *"finitely generated";*

δ_3 : *"singly generated".*

Then the Δ_i are stable transferring isolators, and they generate respectively the radicals:

$r_1 = \beta$: *the prime radical;*

$r_2 = \mathcal{L}$: *the locally nilpotent radical;*

$r_3 = \mathcal{N}$: *the nil radical.*

Proof. Fix any $i \in \{1,2,3\}$. The stableness of Δ_i is an easy consequence of the remarks following condition (χ). To show that Δ_i is tranferring we verify the stronger condition $(P \in \Delta_i(A),\ Q \triangleleft A,\ P \cap Q \neq Q) \Rightarrow P \cap Q \in \Delta_i(Q)$. Let $0 \neq X/(P \cap Q) \triangleleft Q/(P \cap Q)$. Then $Q/(P \cap Q) \cong (Q+P)/P$ gives us the existence of a nonzero ideal $Y/P \cong X/(P \cap Q)$ in $(Q+P)/P$. Let Y^*/P be the ideal generated in A/P by Y/P. Then Y^*/P has a potent δ_i–subring, and hence $(Y^*/P)^3 = (Y^{*^3}+P)/P \neq 0$. Hence, since $P \in \Delta_i(A)$, $(Y^*/P)^3$ has a potent δ_i–subring. Since by the Andrunakievich lemma $(Y^*/P)^3 \subseteq$ $\subseteq Y/P \cong X/(P \cap Q)$, it follows that $P \cap Q \in \Delta_i(Q)$.

Having established that Δ_i is stable and transferring we now prove that the complete quasi–radical r_i generated by Δ_i is a radical. We apply Th. 3. Let once again A be an arbitrary ring. From the lemma we infer that every $P \in \Delta_i(A)$ is a semiprime ideal of A. This ensures that $r_i r_i A \triangleleft A$. Suppose that $r_i^{\prime} A \subset r_i A$. Then there exists a $T \in \Psi(A)$ such that $r_i A \not\subseteq T$. This shows that $\Delta_i(r_i A) \neq \emptyset$, i.e., there is a proper ideal U of $r_i A$ such that every nonzero ideal of $r_i A/U$ has a potent δ_i–subring. It follows by the lemma that U is a semiprime ideal of $r_i A$, and we know that $r_i A := \cap(P \in \Delta_i(A))$ is a semiprime ideal of A. Hence U is a semiprime ideal of A. Let $0 \neq X/U \triangleleft A/U$. If $r_i A \cap X \neq U$ then there is a potent δ_i–subring in $(r_i A \cap X)/U$ and hence in X/U. If $r_i A \cap X = U$ then $(X + r_i A)/r_i A \cong X/U$. Now $X + r_i A \neq r_i A$ and since $r_i((X + r_i A)/r_i A) = 0$ there exists a $T = r_i A \in (X + r_i A)/r_i A$, so that there is a potent δ_i-subring in $((X + r_iA)/r_iA)/(T/r_iA)$, and consequently there is a potent δ_i–subring in $(X+r_iA)/r_iA$. Hence X/U has a potent δ_i–subring. It follows that $U \in \Delta_i(A)$ — a contradiction.

By our construction $r_iA := \cap(P \in \Delta_i(A))$ it follows that $r_iA = A$ if and only if $\Delta_i(A) = \emptyset$. In the case $i = 1$ this means (by the lemma) that $r_1A = A$ if and only if A has no semiprime ideals, i.e. A is a β–radical ring. In the case $i = 2$ we note that a locally nilpotent ring A clearly has $\Delta_2(A) = \emptyset$. On the other hand, let B be a ring with $\Delta_2(B) = = \emptyset$. If B has a potent finitely generated subring $\langle x_1, \dots, x_n\rangle$, (we may assume all x_i are potent elements), we may, in view of the finiteness, select a maximal element in the set $\{C \triangleleft B : \{x_1, \dots, x_n\} \not\subseteq C\}$, say M. But then clearly M must be in $\Delta_2(B)$, contradicting $\Delta_2(B) = \emptyset$. Thus we have that $r_2 = \mathcal{L}$. The same argument applied in the case $i = 3$, together with the fact that a singly generated subring $\langle x\rangle$ of a ring is nilpotent if and only if x is a nilpotent element, ensures that $r_3 = \mathcal{N}$. $\Diamond$

Corollary 4. *The function* Γ_i *defined by*

$$\Gamma_i(A) := \{I \in \Delta_i(A) : I \quad \textit{is a prime ideal}\}$$

is a stable isolator generating the radical r_i *for* $i = 1, 2, 3$.

Proof. One readily sees that Γ_i satisfies condition (χ) and therefore Γ_i is a stable isolator.

Let B be a prime ring such that $r_iB = 0$. This means exactly that every nonzero ideal of B has a potent δ_i–subring. Since $r_i = \beta, \mathcal{L}, \mathcal{N}$ are special radicals, every ring A with $r_iA = 0$ is a subdirect sum of prime rings B_α with $r_iB_\alpha = 0$. Thus for a ring A the condition

$$s_iA := \cap(P \in \Gamma_i(A)) = 0$$

is equivalent to $r_iA = 0$. Moreover, by $\Gamma_i \leq \Delta$ it follows that $r_iX \subseteq s_iX$ for every ring X. Now suppose that $r_iX \neq s_iX$ for a ring X. Then we have

$$s_i(s_iX/r_iX) = r_i(s_iX/r_iX) = 0,$$

which implies that r_iX is an element of the maximal stable isolator generating the quasi–radical s_i. Hence $s_iX \subseteq r_iX$ follows, contradicting the assumption. $\Diamond$

The natural cardinality considerations in Th. 4 seem to explain the imperturbable monopoly that β, $\mathcal{L}$ and $\mathcal{N}$ have maintained on the lower end of the chain of useful concrete radicals.

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ON A FUNCTIONAL EQUATION OCCURRING IN ASTROPHYSICS

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Abstract: In this paper we describe three methods of constructing the general solution of the functional equation $\overline{f(z)f(-1/\overline{z})} = -1$ and we discuss a few examples. The paper ends with a simple uniqueness theorem.

The functional equation

$$f(z)\overline{f(-1/\overline{z})} = -1 \tag{1}$$

occurs in astrophysics (cf. [4]). Here the unknown function f maps the complex plane punctured at zero $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$ into itself and relation (1) ($\overline{z}$ denotes the complex conjugate of z) is assumed to hold for all $z \in \mathbb{C}^*$.

Write

$$h(z) := -1/\overline{z} = -z/|z|^2 , \quad z \in \mathbb{C}^* . \tag{2}$$

The function $h : \mathbb{C}^* \to \mathbb{C}^*$ is an involution

$$h(h(z)) = z , \quad z \in \mathbb{C}^* , \tag{3}$$

without fixed points (of order 1). With the aid of (2) equation (1) can be written in the form

$$f(h(z)) = h(f(z)), \quad z \in \mathbb{C}^*. \tag{4}$$

Relation (4) expresses the permutability of the functions f and h.

In this paper we describe three methods of constructing the general solution of equation (1) and we discuss a few examples. The paper ends with a simple uniqueness theorem.

1. The first method of solving (1) follows the pattern described in [2; Chapter I] (cf. also [1]). Put

$$\begin{cases} D_1 := \{z \in \mathbb{C}^* \mid \operatorname{Im} z > 0\} \cup \{z \in \mathbb{C}^* \mid \operatorname{Re} z > 0,\ \operatorname{Im} z = 0\}, \\ D_2 := \{z \in \mathbb{C}^* \mid \operatorname{Im} z < 0\} \cup \{z \in \mathbb{C}^* \mid \operatorname{Re} z < 0,\ \operatorname{Im} z = 0\}. \end{cases} \tag{5}$$

We have

$$D_1 \cup D_2 = \mathbb{C}^*, \quad D_1 \cap D_2 = \emptyset, \tag{6}$$

and the function h maps (in a one–to–one manner) D_1 onto D_2 and conversely:

$$h(D_1) = D_2, \quad h(D_2) = D_1. \tag{7}$$

Let $F : D_1 \to \mathbb{C}^*$ be a quite arbitrary function and define the function $f : \mathbb{C}^* \to \mathbb{C}^*$ by the formula

$$f(z) = \begin{cases} F(z), & z \in D_1, \\ h(F(h(z))), & z \in D_2. \end{cases} \tag{8}$$

Definition (8) is correct in view of (6) and (7). We are going to show that function (8) satisfies equation (4) (i.e., equation (1)) on $\mathbb{C}^*$. Take an arbitrary $z \in \mathbb{C}^*$. According to (6) either $z \in D_1$ or $z \in D_2$. In the former case we have by (8) $f(z) = F(z)$ so that $h(f(z)) = h(F(z))$, and by (7) $h(z) \in D_2$, whence, again by (8),

$$f(h(z)) = h[F(h(h(z)))] = h(F(z)) = h(f(z))$$

(cf. (3)). Consequently relation (4) holds true.

When $z \in D_2$, then $h(z) \in D_1$, and we obtain using (8) and (3)

$$h(f(z)) = h[h(F(h(z)))] = F(h(z)) = f(h(z)).$$

Thus (4) holds in this case, too.

It is clear that taking in formula (8) all possible functions $F :$ $: D_1 \to \mathbb{C}^*$ we obtain all solutions $f : \mathbb{C}^* \to \mathbb{C}^*$ of equation (4). (In order to get a given solution f of (4) one takes $F = f|D_1$.) Thus we have, since equations (1) and (4) are equivalent.

Proposition 1. *With notation* (2) *and* (5), *for every function* $F : D_1 \to \mathbb{C}^*$ *the function* f *defined by* (8) *satisfies functional equation* (1), *and all the solutions* $f : \mathbb{C}^* \to \mathbb{C}^*$ *of* (1) *may be obtained in this manner.*

Thus formula (8) yields the general solution $f : \mathbb{C}^* \to \mathbb{C}^*$ of equation (1), the function F playing the role of a parameter. We say (see [2] or [3]) that *the general solution of* (1) *depends on an arbitrary function.*

It is readily seen from (8) that equation (1) has a lot of very irregular (e.g., discontinuous or nonmeasurable) solutions: to obtain them it is enough to take in (8) an irregular F. We shall return to the problem of the regularity of solutions of (1) later in this paper. Here we observe only that if the function $F : D_1 \to \mathbb{C}^*$ is continuous on D_1 and, moreover, for real negative z_0 it fulfils the condition

$$\lim_{z \to z_0\,, z \in D_1} F(z) = h(F(-z_0))\,,$$

then the solution f of equation (1) obtained from formula (8) is continuous on $\mathbb{C}^*$. Thus also in the class of the continuous functions $f : \mathbb{C}^* \to \mathbb{C}^*$ the solution of equation (1) depends on an arbitrary function.

Remark 1. In this construction instead of sets (5) we could take arbitrary sets fulfilling conditions (6) and (7) (in the argument we use only these properties, the particular shape of sets (5) is irrelevant), e.g. we could take

$$\begin{cases} D_1 := \{z \in \mathbb{C} \,|\, 0 < |z| < 1\} \cup \{z \in \mathbb{C} \,|\, |z| = 1\,,\ \operatorname{Im} z > 0\} \cup \{1\}\,, \\ D_2 := \{z \in \mathbb{C} \,|\, |z| > 1\} \cup \{z \in \mathbb{C} \,|\, |z| = 1\,,\ \operatorname{Im} z < 0\} \cup \{-1\}\,. \end{cases}$$

The essential thing is that the set D_1 should contain exactly one point of every couple $\{z, h(z)\}$, $z \in \mathbb{C}^*$ (i.e. of every orbit under h contained in $\mathbb{C}^*$) and $D_2 = \mathbb{C}^* \setminus D_1$.

2. The second method of constructing the general solution of (1) is that of the linearization. Its general principles are explained, e.g., in [3; p. 5], but the details must be worked out separately in every particular case.

First we define a function $\sigma : \mathbb{C}^* \to \mathbb{C}^*$ by the formula

$$\sigma(z) = \begin{cases} z\,, & z \in D_1, \\ 1/\overline{z} = z/|z|^2\,, & z \in D_2, \end{cases} \tag{9}$$

where the sets D_1 and D_2 are given by (5). The direct verification shows that σ satisfies for all $z \in \mathbb{C}^*$ the functional relation (the Schröder equation; cf. [2], [3])

$$\sigma[h(z)] = -\sigma(z)\,. \tag{10}$$

Moreover, σ is invertible. Indeed, suppose that for some $u, v \in \mathbb{C}^*$ we have

$$\sigma(u) = \sigma(v)\,. \tag{11}$$

By (5) and (9) we have $\sigma(D_1) = D_1$, $\sigma(D_2) = D_2$ and (11) implies according to (6) that the points u and v must both lie in the same set D_i. In other words, either $u, v \in D_1$, or $u, v \in D_2$. In the former case, in view of (9), relation (11) turns into

$$u = v\,, \tag{12}$$

while in the latter case (11) yields $1/\overline{u} = 1/\overline{v}$ which again is equivalent to (12). Thus for arbitrary $u, v \in \mathbb{C}^*$ relation (11) implies (12), which means that the function σ is invertible, as claimed.

Consequently there exists the function $\sigma^{-1} : \mathbb{C}^* \to \mathbb{C}^*$, inverse to σ, and by virtue of (10) it satisfies on $\mathbb{C}^*$ the functional equation

$$\sigma^{-1}(-z) = h(\sigma^{-1}(z))\,. \tag{13}$$

Let $\psi : \mathbb{C}^* \to \mathbb{C}^*$ be an arbitrary odd function:

$$\psi(-z) = -\psi(z)\,, \quad z \in \mathbb{C}^*\,, \tag{14}$$

and define the function $f : \mathbb{C}^* \to \mathbb{C}^*$ by the formula

$$f(z) = \sigma^{-1}[\psi(\sigma(z))]\,, \quad z \in \mathbb{C}^*\,. \tag{15}$$

Function (15) satisfies equation (4) (or, equivalently, equation (1)) on $\mathbb{C}^*$. In fact, according to (15), (10), (14) and (13), we have for arbitrary $z \in \mathbb{C}^*$

$$\begin{aligned} f(h(z)) &= \sigma^{-1}[\psi(\sigma(h(z)))] = \sigma^{-1}[\psi(-\sigma(z))] = \\ &= \sigma^{-1}[-\psi(\sigma(z))] = h[\sigma^{-1}(\psi(\sigma(z)))] = h(f(z))\,. \end{aligned}$$

Conversely, if a function $f : \mathbb{C}^* \to \mathbb{C}^*$ satisfies equation (4) (i.e., (1)) on $\mathbb{C}^*$, then it can be written in form (15), where $\psi(z) := \sigma[f(\sigma^{-1}(z))]$

is an odd function by virtue of (13), (4) and (10). Thus we have the following

Proposition 2. *With notations* (5) *and* (9), *for every odd function* $\psi : \mathbb{C}^* \to \mathbb{C}^*$, *the function* f *defined by* (15) *satisfies the functional equation* (1), *and all the solutions* $f : \mathbb{C}^* \to \mathbb{C}^*$ *of* (1) *may be obtained in this manner.*

Remark 2. In this construction function (9) could be replaced by an arbitrary other particular invertible solution $\sigma : \mathbb{C}^* \to \mathbb{C}^*$ of equation (10). There exist many such solutions (the general invertible solution $\sigma : \mathbb{C}^* \to \mathbb{C}^*$ of (10) depends on an arbitrary function) and any one of them can be used here. The argument depends only on (10) and on the invertibility of σ and not on the particular shape of function (9). However, we have been unable to find a more regular invertible particular solution of equation (10) on $\mathbb{C}^*$.

Formula (15) yields the general solution of equation (1) on $\mathbb{C}^*$. Unavoidably, also this formula contains an arbitrary (odd) function as a parameter. Formula (15) is more elegant and looks more agreeable than formula (8) but its disadvantage is that — due to the peculiar shape of the function σ — it is rather difficult to deduce from (15) the regularity properties of f. From this point of view the third method of solving (1), which we are now about to explain, seems most promising.

3. The third method of constructing the general solution of equation (1) is not new either (cf., e.g., [2, p. 148]), but I know of no place where it would be explained in a more general setting.

Let $f_0 : \mathbb{C}^* \to \mathbb{C}^*$ be a particular solution of equation (1) on $\mathbb{C}^*$ and let $g : \mathbb{C}^* \to \mathbb{C}^*$ be an arbitrary function. Put

$$f(z) = f_0(z)g(z)/\overline{g(h(z))}, \quad z \in \mathbb{C}^*. \tag{16}$$

We have by (2) and (3), since f_0 satisfies equation (4),

$$\begin{aligned} f(h(z)) &= f_0(h(z))g(h(z))/\overline{g(z)} = h(f_0(z))g(h(z))/\overline{g(z)} = \\ &= -\frac{g(h(z))}{\overline{f_0(z)}\,\overline{g(z)}} = h(f(z)), \end{aligned}$$

which means that f satisfies equation (4) on $\mathbb{C}^*$. Conversely, let f and f_0 be arbitrary solutions of equation (4) on $\mathbb{C}^*$ and let $\varphi : D_1 \to \mathbb{C}^*$ (where the sets D_1 and D_2 are given by (5)) be an arbitrary function fulfilling the condition

$$[\varphi(z)]^2 = f(z)/f_0(z), \quad z \in \mathbb{C}^*. \tag{17}$$

We define the function $g : \mathbb{C}^* \to \mathbb{C}^*$ by the formula

$$g(z) = \begin{cases} \varphi(z), & z \in D_1, \\ 1/\overline{\varphi(h(z))}, & z \in D_2. \end{cases} \tag{18}$$

For $z \in D_1$ we have by (7) $h(z) \in D_2$, and according to (18) and (3)

$$g(z) = \varphi(z), \quad g(h(z)) = 1/\overline{\varphi(z)}, \quad \overline{g(h(z))} = 1/\varphi(z)$$

so that $g(z)/\overline{g(h(z))} = [\varphi(z)]^2$ and by (17)

$$g(z)/\overline{g(h(z))} = f(z)/f_0(z). \tag{19}$$

For $z \in D_2$ we have by (7) $h(z) \in D_1$ and according to (18)

$$g(z) = 1/\overline{\varphi(h(z))}, \quad g(h(z)) = \varphi(h(z)), \quad \overline{g(h(z))} = \overline{\varphi(h(z))},$$

whence we obtain by virtue of (17), and (2), since both f and f_0 satisfy equation (4),

$$g(z)/\overline{g(h(z))} = 1/\left[\overline{\varphi(h(z))}\right]^2 = 1/\overline{[\varphi(h(z))]^2} = \overline{f_0(h(z))}/\overline{f(h(z))} =$$
$$= \overline{h(f_0(z))}/\overline{h(f(z))} = f(z)/f_0(z),$$

i.e. again we get (19). Consequently relation (19), and thus also relation (16), holds for all $z \in \mathbb{C}^*$ and we have proved the following

Proposition 3. *With notation* (2), *if* $f_0 : \mathbb{C}^* \to \mathbb{C}^*$ *is a particular solution of equation* (1), *then for every function* $g : \mathbb{C}^* \to \mathbb{C}^*$ *the function* f *defined by* (16) *satisfies the functional equation* (1), *and all the solutions* $f : \mathbb{C}^* \to \mathbb{C}^*$ *of* (1) *may be obtained in this manner.*

Taking as f_0 the simplest possible particular solution $f_0(z) = z$ of (1), we obtain from (16) the formula

$$f(z) = zg(z)/\overline{g(h(z))}, \quad z \in \mathbb{C}^*. \tag{20}$$

Formula (20) yields the general solution of equation (1) on $\mathbb{C}^*$ and, as was to be expected, it contains an arbitrary function in the role of a parameter.

Remark 3. In each method of solving equation (1) we have used sets (5), but in each instance they played a different role. In the first method sets (5) appeared directly in the formula for the solution, in the second method they were used to construct a particular solution σ of equation (10) and so they appear in formula (15) only indirectly. (The same function σ could also be defined in another way, without appealing to sets (5)). In the third method sets (5) were used in the proof, but not in the formulation of Prop. 3.

4. Now we are going to discuss a number of **examples.**

1. Let $f_0 : \mathbb{C}^* \to \mathbb{C}^*$ be an arbitrary solution of equation (1) on $\mathbb{C}^*$. Taking in (16) $g(z) = c = \text{const} \neq 0$ we obtain

$$f(z) = \eta f_0(z), \quad z \in \mathbb{C}^*, \tag{21}$$

where $\eta = c/\bar{c}$ fulfils the condition

$$|\eta| = 1. \tag{22}$$

Thus, together with f_0 also every function f of form (21), where η fulfils (22), is a solution of (1).

2. In (20) take $g(z) = cz^n$, where $c \neq 0$ is a constant and n is an integer. We obtain

$$f(z) = \eta z^{2n+1}, \quad z \in \mathbb{C}^*, \tag{23}$$

where $\eta = (-1)^n c/\bar{c}$ is a constant fulfilling (22). Functions (23) (with arbitrary $n \in \mathbb{Z}$ and η fulfilling (22)) yield a family of analytic solutions of (1) on $\mathbb{C}^*$ which have a removable singularity or a pole at zero, depending on whether $n \geq 0$ or $n < 0$. (For the converse cf. Section 4).

3. In (20) we take $g(z) = ce^z$, $c \neq 0$. Since $\overline{\exp z} = \exp \bar{z}$, we have $\overline{g(h(z))} = \bar{c}e^{-1/z}$ and

$$f(z) = \eta z e^{z+\frac{1}{z}}, \quad z \in \mathbb{C}^*. \tag{24}$$

Functions (24) (with arbitrary η fulfilling (22)) yield a family of analytic solutions of equation (1) on $\mathbb{C}^*$ which have as essential singularity at zero.

4. Let

$$g(z) = cz^{m_0}(z-u_1)^{m_1} \dots (z-u_p)^{m_p}$$

be a polynomial of degree

$$n = m_0 + m_1 + \dots + m_p, \tag{25}$$

with distinct roots $u_0 = 0, u_1, \dots, u_p$ of multiplicity $m_0 \geq 0$, $m_1 > 0$,

$\ldots, m_p > 0$ respectively, $p > 0$. Write

$$v_1 = h(u_1), \ldots, v_p = h(u_p). \tag{26}$$

Then

$$\overline{g(h(z))} = (-1)^n \,\overline{c}\, \overline{u}_1^{\,m_1} \ldots \overline{u}_p^{\,m_p} z^{-n} (z - v_1)^{m_1} \ldots (z - v_p)^{m_p}$$

and, according to (20),

$$f(z) = \frac{\eta a z^{m_0+n+1}(z-u_1)^{m_1} \ldots (z-u_p)^{m_p}}{(z-v_1)^{m_1} \ldots (z-v_p)^{m_p}} \tag{27}$$

where

$$a = (-1)^{m_1+\cdots+m_p} \left[\overline{u}_1^{\,m_1} \ldots \overline{u}_p^{\,m_p}\right]^{-1} = v_1^{m_1} \ldots v_p^{m_p} \tag{28}$$

and $\eta = (-1)^{n-m_0} c/\,\overline{c}$ is a constant fulfilling (22). For arbitrary distinct $u_1, \ldots, u_p \in \mathbb{C}^*$, arbitrary integers $m_0 \geq 0$, $m_1 > 0, \ldots, m_p > 0$, and arbitrary η fulfilling (22), function (27), where a, n, and $v_1, \ldots, v_p$ are given by (28), (25) and (26), respectively, is a meromorphic solution of equation (1) with poles at $v_1, \ldots, v_p$. (But if some u_i are equal to some v_j with $m_i \geq m_j$ for the corresponding indices i, j, then function (27) has removable singularities at these points v_j).

As a matter of fact functions (27) are not solutions of equation (1) on $\mathbb{C}^*$ in the spirit of the earlier parts of this paper. They do not map $\mathbb{C}^*$ into $\mathbb{C}^*$: they have zeros and poles on $\mathbb{C}^*$. But they satisfy equation (1) on $\mathbb{C}^* \setminus \{u_1, \ldots, u_p, v_1, \ldots, v_p\}$, and even on the whole $\mathbb{C}^*$, in the sense that the product $f(z)\,\overline{f(-1/\overline{z})}$ is equal to -1 everywhere on $\mathbb{C}^*$ except at the points $u_1, \ldots, u_p$, $v_1, \ldots, v_p$, where it has removable singularities.

5. It is easy to check that the functions

$$f(z) = \eta\, \overline{z}^{2n+1}, \quad z \in \mathbb{C}^*, \tag{29}$$

where η fulfils (22) and n is an integer, satisfy equation (1) on $\mathbb{C}^*$. Functions (29) are continuous, but nowhere differentiable on $\mathbb{C}^*$. Similarly the functions

$$f(z) = \eta z^{2n+1}\, \overline{z}^{2m}, \quad z \in \mathbb{C}^*,$$

and

$$f(z) = \eta z^{2n}\, \overline{z}^{2m+1}, \quad z \in \mathbb{C}^*,$$

(obtained from (16) on taking $f_0(z) = (-1)^m \eta z^{2n+1}$, $g(z) = \overline{z}^m$ and $f_0(z) = (-1)^n \eta\, \overline{z}^{2m+1}$, $g(z) = z^n$), where m, n, are integers and η is a constant fulfilling (22), yield families of continuous nondifferentiable solutions of equation (1).

6. Let D_0 denote the set

$$D_0 = \{z \in \mathbb{C}^* \,|\, \operatorname{Re} z\,, \operatorname{Im} z \in \mathbb{Q}\}\,.$$

Both sets D_0 and $\mathbb{C}^* \setminus D_0$ are dense in $\mathbb{C}^*$ and $h(z) \in D_0$ for $z \in D_0$, while for $z \in \mathbb{C}^* \setminus D_0$ also $h(z) \in \mathbb{C}^* \setminus D_0$. Therefore the function $f : \mathbb{C}^* \to \mathbb{C}^*$ defined by

$$f(z) = \begin{cases} f_1(z)\,, & z \in D_0\,, \\ f_2(z)\,, & z \in \mathbb{C}^* \setminus D_0\,, \end{cases} \tag{30}$$

satisfies equation (4) (and thus also equation (1)) on $\mathbb{C}^*$ whenever the functions $f_1 : \mathbb{C}^* \to \mathbb{C}^*$ and $f_2 : \mathbb{C}^* \to \mathbb{C}^*$ do. Taking in particular

$$f_1(z) = z\,, \quad f_2(z) = -z\,, \quad z \in \mathbb{C}^*\,,$$

we obtain from (30)

$$f(z) = \begin{cases} z\,, & z \in D_0\,, \\ -z\,, & z \in \mathbb{C}^* \setminus D_0\,. \end{cases} \tag{31}$$

Function (31) is a measurable, discontinuous (at every point of $\mathbb{C}^*$) solution of equation (1) on $\mathbb{C}^*$.

Such examples could be multiplied. The functions given in examples 3–6 are only a few representatives of solutions of equation (1) in given regularity cases. It is not difficult to show that in each of these classes the general solution of (1) depends on an arbitrary function. The same is true also about nonmeasurable solutions $f : \mathbb{C}^* \to \mathbb{C}^*$ of equation (1). In order to obtain such solutions it is enough to take a nonmeasurable $F : D_1 \to \mathbb{C}^*$ in formula (8).

Therefore the simple uniqueness theorem which we are going to prove in the next section, in spite of the fact that the conditions imposed on f are quite strong, nevertheless seems to be of a considerable interest.

5. As pointed out at the beginning of this paper, the function h given by (2) has no fixed points of order 1. Therefore the conditions ensuring the uniqueness of solutions of equation (1) must have a global character and essential use must be made of the involved structure of the complex plane.

Theorem. *Let $f : \mathbb{C}^* \to \mathbb{C}^*$ be a solution of equation* (1) *on $\mathbb{C}^*$ and suppose that f is analytic on $\mathbb{C}^*$ and has a removable singularity or a pole at the origin. Then f has form* (23)*, where n is an integer and η is a constant fulfilling condition* (22).

Proof. Suppose that an $f : \mathbb{C}^* \to \mathbb{C}^*$ fulfils the conditions of the theorem. Thus there exists an entire function $\varphi : \mathbb{C}^* \to \mathbb{C}^*$ and an integer p (positive, negative, or zero) such that

$$f(z) = z^p \varphi(z), \quad z \in \mathbb{C}^*, \tag{32}$$

and

$$\varphi(0) \neq 0. \tag{33}$$

(f, and hence φ, cannot be zero on $\mathbb{C}^*$ because of (1)). The function $|\varphi|$ is continuous at zero, therefore, in view of (33), there exist positive constants a, ε and r such that

$$0 < a - \varepsilon < |\varphi(z)| < a + \varepsilon, \quad |z| < r. \tag{34}$$

Now we insert (32) into (1) to obtain

$$(-1)^p \varphi(z)\, \overline{\varphi(-1/\overline{z})} = -1, \quad z \in \mathbb{C}^*, \tag{35}$$

that is,

$$\varphi(z) = (-1)^{p-1} / \overline{\varphi(-1/\overline{z})}, \quad z \in \mathbb{C}^*. \tag{36}$$

For $|z| > 1/r$ we have $|-1/\overline{z}| < r$ so that, by virtue of (34),

$$a - \varepsilon < |\varphi(-1/\overline{z})| < a + \varepsilon, \quad |z| > 1/r,$$

and in particular

$$1/\left|\,\overline{\varphi(-1/\overline{z})}\,\right| = 1/\left|\,\varphi(-1/\overline{z})\,\right| < 1/(a-\varepsilon), \quad |z| > 1/r. \tag{37}$$

Relations (37) and (36) imply that the entire function φ is bounded on $\mathbb{C}$ and thus it must be constant:

$$\varphi(z) = \eta = \text{const}\ , \quad z \in \mathbb{C} \tag{38}$$

Inserting (38) into (35) we obtain $|\eta|^2 = (-1)^{p-1}$, which implies (22) and, moreover, shows that $p - 1$ must be an even number:

$$p - 1 = 2n. \tag{39}$$

Formula (23) results now from (32), (38) and (39). $\Diamond$

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S–METRISCHE ZUSAMMENHÄNGE IN ISOTROPEN MANNIGFALTIGKEITEN

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Abstract: In this paper we study some properties of symmetric and quasisymmetric S–metric connections in an isotropic manifold (M, g). Let $\mathcal{X}_0M$ be the set of the isotropic vector fields on M. A linear connection ∇ is called a S–metric connection if $\nabla_{Z_0}g = 0$, $Z_0 \in \mathcal{X}_0M$. After some preliminaries we investigate the problem of existence and uniqueness of S–metric connections in M, and consider connections which have constant curvature K. For $\dim M \geq 3$ it is shown that M admits *a* symmetric S–metric connection with constant curvature $K \neq 0$ iff the metric tensor g is absolutely reducible. The quasisymmetric S–metric connection with curvature $K = 0$ is also an isotropic connection if g is absolutely reducible and semidefinite. Finally, we determine the components of the symmetric S–metric connections with constant curvature in special coordinate systems.

1. Einleitung

Es seien M eine n–dimensionale differenzierbare Mannigfaltigkeit und g ein zweifach kovariantes symmetrisches Tensorfeld auf M, beide von der Klasse C^∞. g heißt eine *r–fach singuläre Riemannsche Metrik* auf M oder eine *Riemannsche Metrik vom Defekt r*, wenn rang $g = n - r$ mit $1 \leq r \leq n$, $r = \text{const}$, auf M. Wir nennen $M^{n(r)} \doteq (M, g)$ eine *r–fach isotrope Mannigfaltigkeit* oder eine *isotrope Mannigfaltigkeit vom Defekt r*. Im folgenden schließen wir isotrope Mannigfaltigkeiten vom Defekt n aus. Sie sind als Untermannigfaltigkeiten einer Mannigfaltigkeit mit regulärer Metrik von Bedeutung.

Es bezeichne $S_p \subset T_pM$ den Nullraum der Bilinearform $g_p = g(p)$ im Tangentialraum T_pM. $S : p \mapsto S_p$ ist eine r–dimensionale Distribution auf M. Es seien weiter $\mathcal{X}M$ die Menge der differenzierbaren Vektorfelder auf M und $\mathcal{X}_0M = \{X \in \mathcal{X}M \mid X(p) \in S_p, p \in M\}$ die Menge der differenzierbaren isotropen Vektorfelder, ferner $\mathcal{F}M$ die Menge der differenzierbaren Funktionen auf M.

Für die Komponenten g_{ij} von g in einem lokalen Koordinatensystem gilt rang $(g_{ij}) = n - r$. g heißt *reduzibel singulär*, wenn jeder Punkt $p \in M$ eine Karte besitzt, so daß[1]

$$(1) \qquad (g_{ij}(x^k)) = \begin{pmatrix} g_{ab}(x^k) & 0 \\ 0 & 0 \end{pmatrix}; \quad \det(g_{ab}) \neq 0.$$

g ist genau dann reduzibel singulär, wenn[2]

$$(2) \qquad [X_0, Y_0] \in \mathcal{X}_0M; \quad X_0, Y_0 \in \mathcal{X}_0M.$$

S ist dann eine involutive Distribution. g heißt *absolut reduzibel singulär*, wenn jeder Punkt $p \in M$ eine Karte besitzt, so daß (1) und $g_{ab} = g_{ab}(x^c)$ gilt. g ist genau dann absolut reduzibel, wenn die Lie–Ableitung von g bezüglich jedes isotropen Vektorfeldes Z_0 verschwindet[3]:

$$(3) \qquad \begin{aligned} &(L_{Z_0}g)(X, Y) = Z_0(g(X, Y)) - g([Z_0, X], Y) - g(X, [Z_0, Y]) = 0; \\ &\qquad\qquad X, Y \in \mathcal{X}M, \quad Z_0 \in \mathcal{X}_0M. \end{aligned}$$

[1] Für die verwendeten Indizes soll im folgenden gelten: $i, j, k, l, m \in \{1, \ldots, n\}$; $a, b, c, d \in \{1, \ldots, n-r\}$; $A, B, C, D \in \{n-r+1, \ldots, n\}$.

[2] Bortolotti [1], p.543.

[3] Bortolotti [1], p.545; Dautcourt [3], p.320.

2. Metrische Zusammenhänge

Ein *metrischer* oder ein *Riemannscher Zusammenhang* ist ein linearer Zusammenhang ∇, für den die kovariante Ableitung des Metriktensors g verschwindet: $\nabla_Z g = 0$, $Z \in \mathcal{X}M$. Metrische Zusammenhänge in isotropen Mannigfaltigkeiten sind schon mehrfach untersucht worden, z. B. Bortolotti [2], Jankiewicz [4], Vogel [6], Oproiu [5]. ∇ ist ein *symmetrischer* Zusammenhang, wenn seine Torsion verschwindet:

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y] = 0; \quad X, Y \in \mathcal{X}M. \tag{4}$$

Die Existenz eines symmetrischen metrischen Zusammenhangs, d. h. eines Zusammenhangs von Levi–Civita, in $M^{n(r)}$ schränkt die Metrik g stark ein. Ein solcher Zusammenhang existiert genau dann, wenn g absolut reduzibel ist[4]. Er ist, anders als im regulären Fall rang $g = n$, durch g allein nicht eindeutig bestimmt.

In Abschwächung von (4) nennen wir einen Zusammenhang *quasisymmetrisch*, wenn

$$T(X,Y) \in \mathcal{X}_0 M; \quad X, Y \in \mathcal{X}M. \tag{5}$$

Quasisymmetrische lineare Zusammenhänge in $M^{n(r)}$ sind von Vogel [7] untersucht worden. Von den Ergebnissen sei erwähnt, daß für die Existenz eines quasisymmetrischen metrischen Zusammenhangs wieder die absolute Reduzibilität von g notwendig und hinreichend ist. Der Zusammenhang ist *quasieindeutig*, d. h. eindeutig bis auf ein isotropes Vektorfeld. Sind ∇, $\overset{0}{\nabla}$ zwei solche Zusammenhänge in $M^{n(r)}$, so gilt

$$\nabla_X Y = \overset{0}{\nabla}_X Y + I(X,Y); \quad I(X,Y) \in \mathcal{X}_0 M.$$

3. S–metrische Zusammenhänge

Ein linearer Zusammenhang ∇ heißt *S–metrisch*, wenn die kovariante Ableitung des Metriktensors g bezüglich jedes isotropen Vektorfeldes Z_0 verschwindet: $\nabla_{Z_0} g = 0$, ausführlich

$$\begin{aligned}(\nabla_{Z_0} g)(X,Y) = Z_0\,(g(X,Y)) - g(\nabla_{Z_0} X, Y) - g(X, \nabla_{Z_0} Y) = 0;\\ X, Y \in \mathcal{X}M, \quad Z_0 \in \mathcal{X}_0 M.\end{aligned} \tag{6}$$

[4] Vogel [6], p.107.

Während identisch $(\nabla_Z g)(X_0, Y_0) = 0$, sind noch die Zusammenhänge mit

$$(7) \quad (\nabla_Z g)(X_0, Y) = -g(\nabla_Z X_0, Y) = 0; \quad Y, Z \in \mathcal{X}M, \ X_0 \in \mathcal{X}_0 M$$

von Interesse, die sogenannten *isotropen* Zusammenhänge[5]. Wegen

$$\nabla_Z X_0 \in \mathcal{X}_0 M; \quad Z \in \mathcal{X}M, \ X_0 \in \mathcal{X}_0 M$$

haben diese Zusammenhänge die Eigenschaft, daß die kovariante Ableitung jedes isotropen Vektorfeldes wieder ein isotropes Vektorfeld ist. Wie schon in [7] gezeigt, existiert ein quasisymmetrischer S–metrischer Zusammenhang *(QS–Zusammenhang)* in $M^{n(r)}$ genau dann, wenn g reduzibel singulär ist. Das gleiche gilt für quasisymmetrische isotrope Zusammenhänge. S–metrische und isotrope Zusammenhänge verdienen ein gewisses Interesse, da jede isotrope Mannigfaltigkeit $M^{n(1)}$ vom Defekt 1 reduzibel singulär ist. Für $n = 3$ sind diese Mannigfaltigkeiten als Untermannigfaltigkeiten der Kodimension 1 in einer 4–dimensionalen Raumzeit von Bedeutung.

Zur lokalen Darstellung verwenden wir ein Koordinatensystem, in dem (1) für die Komponenten des Metriktensors g gilt, ein sog. *σ–Koordinatensystem.* Bezeichnet man die Komponenten des linearen Zusammenhangs mit Λ^i_{jk}, so gilt nach (6) für einen S–metrischen Zusammenhang

$$(8) \qquad 2\Gamma_{bc|C} + \Lambda^a_{Cb} g_{ac} + \Lambda^a_{Cc} g_{ba} = 0, \quad \Lambda^a_{BC} = 0,$$

nach (7) für einen isotropen Zusammenhang

$$(9) \qquad \Lambda^a_{jC} = 0$$

und nach (5) für einen quasisymmetrischen Zusammenhang

$$(10) \qquad \Lambda^a_{jk} - \Lambda^a_{kj} = 0.$$

Dabei sind $\Gamma_{ij|k} = \frac{1}{2}\left(\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij}\right)$ die *Christoffelsymbole 1. Art* von g_{ij}.

Um zu einer Aussage über die Eindeutigkeit der QS–Zusammenhänge zu kommen, formen wir (6) mit Hilfe der Lie-Ableitung (3) um, beachten dabei (5) und erhalten

$$(11) \qquad (L_{Z_0} g)(X, Y) - g(\nabla_X Z_0, Y) - g(X, \nabla_Y Z_0) = 0.$$

[5] [7], p.16.

Alle 3 Terme sind jetzt $\mathcal{F}M$–linear in Z_0, X, Y, im Gegensatz zu den 3 Termen in (6).

Definition. ∇ heißt ein *spezieller QS–Zusammenhang*, wenn

$$g(\nabla_X Z_0, Y) = \tfrac{1}{2}(L_{Z_0} g)(X,Y), \quad T(X,Y) \in \mathcal{X}_0 M; \qquad X, Y \in \mathcal{X}M, \quad Z_0 \in \mathcal{X}_0 M. \tag{12}$$

In einem σ–Koordinatensystem gilt für die Komponenten eines speziellen QS–Zusammenhangs nach (8), (10), (12)

$$\Lambda^a_{bc} = \Lambda^a_{cb}, \quad \Lambda^a_{bC} = \Lambda^a_{Cb} = -\Gamma_{bc|C} g^{ca}, \quad \Lambda^a_{BC} = \Lambda^a_{CB} = 0. \tag{13}$$

Dabei sind g^{ca} die Elemente der inversen Matrix zu (g_{ab}). Λ^a_{bc} und die nicht genannten Komponenten Λ^A_{jk} können beliebig gewählt werden.

Aus (12) ergibt sich, daß $\nabla_X Z_0$, die kovariante Ableitung eines isotropen Vektorfeldes, bis auf ein isotropes Vektorfeld eindeutig bestimmt ist. Um den Zusammenhang ∇ genauer zu fixieren, hat man noch gewisse Freiheiten. Wir betrachten z. B. eine zur involutiven Distribution S komplementäre Distribution $H : p \mapsto H_p$, die nicht involutiv sein muß. Eine solche Distribution H existiert immer und kann z. B. definiert werden als das orthogonale Komplement zu S bezüglich einer beliebigen auf M erklärten positiv definiten (regulären) Riemannschen Metrik. Es sei $\mathcal{X}_1 M$ die Menge der *horizontalen* Vektorfelder, d. h. $\mathcal{X}_1 M = \{X \in \mathcal{X}M | X(p) \in H_p,\ p \in M\}$. Für den speziellen QS–Zusammenhang (12) fordern wir noch

$$\nabla_{X_1} Y_1 \in \mathcal{X}_1 M, \tag{14}$$

$$(\nabla_{Z_1} g)(X_1, Y_1) = Z_1(g(X_1, Y_1)) - g(\nabla_{Z_1} X_1, Y_1) - g(X_1, \nabla_{Z_1} Y_1) = 0; \qquad X_1, Y_1, Z_1 \in \mathcal{X}_1 M, \tag{15}$$

d. h. die Einschränkung von ∇ auf H sei ein quasisymmetrischer metrischer Zusammenhang. Aus (15) und der Quasisymmetrie ergibt sich nach bekannten Regeln die Formel von Koszul

$$\begin{aligned} g(\nabla_{X_1} Y_1, Z_1) = \frac{1}{2}\Big(& X_1(g(Y_1, Z_1)) + Y_1(g(X_1, Z_1)) - Z_1(g(X_1, Y_1)) \\ & + g(Z_1, [X_1, Y_1]) - g(X_1, [Y_1, Z_1]) - g(Y_1, [X_1, Z_1])\Big). \end{aligned} \tag{16}$$

Umgekehrt folgt aus (14) und (16), daß $\nabla_{X_1} Y_1$ eindeutig bestimmt ist und die Eigenschaften eines auf H quasisymmetrischen metrischen Zusammenhangs besitzt.

Ein Zusammenhang, der (12), (14), (15) genügt, heiße ein *spezieller QSH-Zusammenhang.* Aus (5), (14) und

$$T(X_1, Y_1) = \nabla_{X_1} Y_1 - \nabla_{Y_1} X_1 - [X_1, Y_1]$$

folgt $[X_1, Y_1] \in \mathcal{X}_1 M$, wenn $T(X_1, Y_1) = 0$, und umgekehrt. Das heißt, ein spezieller QSH-Zusammenhang ist auf H genau dann symmetrisch, wenn H eine involutive Distribution ist.

4. Existenz und Eindeutigkeit

Zum Nachweis der Existenz konstruieren wir zunächst in einer Umgebung jedes Punktes von M einen Zusammenhang, der (12), (14), (15) erfüllt. Dazu verwenden wir für den betrachteten Punkt p_0 ein σ–Koordinatensystem (U_0, φ_0) und wählen für die Komponenten Λ^i_{jk} außer (13) noch

$$\Lambda^A_{bC} = \Lambda^A_{Cb} = 0, \quad \Lambda^A_{BC} = \Lambda^A_{CB} = 0. \tag{17}$$

Sind $X_1 = \xi^i_1 \partial_i$, $Y_1 = \eta^i_1 \partial_i$, $Z_1 = \zeta^i_1 \partial_i$ die Darstellungen der Vektorfelder X_1, Y_1, Z_1 in U_0, so lautet (14)

$$(\xi^j_1\, \partial_j\, \eta^i_1 + \Lambda^i_{jk}\, \xi^j_1\, \eta^k_1)\, \partial_i \in \mathcal{X}_1 U_0 \tag{18}$$

und (16)

$$\Lambda^a_{jk} g_{ac}\, \xi^j_1\, \eta^k_1\, \zeta^c_1 = \Gamma_{jk|l}\, \xi^j_1\, \eta^k_1 \zeta^l_1. \tag{19}$$

Wir zeigen, daß sämtliche Komponenten Λ^i_{jk} in U_0 durch (13), (17), (18), (19) eindeutig bestimmt sind. Die Unterräume S_p, $p \in U_0$, werden bei Verwendung eines σ–Koordinatensystems von den Basisvektoren $\partial_{n-r+1}, \ldots, \partial_n$ aufgespannt, und die Unterräume H_p, $p \in U_0$, mögen von den Vektoren $e_1, \ldots, e_{n-r}$ aufgespannt werden. Dabei lassen sich die e_a wegen der Komplementarität von S_p, H_p durch die Basisvektoren ∂_i von $T_p M$ in der Form darstellen

$$e_a = \partial_a + \lambda^A_a \partial_A.$$

Für die Komponenten ξ^A_1 eines Vektors $X_1 = \xi^i_1 \partial_i \in \mathcal{X}_1 U_0$ ist dann $\xi^A_1 = \lambda^A_a \xi^a_1$ zu setzen. Aus (18), (19) ergeben sich, unter Verwendung von (13), (17), die noch fehlenden Komponenten $\Lambda^a_{bc}, \Lambda^A_{bc}$ zu

$$\begin{aligned} \Lambda^a_{bc} = \Lambda^a_{cb} &= \Gamma_{bc|d} g^{da} + \Gamma_{bc|D} \lambda^D_d g^{da}, \\ \Lambda^A_{bc} &= (\Gamma_{bc|d} + \Gamma_{bc|D}\lambda^D_d - \Gamma_{bd|D}\lambda^D_c - \Gamma_{cd|D}\lambda^D_b)\lambda^A_a g^{da} \\ &\quad -\partial_b \lambda^A_c - \lambda^B_b \partial_B \lambda^A_c. \end{aligned} \tag{20}$$

Ist H nicht involutiv, so ist nach einer früheren Bemerkung $\Lambda^A_{bc} \neq \Lambda^A_{cb}$.

Wie gezeigt, gibt es eine offene Überdeckung $(U_\alpha)_{\alpha \in A}$ von $M^{n(r)}$ derart, daß auf jeder Umgebung U_α ein spezieller QSH–Zusammenhang $\overset{\alpha}{\nabla}{}'$ erklärt ist. Einen Zusammenhang in $M^{n(r)}$ erhält man mittels Zerlegung der Eins $(\varphi_\alpha)_{\alpha \in A}$, Tr $\varphi_\alpha \subset U_\alpha$. Man definiert zunächst für jedes α die Abbildung $\overset{\alpha}{\nabla}$: $\mathcal{X}M \times \mathcal{X}M \to \mathcal{X}M$ durch

$$\overset{\alpha}{\nabla}_X Y_{|p} = \varphi_\alpha(p)\, \overset{\alpha}{\nabla}{}'_X Y_{|p} \quad \text{für } p \in U_\alpha, \qquad \overset{\alpha}{\nabla}_X Y_{|p} = 0 \quad \text{für } p \notin U_\alpha.$$

Man sieht leicht, daß $\nabla : \mathcal{X}M \times \mathcal{X}M \to \mathcal{X}M$ mit $\nabla_X Y_{|p} = \sum_\alpha \overset{\alpha}{\nabla}_X Y_{|p}$ ein linearer Zusammenhang ist, der (12), (14), (15) erfüllt, d. h. ∇ ist ein spezieller QSH–Zusammenhang.

Zur Frage nach der Eindeutigkeit betrachten wir zwei Zusammenhänge $\nabla, \overset{\circ}{\nabla}$, die (12), (14), (15) genügen. Setzt man

$$\nabla_X Y = \overset{\circ}{\nabla}_X Y + I(X, Y),$$

so ist $I(X, Y)$ ein in beiden Argumenten $\mathcal{F}M$–lineares isotropes Vektorfeld

$$I(X, Y) \in \mathcal{X}_0 M; \quad X, Y \in \mathcal{X}M, \tag{21}$$

für das noch

$$I(X_1, Y_1) = 0; \quad X_1, Y_1 \in \mathcal{X}_1 M \tag{22}$$

gilt. Ist umgekehrt $\overset{\circ}{\nabla}$ ein spezieller QSH–Zusammenhang und $I(X, Y)$ ein $\mathcal{F}M$–lineares Vektorfeld mit (21), (22), so ist auch ∇ ein spezieller QSH–Zusammenhang.

Dieser Zusammenhang kann mit Hilfe von r linear unabhängigen isotropen Vektorfeldern $Z_{(n-r+1)}, \ldots, Z_{(n)}$, die gegeben sind, sogar eindeutig gemacht werden. Wie man leicht zeigt, gibt es genau einen speziellen QSH–Zusammenhang ∇, für den $T(X_0, Y_1) = 0$, $X_0 \in \mathcal{X}_0 M$, $Y_1 \in \mathcal{X}_1 M$ und

$$\nabla_X Z_{(A)} \in \mathcal{X}_1 M; \quad X \in \mathcal{X}M.$$

∇ ist genau dann ein symmetrischer Zusammenhang, wenn H eine involutive Distribution ist und

$$[Z_{(A)}, Z_{(B)}] \in \mathcal{X}_1 M.$$

5. S–metrische Zusammenhänge mit konstanter Krümmung $K \neq 0$

Für einen symmetrischen S–metrischen Zusammenhang (*SS–Zusammenhang*) beweisen wir

Satz 1. *Sei $n - r > 1$ oder $r > 1$. Ist ∇ ein SS–Zusammenhang mit konstanter Krümmung $K \neq 0$, so ist ∇ auch ein isotroper Zusammenhang und g absolut reduzibel.*

Beweis. Es sei $R(X,Y,Z) = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$ die Krümmung des Zusammenhangs ∇ und $R(X,Y,Z,\omega) = \omega(R(X,Y,Z))$ der dreifach kovariante, einfach kontravariante Krümmungstensor. Ist ∇ symmetrisch, so besteht die Bianchi-Identität

$$(23)\quad \begin{gathered}(\nabla_U R)(X,Y,Z,\omega) + (\nabla_X R)(Y,U,Z,\omega) + (\nabla_Y R)(U,X,Z,\omega) = 0;\\ U,X,Y,Z \in \mathcal{X}M, \quad \omega \in \mathcal{X}^*M,\end{gathered}$$

$\mathcal{X}^*M$ die Menge der differenzierbaren Kovektorfelder auf M. Wir setzen jetzt voraus, daß ∇ von konstanter Krümmung K sei:

$$(24)\quad R(X,Y,Z,\omega) = K \cdot \big(\omega(X)g(Y,Z) - \omega(Y)g(X,Z)\big).$$

Nach einer kleinen Rechnung erhält man aus (23), (24)

$$(25)\quad \begin{aligned} K \cdot \Big\{&\omega(X)\big((\nabla_U g)(Y,Z) - (\nabla_Y g)(U,Z)\big)\\ &+ \omega(Y)\big((\nabla_X g)(U,Z) - (\nabla_U g)(X,Z)\big)\\ &+ \omega(U)\big((\nabla_Y g)(X,Z) - (\nabla_X g)(Y,Z)\big)\Big\} = 0.\end{aligned}$$

Mit $K \neq 0$ und $X = X_0 \in \mathcal{X}_0M$, $Y = Y_0 \in \mathcal{X}_0M$ wird

$$\begin{aligned} &\omega(X_0)\left((\nabla_U g)(Y_0,Z) - (\nabla_{Y_0} g)(U,Z)\right)\\ +\, &\omega(Y_0)\left((\nabla_{X_0} g)(U,Z) - (\nabla_U g)(X_0,Z)\right)\\ +\, &\omega(U)\left((\nabla_{Y_0} g)(X_0,Z) - (\nabla_{X_0} g)(Y_0,Z)\right) = 0.\end{aligned}$$

Für S–metrische Zusammenhänge ist $\nabla_{X_0} g = \nabla_{Y_0} g = 0$ und es bleibt

$$\omega(X_0)(\nabla_U g)(Y_0,Z) - \omega(Y_0)(\nabla_U g)(X_0,Z) = 0$$

und nach einer kleinen Umformung

$$-\omega(X_0)g(\nabla_U Y_0, Z) + \omega(Y_0)g(\nabla_U X_0, Z) = 0.$$

Durch Verjüngung über ω, Y erhält man hieraus $(-1+r)g(\nabla_U X_0, Z) = 0$, für $r > 1$ also

$$g(\nabla_U X_0, Z) = 0. \tag{26}$$

Setzt man in (25) $K \neq 0$, $U = U_0 \in \mathcal{X}_0 M$, $\nabla_{U_0} g = 0$, so wird

$$\begin{aligned} -\omega(X)(\nabla_Y g)(U_0, Z) + \omega(Y)(\nabla_X g)(U_0, Z) & \\ + \omega(U_0)\left((\nabla_Y g)(X, Z) - (\nabla_X g)(Y, Z)\right) &= 0, \end{aligned}$$

und nach einer kleinen Umformung

$$\begin{aligned} \omega(X) g(\nabla_Y U_0, Z) - \omega(Y) g(\nabla_X U_0, Z) & \\ + \omega(U_0)\left((\nabla_Y g)(X, Z) - (\nabla_X g)(Y, Z)\right) &= 0. \end{aligned} \tag{27}$$

Nun sei $\omega = \omega_0 \in \mathcal{X}_0^* M$ ein isotropes Kovektorfeld, d. h. $\omega_0(U_0) = 0$. Dann geht (27) über in

$$\omega_0(X) g(\nabla_Y U_0, Z) - \omega_0(Y) g(\nabla_X U_0, Z) = 0.$$

Hieraus erhält man $(1 - (n - r))\, g(\nabla_X U_0, Z) = 0$, wenn man über ω, Y verjüngt, und für $n - r > 1$ wieder (26). ∇ ist nach (7) ein isotroper Zusammenhang. Besitzt $M^{n(r)}$ einen quasisymmetrischen S–metrischen isotropen Zusammenhang, so ist nach (11) $L_{Z_0} g = 0$ und nach (3) g absolut reduzibel singulär. $\diamondsuit$

In einem Koordinatensystem lautet (24)

$$R^i_{jkl} = \partial_j \Lambda^i_{kl} - \partial_k \Lambda^i_{jl} + \Lambda^i_{jm} \Lambda^m_{kl} - \Lambda^i_{km} \Lambda^m_{jl} = K\left(\delta^i_j g_{kl} - \delta^i_k g_{jl}\right), \tag{28}$$

wo R^i_{jkl} die Komponenten des Krümmungstensors R sind. Ist $n - r > 1$ oder $r > 1$, so ist g nach Satz 1 absolut reduzibel. Für die Komponenten eines SS–Zusammenhangs mit konstanter Krümmung $K \neq 0$ zeigt man dann in einem geeigneten σ –Koordinatensystem

$$\Lambda^i_{bc} = K g_{bc} x^i \,, \quad \Lambda^i_{jC} = \Lambda^i_{Cj} = 0, \tag{29}$$

wobei die Komponenten des Metriktensors den Bedingungen

$$g_{iB} = 0, \quad \partial_A g_{bc} = 0, \quad \partial_a g_{bc} - \partial_b g_{ac} + \left(g_{ad} g_{bc} - g_{bd} g_{ac}\right) K x^d = 0 \tag{30}$$

genügen. Von den in (30) stehenden Differentialgleichungen lassen sich leicht sämtliche *isothermen* Lösungen $g_{bc} = F^{-1}(x^i)\, \delta_{bc}$ angeben. Ist der

Zusammenhang ∇ sogar metrisch, so kommt noch $\Lambda^a_{bc} = \Gamma_{bc|d}g^{da}$ hinzu. Setzt man dies in (29.1) ein, so ergibt sich eine zusätzliche Bedingung für g_{bc}, nämlich

$$K g_{bc} g_{ad} x^a = \Gamma_{bc|d}. \tag{31}$$

Mit (31) sind die Differentialgleichungen in (30) von selbst erfüllt.

Für $n - r = 1$, $r = 1$, d. h. $n = 2$, $M^{2(1)}$, muß g nicht absolut reduzibel sein. Es lassen sich leicht sämtliche Komponenten Λ^i_{jk}, g_{ij} angeben. Ist g absolut reduzibel, so ist ∇ auch ein isotroper Zusammenhang, und umgekehrt. In diesem Fall gelten (29), (30.1), (30.2) für $r = 1$, $n = 2$. Die Differentialgleichung (30.3) ist identisch erfüllt.

6. S–metrische Zusammenhänge mit verschwindender Krümmung K

Ist ∇ ein spezieller QS–Zusammenhang und g absolut reduzibel, so ist nach (3), (12) ∇ auch ein isotroper Zusammenhang. Wir beweisen

Satz 2. *Sei ∇ ein QS-Zusammenhang mit verschwindender Krümmung K, und sei g absolut reduzibel und semidefinit. Dann ist ∇ auch ein isotroper Zusammenhang.*

Beweis. Für ∇ gelten nach (6) $\nabla_{Z_0} g = 0$, nach (5) $T(X,Y) \in \mathcal{X}_0 M$ und für g nach (3) $L_{Z_0} g = 0$. Nach einigen Umformungen erhält man hieraus

$$g(\nabla_X Z_0, Y) + g(X, \nabla_Y Z_0) = 0; \quad X, Y \in \mathcal{X}M,\ Z_0 \in \mathcal{X}_0 M \tag{32}$$

und speziell für $X = Y$

$$g(\nabla_X Z_0, X) = 0. \tag{33}$$

Durch Ableiten nach Z_0 entsteht

$$(\nabla_{Z_0} g)(\nabla_X Z_0, X) + g(\nabla_{Z_0} \nabla_X Z_0, X) + g(\nabla_X Z_0, \nabla_{Z_0} X) = 0,$$

und mit $\nabla_{Z_0} g = 0$ wird

$$g\,(\nabla_{Z_0} \nabla_X Z_0, X) + g\,(\nabla_X Z_0, \nabla_{Z_0} X) = 0. \tag{34}$$

Da die Krümmung verschwindet, ist

$$\nabla_{Z_0} \nabla_X Z_0 - \nabla_X \nabla_{Z_0} Z_0 - \nabla_{[Z_0, X]} Z_0 = 0.$$

Damit geht (34) über in

(35) $g\left(\nabla_X \nabla_{Z_0} Z_0, X\right) + g\left(\nabla_{[Z_0,X]} Z_0, X\right) + g\left(\nabla_X Z_0, \nabla_{Z_0} X\right) = 0.$

Setzt man in (32) $[Z_0, X]$ an Stelle von X, so erhält man

$$g\left(\nabla_{[Z_0,X]} Z_0, Y\right) + g\left([Z_0, X], \nabla_Y Z_0\right) = 0$$

und speziell für $X = Y$

(36) $$g\left(\nabla_{[Z_0,X]} Z_0, X\right) + g\left([Z_0, X], \nabla_X Z_0\right) = 0.$$

Nach (35), (36) wird

$$g(\nabla_X \nabla_{Z_0} Z_0, X) + g\left(\nabla_X Z_0, \nabla_{Z_0} X - [Z_0, X]\right) = 0.$$

Es ist $\nabla_{Z_0} X - \nabla_X Z_0 - [Z_0, X] = T(Z_0, X) \in \mathcal{X}_0 M$, so daß

(37) $$g\left(\nabla_X \nabla_{Z_0} Z_0, X\right) + g\left(\nabla_X Z_0, \nabla_X Z_0\right) = 0.$$

Aus (32) folgt für $Y = Z_0$, daß $V_0 := \nabla_{Z_0} Z_0 \in \mathcal{X}_0 M$. Nach (33) ist $g\left(\nabla_X V_0, X\right) = 0$, also

$$g(\nabla_X \nabla_{Z_0} Z_0, X) = 0,$$

und von (37) bleibt noch

$$g(\nabla_X Z_0, \nabla_X Z_0) = 0.$$

Wegen der Semidefinitheit von g ist $\nabla_X Z_0 \in \mathcal{X}_0 M$ und damit ∇ ein isotroper Zusammenhang. ◇

Im folgenden sei $n - r \geq 1$, $r \geq 1$ und ∇ ein SS–Zusammenhang mit verschwindender Krümmung K. In einem Koordinatensystem gilt (28) mit $K = 0$. Leider gibt es im allgemeinen kein σ–Koordinatensystem, in dem sämtliche Komponenten des Zusammenhangs verschwinden. Wie man leicht sieht, gibt es ein σ–Koordinatensystem (U, φ), $0 \in \varphi(U)$, in dem außer $g_{iB} = 0$ noch

(38) $$\Lambda^a_{bc}\big|_{x^A=0} = 0, \quad \Lambda^A_{jk} = 0$$

ist. Durch eine Koordinatentransformation $\overline{\varphi} \circ \varphi^{-1}$ der Gestalt $\overline{x}^a = \psi^a(x^b, x^B)$, $\overline{x}^A = \psi^A(x^B)$, wo ψ^a, ψ^A einem Differentialgleichungssystem

2. Ordnung genügen, läßt sich erreichen, daß im neuen Koordinatensystem $(\overline{U}, \overline{\varphi})$ nun $\overline{\Lambda}^{i}_{jk} = 0$ gilt. Dabei können die Anfangsbedingungen

$$\overline{x}^i(0) = 0, \quad \frac{\partial \overline{x}^i}{\partial x^j}(0) = \delta^i_j \tag{39}$$

gewählt werden. Im allgemeinen gilt nicht $\overline{g}_{iB} = 0$, d. h. $(\overline{U}, \overline{\varphi})$ ist kein σ–Koordinatensystem.

Ausgehend von $\overline{\Lambda}^{i}_{jk} = 0$ berechnen sich die Komponenten Λ^{i}_{jk} im Koordinatensystem (U, φ) bekanntlich nach

$$\Lambda^{i}_{jk} = \frac{\partial^2 \overline{x}^l}{\partial x^j \partial x^k} \frac{\partial x^i}{\partial \overline{x}^l}. \tag{40}$$

Wegen den Bedingungen (8.2), (38) für Λ^{i}_{jk} in (U, φ) und mit (39) ist $\overline{\varphi} \circ \varphi^{-1}$ von der Gestalt

$$\begin{aligned} \overline{x}^a &= x^a + \varphi^a_B(x^b) x^B \\ \overline{x}^A &= x^A. \end{aligned} \tag{41}$$

Die Λ^{i}_{jk} in (40) werden damit zu

$$\Lambda^{a}_{bc} = \frac{\partial^2 \varphi^d_B}{\partial x^b \partial x^c} x^B \frac{\partial x^a}{\partial \overline{x}^d}, \quad \Lambda^{a}_{bC} = \frac{\partial \varphi^d_C}{\partial x^b} \frac{\partial x^a}{\partial \overline{x}^d}, \quad \Lambda^{a}_{BC} = 0, \quad \Lambda^{A}_{jk} = 0. \tag{42}$$

In (U, φ) gelten für g_{ij}, Λ^{i}_{jk} die Beziehungen (1), (8.1), (10), (28) mit $K = 0$, und (38). Daraus folgt nach einer kleinen Rechnung, daß die Komponenten g_{ab} des Metriktensors g von der Form sind

$$g_{ab}\left(x^c, x^C\right) = G_{ABab}(x^c) x^A x^B + G_{Aab}(x^c) x^A + G_{ab}(x^c). \tag{43}$$

Setzt man nun (40), (41), (43) in (8.1) ein, so erhält man

$$\begin{aligned} G_{Aab} &= G_{ad} \frac{\partial \varphi^d_A}{\partial x^b} + G_{bd} \frac{\partial \varphi^d_A}{\partial x^a}, \\ G_{ABab} &= \frac{1}{2} G_{cd} \left(\frac{\partial \varphi^c_A}{\partial x^a} \frac{\partial \varphi^d_B}{\partial x^b} + \frac{\partial \varphi^c_B}{\partial x^a} \frac{\partial \varphi^d_A}{\partial x^b} \right). \end{aligned} \tag{44}$$

Umgekehrt sind Λ^{i}_{jk} nach (41), (42) die Komponenten eines SS–Zusammenhangs mit $K = 0$ in (U, φ), wenn die Komponenten g_{ij} des Metriktensors den Gleichungen (1), (43), (44) genügen.

Wir wollen noch zwei Folgerungen ziehen. Ist ∇ auch ein isotroper Zusammenhang, so ist nach (9) $\Lambda^a_{jC} = 0$, nach (41), (42) $\dfrac{\partial \varphi^d_C}{\partial x^b} = 0$ und $\Lambda^a_{bc} = 0$. Alle Komponenten Λ^i_{jk} verschwinden in (U, φ). Nach (43), (44) ist $g_{ab} = G_{ab}$, d. h. g ist absolut reduzibel.

Aus (44) folgt speziell

$$G_{AAaa} = G_{cd} \frac{\partial \varphi^c_A}{\partial x^a} \frac{\partial \varphi^d_A}{\partial x^a}. \tag{45}$$

Ist g absolut reduzibel und semidefinit, so ist $G_{AAaa} = 0$, $\dfrac{\partial \varphi^c_A}{\partial x^a} = 0$, nach (42) $\Lambda^a_{jC} = 0$ und nach (9) ∇ ein isotroper Zusammenhang. Damit ist Satz 2 für SS-Zusammenhänge nochmals bestätigt.

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H–INTEGRAL NEAR–RINGS

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Abstract: H - integral near - rings are intended to fill the wide gap between the disparate types of integral near - rings on one hand and near - rings with large annihilator ideals (zero - near - rings at the extreme end) on the other hand. If H is a subset of a near - ring N, N is said to be *H - integral* if H has no divisors of zero and $N^2 \subseteq H$. After preliminary results and some motivating examples are presented, we show that such a near - ring N "consists" of an ideal K with $K^2 = 0$ and an integral near - ring; if the latter is finite, N is a semidirect sum of these two parts. This gives rise to a construction method to obtain a large class of H - integral near - rings in an easy way. The last section considers distributively generated H - integral near - rings. In this case and if K has finite index, $N/_K$ is a finite field.

1. Basic facts

In this paper, we consider *left near - rings* $(N,+,.)$, hence $(N,+)$

is a group (not necessarily abelian), $(N,.)$ a semigroup and $n_1(n_2+n_3) = n_1n_2+n_1n_3$ for all $n_1,n_2,n_3 \in N$. See [4] or [5] for the general theory of near-rings. For $n \in N$ and $S \subseteq N$ we use the notations $nS := \{ns|s \in S\}$ and $S^2 := \{s_1s_2|s_1,s_2 \in S\}$ throughout the paper. A subset S of N is called *integral*, if S has no non-zero divisors of zero. N_0, N_c denote the zero-symmetric (constant) parts of N, respectively.

Definition 1.1. If H is an integral subset of a near-ring N with $N^2 \subseteq H$ then N is called *H-integral.*

If N is H-integral with $H = \{0\}$ then N has zero multiplication and may be considered as "known" from the near-ring point of view. If, on the other extreme, $H = N$ then N is an integral near-ring. Again, this case is well-studied (cf. e.g. [5], section 9b2). Hence in the sequel we mainly restrict ourselves to the study of H-integral near-rings with $\{0\} \neq H \neq N$. Note that $0 \in H$ for each H-integral near-ring, as well as $H^2 \subseteq H$; however, H need not be closed under addition, even if $(N,.)$ is commutative (cf. [5], 29 and 30 on p. 411 for such cases with $(N,+) = S_3$, the symmetric group of order 6).

A near-ring N may be H-integral for more than one H. If N^2 is integral, for instance, then N is H-integral for each H between N^2 and N. More precisely we have

Proposition 1.2. *Let $\mathcal{H}$ be the set of all subsets H of N such that N is H-monogenic. Then $\cap\mathcal{H}$ and this is the smallest element of $\mathcal{H}$, while $\cup\mathcal{H}$ is the biggest one.*

Proof. $\mathcal{H}$ is clearly closed w.r.t. intersections, hence $\cap\mathcal{H}$ is the smallest element in $\mathcal{H}$. But $\mathcal{H}$ is also closed under unions: If $h_1 \in H_1 \in \mathcal{H}$, $h_2 \in H_2 \in \mathcal{H}$, $h_1h_2 = 0$ implies $(h_1h_1)h_2 = h_1(h_1h_2) = h_10 = 0$. Since $h_1h_1 \in N^2 \subseteq H_2$, we get either $h_2 = 0$ or $h_1h_1 = 0$ (but then $h_1 = 0$). So $\cup\mathcal{H}$ is the greatest element in $\mathcal{H}$. $\Diamond$

We now give two examples of H-integral near-rings.

Example 1.3. Let N_1 be an arbitrary, N_2 an integral near-ring. Define, in $N := N_1 \times N_2$, $(x,y) \cdot (x',y') := (0,yy')$, $+$ component-wise. Then N is $H := \{0\} \times N_2$-integral.

In other cases, however, N is not so simply composed of an integral and an arbitrary part, even if N is commutative:

Example 1.4. Let $(G,+)$ be a non-abelian group and $K \leq G$ such that $G/_K$ is cyclic of prime or infinite order. Let $x+K$ be a generator of $G/_K$. If $g_1,g_2 \in G$, there are integers n_1,n_2 such that $g_i \in n_ix+K$ $(i=1,2)$. Define $g_1{}^*g_2 := (n_1n_2)x$. By [2], Th. 2.1, $(G,+,^*)$ is a commutative near-ring. It is straightforward to see that G is H-integral with $H = \langle x \rangle$. Note that if x is of composite order, G would not be $\langle x \rangle$-integral.

Not all non-trivial distributive near-rings are H-integral, since some are nilpotent, a property which no H-integral near-ring with $H \neq \{0\}$ can have.

For a subset S of a near-ring N, we denote its annihilator $\{a \in \in N | Sa = 0\}$ by $(0 : S)$, while $[0 : S]$ denotes the two-sided annihilator $\{a \in (0 : S) | aS = 0\}$. Also, let $S^* := S \setminus \{0\}$. We now list a number of properties of H-integral near-rings, some being technical (but necessary), some seem to be of independent interest.

Theorem 1.5. *Let N be H-integral and N_0 its zero-symmetric part. Suppose $N_0^2 \neq \{0\}$.*

(1) *For each $h \in H^*$, $(0 : h) = (0 : N_0)$; hence $K := (0 : h)$ is the same for each non-zero $h \in H$, and K is an ideal in N.*
(2) *$H \cap K = (hN) \cap K = \{0\}$ for all $h \in H^*$.*
(3) *$K \subseteq N_0$.*
(4) *For each $n \in N$, $n \in K \iff n$ is nilpotent $\iff n^2 = 0$. Each nilpotent element is therefore zero-symmetric.*
(5) *For each $n, m \in N$, $nm = 0 \iff [(n \in K,\ m \in N_0)$ or $m \in \in K] \iff nm \in K$.*
(6) *N_0 has the IFP (insertion-of-factors property).*
(7) *If $x \in N \setminus K$, $xn \equiv xm (\operatorname{mod} K) \iff n \equiv m (\operatorname{mod} K)$.*
(8) *K is a prime ideal.*
(9) *$N/_K$ is an integral and prime near-ring which is N-isomorphic to hN for each $h \in H^*$.*
(10) *If $\mathcal{P}(N)$ and $\mathcal{N}(N)$ denote the prime and the nil radical of N then $\mathcal{P}(N) = \mathcal{N}(N) = K$.*
(11) *If $N = N_0$ has the* DCC *on N-subgroups, too, then K also coincides with all Jacobson-type radicals $J_v(N)$ $(v = 0, 1/2, 1, 2)$.*
(12) *If N is not integral, it is never $\mathcal{P}$-, $\mathcal{N}$-, ..., J_2-semisimple.*
(13) *For each $S \subseteq N$, $(0 : S) = K$ or $(0 : S) = N$. Hence each annihilator right ideal is in fact an ideal (N is "almost small" [5], 9.11).*
(14) *If N is planar then N is integral.*

Proof. (1): We first show that $(0 : h) \subseteq (0 : N_0)$. Take $k \in (0 : h)$ and $0 \neq mm' \in N_0^2$. Then for each $n_0 \in N_0$, $hkn_0 = 0$, whence $kn_0 = 0$, since both h and kn_0 are in H. So $kN_0 = 0$. Also, $(0k)(mm') =$

$= 0(km)m' = 0m' = 0$, and since $mm' \neq 0$ we get $0k = 0$. So $n_0kn_0k = = 0k = 0$, and $N_0k = 0$ is shown. Conversely, let $k \in (0 : N_0)$. Then for each $n_0 \in N_0$ we get $kn_0kn_0 = 0$, so $kN_0 = 0$. Hence $hkmm' = 0$, from which we deduce that $hk = 0$.

(2): Since $hN \subseteq H$, we consider $k \in H \cap K$. If $H \neq \{0\}$, take $h \in H^*$. By (1), we can write K as $K = (0 : h)$, so $h^2 = 0$, hence $h = 0$.

(3): Follows from the proof of (1).

(4): By (3), $K \subseteq N_0$, and each $k \in K$ has $k^2 = 0$ by (1). Conversely, suppose that $n^r = 0$ some $r \in \mathbb{N}$. Then $n^{2^r} = 0$; hence it sufficies to show that if some $a \in N$ fulfills $a^2 = 0$, then $a \in K$. If $n_0a \neq 0$ for some $n_0 \in N_0$ then $n_0aan = 0$ for all $n \in N_0$, hence $aN_0 = \{0\}$. As in the proof of (1), we see that $N_0a = 0$, so anyhow $a \in K$.

(5): If $nm = 0$, take an arbitrary $n_0' \in N_0$. Then $nnmn_0' = 0$, so either $n^2 = 0$ and hence $n \in K$ by (4), or $n^2 \neq 0$, then $mN_0 = 0$. In the first case, write $m = m_0 + m_c \in N_0 + N_c$, $0 = nm = nm_0 + nm_c = = nm_0 + m_c$. Now $nm_0 \in H \cap K = \{0\}$, so $m_c = 0$ and $m \in N_0$. In the second case, take $ab \in N_0^2$, $ab \neq 0$. Then for each $c \in N$ we get $cmab = 0$ and hence $N_0m = 0$. This shows that $m \in (0 : N) = K$. Conversely. suppose that ($n \in K$, $m \in N_0$) or $m \in K$. In both cases, $nm \in H \cap K$ (since K is an ideal), so $nm = 0$ by (2). Finally, the second equivalence follows from (2), too.

(6) If $nm = 0$ then $n \in K$, or $m \in K$ by (5). Hence $nxm = 0$ for all $x \in N_0$, since $nxm \in H \cap K$.

(7) $xn \equiv xm (\operatorname{mod} K) \Rightarrow x(n-m) = xn - xm \equiv 0 \Rightarrow n - m \in K$ by (5) Conversely, $n - m \in K \Rightarrow xn - xm = x(n-m) \in K$, since K is an ideal

(8) Let I, J be ideals of N with $I \cdot J \subseteq K$. Then $I \cdot J \subseteq H \cap K$, so $I \cdot J = \{0\}$ Suppose $I \subseteq K$, and take $i \in I \setminus K$. For each $j \in J$, $ij = 0 = i0$, by (7), $j \in K$ hence $J \subseteq K$.

(9): If $h \in H^*$, $\phi : N \to hN$, $n \to hn$ is an N-epimorphism with kernel $(0 : h) = K$ $N/_K$ is integral by (5) and prime by (8).

(10): The intersection $\mathcal{P}(N)$ of all prime ideals of N is contained in K by (8). Conversely, if P is a prime ideal then $K \subseteq P$ because of $K \cdot K = \{0\} \subseteq P$ Hence $K = \mathcal{P}(N)$. By (4) and (5), K contains all nil ideals, and hence also their sum $\mathcal{N}(N)$. On the other hand, K itself is nil and hence $K = \mathcal{N}(N)$.

(11): Follows from [5], 5.61, while

(12): is a consequence of (10) and the fact that $\mathcal{P}(N) \subseteq J_2(N)$ always holds.

(13): If $n \in K$ then $nN_0 = \{0\}$ by (1) and (3). Hence $N_0 \subseteq (0:n)$. If $n' = n'_0 + n'_c \in (0:n)$ then $0 = nn' = nn'_0 + nn'_c = 0 + n'_c$. Hence $(0:n) = N_0$. If, on the other hand, $n \notin K$ then $a \in (0:n)$ implies $na = 0$, consequently $a \in K$ by (5), so $(0:n) \subseteq K$. But also $nK = 0$ by (5), so $(0:n) = K$. So all $(0:n)$ are either $= K$ or $= N$, and the same applies to all $(0:S)$.

(14): A planar near-ring N fulfills $N^2 = N$ by [5], 8.102. Hence $H = N$, and N is integral. $\Diamond$

Although for all $h_1, h_2 \in H^*$, the near-rings h_1N and h_2N are integral and N-isomorphic, they are not necessarily equal ([5], no. 37 on p. 411), nor are they always near-integral domains ([7], no. 74 on p. 112).

The condition $N_0^2 \neq \{0\}$ in Th. 1.5 is indispensable: Define on $N := \mathbb{Z} \times \mathbb{Z}$ (with componentwise addition) $(a,b)\cdot(c,d) := (0, 3\bar{b}c + d)$, where $\bar{b}$ denotes the remainder $\in \{0,1,2,\}$ of b after division by 3. N becomes so a near-ring with $N_0 = \mathbb{Z} \times \{0\}$, $N_c = \{0\} \times \mathbb{Z}$, $N^2 = N_c$. If we take $H := N_c \cup \{(1,1)\}$, N can be checked to be H-integral. $((0,0):N) = N_0$, but $((0,0):(1,1))$ also contains, for instance, the element $(-1,3)$, since $(1,1)(-1,3) = (0, 3\cdot 1\cdot(-1)+3) = (0,0)$. Therefore we adapt for the rest of this paper the

Convention: All near-rings have $N_0^2 \neq \{0\}$. So all H-integral near-rings have $H \neq \{0\}$.

2. Decompositions and constructions

In (9) of Th. 1.5 we have seen that an H-integral near-ring N is an extension of K by hN (h any element of H^*). In fact, we often can get even more:

Theorem 2.1. *Let K be H-integral such that $N/_K$ is not (group-) isomorphic to one of its proper subgroups. Then $(N,+)$ is a semidirect sum of K and hN (h any element in H^*).*

Proof. All that remains to be shown after Th. 1.5 is that $N = hN + K$. By the first isomorphism theorem for groups, $(hN+K)/_K \cong hN/_{(hN\cap K)} = hN/_{\{0\}} \cong hN \cong N/_K$, hence $N/_K = (hN+K)/_K$, so $N = hN + K$ as desired. $\Diamond$

Note that the assumption on $N/_K$ in Th. 2.1 is trivially fulfilled if $N/_K$ is finite. This theorem has a lot of consequences. For that, call a near-ring N *almost constant* if N is constant or $0m = 0$, $nm = m$ for all $n \neq 0$.

Corollaries 2.2. *Let N be H-integral and $N/_K$ finite.*

(i) *For each $h \in H^*$, hN is (as a near-ring!) isomorphic to $N/_K$. Hence all h_iN ($h_i \in H^*$) are pairwise isomorphic near-rings.*

(ii) *N has no non-zero nilpotent elements iff N is integral.*

(iii) *If hN is not almost constant then $(N,+)$ is nilpotent iff $(K,+)$ is nilpotent.*

Proof. (i): Since $(N,+)$ is a semidirect sum of K and hN (for $h \in H^*$), the map $\phi : N \to hN$, $x = k + hn \to hn$ is a (well-defined) group epimorphism. For $x, x' \in N$ $x = k + hn$, $x' = k' + hn'$ ($k, k' \in K$, $n, n' \in N$) we get $xx' = (k+hn)(k'+hn') = (k+hn)k' + (k+hn)hn' - -hnhn' + hnhn' = k'' + hnhn'$ for a suitable $k'' \in K$ (because K is an ideal of N). Hence $\phi(xx') = \phi(x)\phi(x')$, $\operatorname{Ker}\phi = K$; and we are done.

(ii): If N has no non-zero nilpotent elements then $K = \{0\}$, so $N = hN \subseteq N^2 \subseteq H$, so N is integral. The converse is clear.

(iii): By [5], 9.45 and 9.51 $(hN,+)$ is nilpotent if $h \in H^*$. So by Th. 2.1 (or by [6], p. 382), $(N,+)$ is nilpotent iff $(K,+)$ is. $\Diamond$

Let us remark that (iii) cannot be improved: Take any group $(G,+)$ and define $g * g' := g'$ for all $g, g' \in G$. Then $(G,+,^*)$ is H-integral for $H = G$, and $hG = H = G$ for all $h \in H$, $K = \{0\}$. We also remark that the proof of (i) in Cor. 2.2 shows that for all $a, a' \in hN$ and $k, k' \in K$, $(k+a)(k'+a') \equiv aa' (\operatorname{mod} K)$.

Corollary 2.3. *Let N be H-integral, $h \in H^*$, hH a finite ideal of N. Then $N \cong K \oplus hN$ (the direct sum in the near-ring sense).*

Proof. hN is then normal, hence $(N,+) = K + hN$. Also, if $x = k + hn$, $x' = k' + hn'$ are "typical" elements of N, then $xx' = (k+hn)(k'+hn') = = (k+hn)k' + (k+hn)hn' = (k+hn)hn' = (k+hn)hn' - hnhn' + hnhn' = = hnhn' = kk' + hnhn'$ (since $(k+hn)hn' - hnhn' \in K \cap hN = \{0\}$). Hence the result. $\Diamond$

Now we show that the semidirect decomposition in Th. 2.1 is in some sense the only decomposition of that kind.

Theorem 2.4. *Let N be H-integral, $h \in H^*$, A a nilpotent ideal of N, B an integral N-subgroup of N. If $(N,+)$ is a semidirect sum of A and B then $A = K$ and $(B,+,.) \cong (hN,+,.)$.*

Proof. By (10) of Th. 1.5, $A \subseteq K \subseteq N_0$. Conversely, if $k \in K$

then $k = a + b$ ($a \in A$, $b \in B$). Now $0 = ak = a^2 + ab = ab$, hence $baba = b0a = 0a = 0$. But $ba \in BN \subseteq H$, so $ba = 0$ as well. Hence $0 = bk = ba + b^2$, whence $b^2 = 0$, hence $b = 0$ and $A = K$. As in the proof of Cor. 2.2 (i), $N/_K \cong B$ (as near - rings). Since $N/_K \cong hN$ as well, we have the desired result. ◊

We turn to construction methods for H - integral near - rings. The first one comes from Th. 2.1 and contains both Examples 1.3 and 1.4 as special cases:

Construction Method 1. Take any near - ring N_1, an integral near - ring N_2, and a semidirect sum $(N, +)$ of $(N_1, +)$ (normal) and $(N_2, +)$. Define in $N : (n_1 + n_2) \cdot (n_1' + n_2') := n_2 n_2'$. Then $(N, +, .)$ is H - integral for each H such that $N_2 \subseteq H \subseteq \{n_1 + n_2 | n_1 \neq 0\}$.

A special case of this construction is supplied by a method due to G. Ferrero [1].

Construction Method 2. Let $(G, +)$ be a group which is a semidirect sum of the normal subgroup K and the finite subgroup A. Let Φ be a fixed - point - free group of automorphisms of A, and R a (complete) system of representatives of the orbits of A^* under Φ. If $x = k + a$, $x' = k' + a'$ are in G, define $x \cdot y = 0$ if $a = 0$ and $x \cdot y = \phi(a')$ if a is in the orbit of $r \in R$ and $f(r) = a$ with $f \in F$. Then $(G, +, .)$ is H - integral with $H = \{k + a | k \in K, \ a \in A^*\} \cup \{0\}$, $K = (0 : G)$, $G/_K \cong A$, R =set of all left identities of $(A, .)$.

Note that the Method 2 works because this construction gives an integral near - ring $(A, +, .)$ and $(k + a)(k' + a') = aa'$ as in Method 1. That R is the set of left identities of $(A, .)$ is straightforward.

3. Distributively generated H - integral near - rings

In this final section, we briefly discuss the special class of d.g. H - integral near - rings. Let N'' be the second commutator subgroup of $(N, +)$. We will use the following

Lemma 3.1. ("Itô's Theorem", [3]) *If a group G is the sum of two abelian subgroups, then $G'' = \{0\}$.*

Theorem 3.2. *Let N be a d.g. near - ring such that $K \neq N$ has finite index in N. Then $N/_K$ is a finite field. If, moreover, $(K, +)$ is abelian then $N'' = \{0\}$.*

Proof. Recall that Th. 2.1 is applicable; N is zero - symmetric because it is d.g. If $d = k + hn$ ($k \in K$, $h \in H^*$, $n \in N$) is distributive then by Theorem 1.5 (1), hn is distributive, too.So hN is again d.g., and by [5], 9.48 (d), hN (and the isomorphic copy $N/_K$) are fields. In particular, $(hN, +)$ is abelian. If $(K, +)$ is abelian too, we can apply Itô's Theorem 3.1. $\Diamond$

Surprisingly enough, it possible for $(N, +)$ to be non-nilpotent, even if N is "almost a ring": the near-ring N on p. 411 of [5], no. 29 is a distributive, commutative and anticommutative H-integral near-ring with $hN \cong GF(2)$, K cyclic of order 3 and $(N, +) =$ the non-nilpotent group S_3.

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A GENERALIZATION OF CARISTI'S FIXED POINT THEOREM

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Abstract: General common fixed and periodic point theorems are proven for a class of selfmaps of a quasi–metric space which satisfy the contractive conditions (1), or (7), or (8), or (10) below. Presented theorems generalize and extend Caristi's Theorem [2]. Two examples are constructed to show that an introduced class of selfmaps is indeed wider than a class of selfmaps which satisfy Caristi's contractive definition (C) below.

1. Introduction.

Let X be a non–void set and $T : X \to X$ a selfmap. A point $x \in X$ is called a periodic point for T iff there exists a positive integer k such that $T^k x = x$. If $k = 1$, then x is called a fixed point for T.

J. Caristi [2] proved the following an important contraction fixed point theorem.

Theorem 1 (Caristi [2]). *Suppose $T : X \to X$ and $\phi : X \to [0, \infty)$, where X is a complete metric space and ϕ is lower semi–continuous. If for each x in X*

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$$\text{(C)} \qquad d(x,Tx) \le \phi(x) - \phi(Tx)\,,$$

then T has a fixed point.

Th. 1 is sometimes called a Caristi–Kirk–Browder theorem (see [5]). Recently A. Bollenbacker and T. Hicks [1] revisited Th. 1. Various proofs of Th. 1 were presented later in [11, 13, 15]. It is known that Caristi's theorem is essentially equivalent to Ekelend's variational principle [5]. Up to new many extensions of Caristi's result have been obtained [6, 7, 8, 9].

The purpose of this paper is to introduce and investigate a class of selfmaps which satisfy a contractive condition weaker than (C) and still have a fixed or periodic point.

2. Main results.

We begin with some notation needed in the sequel. A pair (X,d) of a set X and a mapping d from $X \times X$ into the real numbers is said to be a *quasi-metric space* iff for all $x, y, z \in X$:

(1) $d(x,y) \ge 0$ and $d(x,y) = 0$ iff $x = y$,
(2) $d(x,z) \le d(x,y) + d(y,z)$.

Let $d_x : X \to [0, +\infty)$ be defined by $d_x(y) = d(x,y)$. Let $\mathbb{N}$ denotes the set of all positive integers.

A sequence $\{x_n\}$ in X is said to be a *left k–Cauchy* sequence if for each $k \in \mathbb{N}$ there is one N_k such that $d(x_n, x_m) < 1/k$ for all $m \ge n \ge N_k$. A quasi–metric space is a *left k–sequentially complete* if each left k–Cauchy sequence is convergent (compare [12, 14]).

Now we are in position to state the following result.

Theorem 2.1. *Let (X,d) be a left k-complete quasi–metric space and let for each $x \in X$ a function d_x be lower semi–continuous (l.s.c) on X. Let F be a family of mappings $f : X \to X$. If there exists l.s.c. function $\phi : X \to [0,\infty)$ such that for each $x \in X$:*

$$(1) \qquad d(x,fx) \le \phi(x) - \phi(fx) \text{ for all } f \in F\,,$$

then for each $x \in X$ there is a common fixed point u of F such that

$$d(x,u) \le \phi(x) - s\,, \text{ where } s = \inf\{\phi(x) : x \in X\}\,.$$

Proof. For any $x \in X$ denote

$$S(x) = \{y \in X : d(x,y) \leq \phi(x) - \phi(y)\},$$
$$a(x) = \inf\{\phi(y) : y \in S(x)\}.$$

As $x \in S(x)$, $S(x)$ is not empty and $0 \leq a(x) \leq \phi(x)$.

Let $x \in X$ be arbitrary. Put $x_1 = x$. Now we shall choose a sequence $\{x_n\}$ in X as follows: when $x_1, x_2, \ldots, x_n$ have been chosen, choose $x_{n+1} \in S(x_n)$ such that $\phi(x_{n+1}) \leq a(x_n) + 1/n$. In doing so, one obtains a sequence $\{x_n\}$ such that

(2) $d(x_n, x_{n+1}) \leq \phi(x_n) - \phi(x_{n+1}); \quad a(x_n) \leq \phi(x_{n+1}) \leq a(x_n) + 1/n.$

Then, as $\{\phi(x_n)\}$ is a decreasing sequence of reals, there is some $a \geq 0$ such that

(3) $$a = \lim{}_n \phi(x_n) = \lim{}_n a(x_n).$$

Let now $k \in \mathbb{N}$ be arbitrary. From (3) there exists some N_k such that $\phi(x_n) < a + 1/k$ for $n = N_k$. Thus, by monotonocity of $\{\phi(x_n)\}$ for $m \geq n \geq N_k$ we have $a \leq \phi(x_m) \leq \phi(x_n) < a + 1/k$ and hence

(4) $$\phi(x_n) - \phi(x_m) < 1/k \text{ for all } m \geq n \geq N_k.$$

From (ii) and (2) we get

(5) $$d(x_n, x_m) \leq \sum_{s=n}^{m-1} d(x_s, x_{s+1}) \leq \phi(x_n) - \phi(x_m).$$

Then by (4) we have

$$d(x_n, x_m) < 1/k \text{ for all } m \geq n \geq N_k.$$

Therefore, $\{x_n\}$ is a left k–Cauchy sequence and, by completeness of X, it converges to some $u \in X$.

Since d_x and ϕ are l.s.c. functions, by (5) we have

$$\begin{aligned} d(x_n, u) &\leq \lim{}_m \inf d(x_n, x_m) \leq \lim{}_m \sup d(x_n, x_m) \leq \\ &\leq \phi(x_n) + \lim{}_m \sup[-\phi(x_m)] = \phi(x_n) - \lim{}_m \inf \phi(x_m) \leq \\ &\leq \phi(x_n) - \phi(u). \end{aligned}$$

Thus $u \in S(x_n)$ for all $n \in \mathbb{N}$ and hence $a(x_n) \leq \phi(u)$. So by (3), $a \leq \phi(u)$. On the other hand, by l.s.c. of ϕ and (3), we have $\phi(u) \leq \leq \lim{}_n \inf \phi(x_n) = a$. Therefore, $\phi(u) = a$.

Now we shall show that $fu = u$ for all $f \in F$. Suppose not and let $f \in F$ be such that $fu \neq u$. Then (1) implies $\phi(fu) < \phi(u) = a$. Hence, by (3), there is a $n \in \mathbb{N}$ such that

$$\phi(fu) < a(x_n). \tag{6}$$

Since $u \in S(x_n)$ for all $n \in \mathbb{N}$, we have

$$d(x_n, fu) \leq d(x_n, u) + d(u, fu) \leq [\phi(x_n) - \phi(u)] + [\phi(u) - \phi(fu)] = \\ = \phi(x_n) - \phi(fu).$$

Hence we conclude that $fu \in S(x_n)$. Hence $\phi(fu) \geq a(x_n)$, which is a contradiction with (6). Therefore, $fu = u$ for all $f \in F$. Since $u \in S(x_n)$ implies

$$d(x_n, u) \leq \phi(x_n) - \phi(u) \leq \phi(x) - \inf\{\phi(y) : y \in X\} = \phi(x) - s. \quad \Diamond$$

The following result contains the above theorem.

Theorem 2.2. *Let E be a set, (X, d) as in Th. 2.1, $g : E \to X$ a surjective mapping and $F = \{f\}$ a family of arbitrary mappings $f : E \to X$. If there exists a l.s.c. function $\phi : X \to [0, \infty)$, such that*

$$d(ga, fa) \leq \phi(ga) - \phi(fa) \text{ for all } f \in F \tag{7}$$

and each $a \in E$, then g and F has a common coincidence point, that is, for some $v \in E$ $gv = fv$ for all $f \in F$.

Proof. Let $x \in X$ be arbitrary and $u \in X$ as in Th. 2.1. Since g is surjective, for each $x \in X$ there is some $a = a(x)$ such that $ga = x$. Let $f \in F$ be a fixed mapping. Define by f a mapping $h = h(f)$ of X into itself such that $hx = fa$, where $a = a(x)$, that is, $ga = x$. Let H be a family of all mappings $h = h(f)$. Then (7) implies

$$d(x, hx) \leq \phi(x) - \phi(hx) \text{ for all } h \in H. \tag{8}$$

Thus, by Th. 2.1, $u = hu$ for all $h \in H$. Hence $gv = fv$ for all $f \in F$, where $v = v(u)$ is such that $gv = u$. $\Diamond$

The following result is related to periodic points.

Theorem 2.3. *Let (X, d) and ϕ be as in Th. 2.1. Let $T : X \to X$ be an arbitrary mapping. If for each $x \in X$ there is $n(x)$ in $\mathbb{N}$ such that*

$$d(x, T^{n(x)}x) \leq \phi(x) - \phi(T^{n(x)}x), \tag{9}$$

then T has a periodic point.

Proof. Define $f : X \to X$ by $fx = T^{n(x)}x$. Then by Th. 2.1 (with F singleton) $fu = u$ for some $u \in X$. Hence $T^{n(x)}u = u$ that is, u is a periodic point of T. $\Diamond$

Remark 2.1. Example 2 below shows that a periodic point in Th. 2.3 need not be a fixed point. Therefore, one must add some hypothesis in order to ensure that T possesses a fixed point.

Theorem 2.4. *Let (X, d) and ϕ be as in Th. 2.1 and let $T : X \to X$ be a mapping. If for each $x \in X$, with $Tx \neq x$, there is $n(x) \in \mathbb{N}$ and a real number $C(x) > 0$ such that*

$$\max\{d(x, T^{n(x)}x), C(x) \cdot d(x, Tx)\} \leq \phi(x) - \phi(T^{n(x)}x), \tag{10}$$

then T has a fixed point.

Proof. If we suppose that $T^n x \neq x$ for all $n \in \mathbb{N}$, then we can choose $C(x)$ such that (10) reduces to (9). Then by the proof of Th. 2.3 $T^{n(x)}u = u$ for some $u \in X$. Therefore, from (10) we have

$$\max\{0, C(u) \cdot d(u, Tu)\} \leq \phi(u) - \phi(u) = 0 .$$

If we suppose that $u \neq Tu$, then $C(u) > 0$ and so we have $C(u) \cdot d(u, Tu) \leq 0$, a contradiction. Therefore $Tu = u$. $\Diamond$

Remark 2.2. It is clear that if T satisfies (C), then T satisfies (10) with $n(x) = 1$ and, for instance, $C(x) = 1$. Therefore, Th. 1 is a special case of Th. 2.1, even if (X, d) in Th. 2.1 is a metric space. Example 1 below shows that Th. 2.1 is a proper generalization of Caristi's Th. 1.

Remark 2.3. In [14] is given an example of a quasi–metric space (X, d) with d_x continuous for each x that is not metrizable.

3. Examples.

1. Let $X = \{0\} \cup \{\pm 1/n : n = 1, 2, \dots\}$ with the usual metric. Define $T : X \to X$ by $T(1/n) = -1/(n+1)$, $T(-1/n) = 1/(n+1)$ and $T(0) = 0$. Define $\phi : X \to [0, +\infty)$ by $\phi(x) = d(x, Tx)$. Then for $x = \pm 1/n$ we have

$$d(x, Tx) = 1/n + 1/(n+1) : \; d(x, T^2x) = 1/n = 1/(n+2) .$$

Hence

$$d(x, T^2x) = 1/n - 1/(n+2) < 1/n + 1/(n+1) - \\ -[1/(n+2) + 1/(n+3)] = \phi(x) - \phi(T^2x) .$$

Since for each $x = \pm 1/n$ we can choose $C(\pm 1/n) \leq 2(n+1)/(n+2)^2$, we conclude that T satisfies (10) for each x in X with $n(x) = 2$ (and $n(0) = 1$). As X is a complete metric space and $\phi(x) = |x|+|x|/(1+|x|)$ is continuous on X, we conclude that Th. 2.4 can be applied and $x = 0$ is a fixed point.

To show that Caristi's theorem is not applicable, we shall show that there is not a function $\phi : X \to [0, \infty)$ such that T satisfies (C). We pointed put [4] that such a function exists if and only if the series $\sum_{n=0}^{\infty} d(T^n x, T^{n+1}x)$ converges for all $x \in X$. Since in our example for any fixed $x = \pm 1/m_0$ we have

$$d(T^n x, T^{n+1}x) = 1/(n + m_0) + 1/(n + 1 + m_0) > 2/(n + 1 + m_0),$$

we conclude that the above series is divergent and hence there is no function ϕ such that (C) holds for any $x = \pm 1/n$ in X.

2. Let $X = [-2, -1] \cup [1, 2]$ with the usual metric. Define $T :$ $: X \to X$ by $Tx = -x$. Then T satisfies (9) with $n(x) = 2$ for any (continuous) function $\phi : X \to [0, +\infty)$.

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GENERALIZED MAPPINGS BETWEEN FUZZY TOPOLOGICAL SPACES

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Abstract: In a previous paper [8] we introduced and studied the concept of φ-operation on a fuzzy topology τ on a set X. In this paper we introduce the concept of fuzzy $\varphi\psi$-continuous mappings which generalizes most forms of fuzzy continuity. Also we introduce the concept of fuzzy $\varphi\psi$-open (fuzzy $\varphi\psi$-closed) mappings in which fuzzy open (fuzzy closed) and fuzzy homeomorphism, fuzzy θ-open (fuzzy θ-closed) and fuzzy δ-open (fuzzy

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δ - closed) become special cases. Then we introduce the concept of fuzzy $\varphi\psi$ - homeomorphism, generalizing the concepts of fuzzy homeomorphism, fuzzy θ - homeomorphism and fuzzy δ - homeomorphism. Finally, we prove that these concepts are good extensions.

1. Introduction

In order to unify several characterizations and properties of some fuzzy topological concepts and their weaker and stronger forms, in [8] we introduced and studied the concept of an operation φ on a fuzzy topology τ on a set X. Then we introduced the concepts of φ - closure (φ - interior) of fuzzy sets and φ - closed (φ - open) fuzzy sets. We showed that the collection of φ - open fuzzy sets plays a significant role in the context of fuzzy topology in a natural way analogous to that of the φ - open sets in general topology [5, 9].

In this paper, we introduce the concept of fuzzy $\varphi\psi$ - continuous mappings to unify several characterizations and properties of fuzzy continuity, fuzzy θ - continuity, fuzzy δ - continuity, fuzzy weak - continuity, fuzzy strong θ - continuity, fuzzy almost - continuity, fuzzy almost strong θ - continuity, fuzzy super continuity and fuzzy weak θ - continuity. Then we introduce and study the concepts of fuzzy $\varphi\psi$ - open and fuzzy $\varphi\psi$ - closed mappings. After that we introduce the concept of fuzzy $\varphi\psi$ - homeomorphism, generalizing the concepts of fuzzy homeomorphism, fuzzy θ - homeomorphism and fuzzy δ - homeomorphism. Several characterizations of these mappings are investigated. Finally, Lowen's good extension criterion is used to test all concepts mentioned above.

2. Preliminaries

The class of all fuzzy sets in a universe X will denoted by I^X. Fuzzy sets of X will be denoted by Greek letters as μ, ν, η, etc. Crisp subsets of X will be denoted by capital letters as A, B, C, etc. The value of a fuzzy set μ at the element x of X will be denoted by $\mu(x)$. Fuzzy singletons [10] will be denoted by x_ε, y_ν, z_ρ. The class of all fuzzy singletons will be denoted by $S(X)$. Hence $x_\varepsilon \subseteq \mu$ means $\varepsilon \in]0,1]$ and $\varepsilon \leq \mu(x)$. The definitions and results in a fuzzy topological space (fts, for short) due to Chang [2] have already been standardized. For two fuzzy sets μ and ν, we shall write $\mu q \nu$ (resp. $\mu \overline{q} \nu$) to mean that μ is quasi-coincident (resp. not quasi-coincident) with ν [13]. Let $\mu \in I^X$ and $x_\varepsilon \in S(X)$, by $N_Q(x_\varepsilon)$, $\mathrm{int}(\mu)$, $\mathrm{cl}(\mu)$ and $\mathrm{co}(\mu)$, we mean, the family of all open q-neighbourhoods of x_ε, the interior of μ, the closure of μ and the complement of μ.

Proposition 2.1 [8]. *Let $\mu, \nu \in I^X$ and $\{\mu_j : j \in J\} \subseteq I^X$, then:*

(1) $\mu q \nu \Longrightarrow \mu \cap \nu \neq 0$;

(2) $\mu q \nu \Longleftrightarrow (\exists x_\varepsilon \in S(X))(x_\varepsilon \subseteq \mu$ *and* $x_\varepsilon q \nu)$;

(3) $(\forall (x,y) \in X^2)(\forall (\varepsilon, \nu) \in (]0,1])^2)(x \neq y \Longrightarrow x_\varepsilon \overline{q} y_\nu)$;

(4) $x_\varepsilon \overline{q} \mu \Longleftrightarrow x_\varepsilon \subseteq \mathrm{co}(\mu)$;

(5) $\mu \overline{q} \,\mathrm{co}(\mu)$;

(6) $\mu \subseteq \nu \Longleftrightarrow (\forall x_\varepsilon \in S(X))(x_\varepsilon \subseteq \mu \Longrightarrow x_\varepsilon \subseteq \nu) \Longleftrightarrow (\forall x_\varepsilon \in \in S(X))(x_\varepsilon q \mu \Longrightarrow x_\varepsilon q \nu)$.

Definition 2.2 [4]. For $\mu \in I^X$ we define

(1) $\mu_\alpha = \{x | x \in X$ and $\mu(x) \geq \alpha\}$ as the *weak α-cut* of μ, where $\alpha \in]0,1]$;

(2) $\mu_{\overline{\alpha}} = \{x | x \in X$ and $\mu(x) > \alpha\}$ as the *strong α-cut* of μ, where $\alpha \in [0,1[$.

The strong 0-cut of μ is called the *support* of μ and is denoted as $\mathrm{supp}(\mu)$.

Definition 2.3 [4]. Let (X,T) be an ordinary topological space. The set of all lower semicontinuous functions from (X,T) into the closed unit interval equipped with the usual topology constitutes a fuzzy topology on X that is called the *induced fuzzy topology* associated with (X,T) and is denoted as $(X, \omega(T))$.

Lemma 2.4 [4]. *Let (X,T) be an ordinary topological space, $\mu \in I^X$ and $A \in 2^X$. Then we have:*

(1) $\mu \in \omega(T) \Longleftrightarrow (\forall \alpha \in [0,1[)(\mu_{\overline{\alpha}} \in T)$;

(2) $\mu \in \omega(T)' \Longleftrightarrow (\forall \alpha \in]0,1])(\mu_\alpha \in T')$;

(3) $A \in T \Longleftrightarrow 1_A \in \omega(T)$;

(4) $A \in T' \Longleftrightarrow 1_A \in \omega(T)'$;

(5) $\mathrm{cl}(1_A) = 1_{\mathrm{cl}(A)}$, *where 1_A denotes the characteristic mapping of $A \subseteq X$.*

Definition 2.5 [8]. Let (X, τ) be a fts. A mapping $\varphi : \tau \to I^X$ such that $(\forall \mu \in \tau)(\mu \subseteq \mu^\varphi)$, where μ^φ denotes the value of φ at μ, is called an *operation on τ*. The family of all operations on a fuzzy topology τ on a set X is denoted by $O_{(X,\tau)}$.

Examples 2.6. The mapping $\varphi : \tau \to I^X$ defined by:

(1) $(\forall \mu \in \tau)(\mu^{\varphi} = \mu)$, is an operation on τ, the so-called *identity operation* $\imath$;
(2) $(\forall \mu \in \tau)(\mu^{\varphi} = \operatorname{cl}(\mu))$, is an operation on τ, the so-called *closure operation* cl;
(3) $(\forall \mu \in \tau)(\mu^{\varphi} = \operatorname{int}(\operatorname{cl}(\mu)))$, is an operation on τ, the so-called *interior-closure operation* int ∘ cl.

Definition 2.7 [8]. An operation $\varphi \in O_{(X,\tau)}$ is said to be:
(1) *regular* $\iff (\forall x_\varepsilon \in S(X))(\forall (\nu,\eta) \in N_Q^2(x_\varepsilon))(\exists \rho \in N_Q(x_\varepsilon))(\rho^{\varphi} \subseteq \subseteq \nu^{\varphi} \cap \eta^{\varphi})$;
(2) *monotone* $\iff (\forall (\nu,\eta) \in \tau^2)(\nu \subseteq \eta \Longrightarrow \nu^{\varphi} \subseteq \eta^{\varphi})$.

It follows immediately that every monotone operation is regular, but the converse may not be true [8].

Definition 2.8 [8]. Let (X,τ) be a fts. The mapping $\varphi^{\sim} : \tau' \to I^X$ is called an *operation on* τ' iff $(\forall \lambda \in \tau')(\lambda \supseteq \lambda^{\varphi^{\sim}})$, where τ' denotes the family of all closed fuzzy sets of X. The family of all operations on τ' on a set X is denoted by $O_{(X,\tau')}$.

Definition 2.9 [8]. The operations $\varphi \in O_{(X,\tau)}$ and $\varphi^{\sim} \in O_{(X,\tau')}$ are said to be *dual* iff $(\forall \nu \in \tau)(\operatorname{co}(\nu^{\varphi}) = (\operatorname{co}(\nu))^{\varphi^{\sim}})$. Equivalently, φ and $\varphi^{\sim}$ are dual iff $(\forall \lambda \in \tau')((\operatorname{co}(\lambda))^{\varphi} = \operatorname{co}(\lambda^{\varphi^{\sim}}))$.

Definition 2.10 [8]. Let (X,τ) be a fts, $\varphi \in O_{(X,\tau)}$ and $\mu \in I^X$. Then:
(1) the φ-*closure* of μ, denoted by $\operatorname{cl}_{\varphi}(\mu)$, is given by:

$$x_\varepsilon \subseteq \operatorname{cl}_{\varphi}(\mu) \iff (\forall \eta \in N_Q(x_\varepsilon))(\eta^{\varphi} q \mu);$$

(2) the φ-*interior* of μ, denoted by $\operatorname{int}_{\varphi}(\mu)$, is given by:

$$x_\varepsilon q \operatorname{int}_{\varphi}(\mu) \iff (\exists \eta \in N_Q(x_\varepsilon))(\eta^{\varphi} \subseteq \mu).$$

Definition 2.11 [8]. Let (X,τ) be a fts, $\varphi \in O_{(X,\tau)}$ and $\mu \in I^X$. Then:
(1) μ is called φ-*closed* $\iff \operatorname{cl}_{\varphi}(\mu) = \mu$;
(2) μ is called φ-*open* $\iff \operatorname{int}_{\varphi}(\mu) = \mu$;
(3) μ is φ-open iff $\operatorname{co}(\mu)$ is φ-closed.

Theorem 2.12 [8]. *Let (X,τ) be a fts and $\varphi \in O_{(X,\tau)}$. If φ is regular, then the family of all φ-open fuzzy sets forms a fuzzy topology on X and is denoted by τ_{φ}. Moreover, $\tau_{\varphi} \subseteq \tau$.*

Definition 2.13 [8] Let (X,τ) be a fts, $\varphi \in O_{(X,\tau)}$ and $\mu \in I^X$. Then μ is called an $\varphi.q$-*neighbourhood* of $x_\varepsilon \iff (\exists \nu \in N_Q(x_\varepsilon))(\nu^{\varphi} \subseteq \mu)$.

Theorem 2.14 [8]. $(\forall \mu \in I^X)$ *(μ is φ-open in $(X,\tau) \iff \mu$ is open in (X,τ^{φ})).*

Definition 2.15 [8] A fts (X,τ) is called:

(1) $\varphi.FT_1$ iff for any $x_\varepsilon, y_\nu \in S(X)$ and $x \neq y$, $(\exists\mu \in N_Q(x_\varepsilon))(\exists\eta \in N_Q(y_\nu))(y_\nu \,\overline{q}\mu^\varphi$ and $x_\varepsilon \,\overline{q}\eta^\varphi)$;
(2) $\varphi.FT_2$ or *F - Hausdorff* iff for any $x_\varepsilon, y_\nu \in S(X)$ and $x \neq y$, $(\exists\mu \in N_Q(x_\varepsilon))(\exists\eta \in N_Q(y_\nu))(\mu^\varphi \cap \eta^\varphi = \emptyset)$;
(3) $\varphi.FR_2$ or *R - regular* iff $(\forall x_\varepsilon \in S(X))(\forall\mu \in N_Q(x_\varepsilon))(\exists\eta \in N_Q(x_\varepsilon))(\eta^\varphi \subseteq \mu)$.

Theorem 2.16 [8]. *A fts (X, τ) is $\varphi.FR_2$ iff $\tau = \tau_\varphi$.*

3. Fuzzy $\varphi\psi$ – continuous mappings

In the remainder of this paper, by (X, τ, φ) and (Y, Δ, ψ) we mean (X, τ) and (Y, Δ) are fts's, φ and ψ are operations on τ and Δ respectively.

Definition 3.1. A mapping f from (X, τ, φ) into (Y, Δ, ψ) is called *$F.\varphi\psi$ - continuous* iff $(\forall x_\varepsilon \in S(X))(\forall\eta \in N_Q(f(x_\varepsilon)))(\exists\nu \in N_Q(x_\varepsilon))(f(\nu^\varphi) \subseteq \eta^\psi)$.

Examples 3.2.
(1) For $\varphi = \imath = \psi$, $F.\varphi\psi$ - continuity coincides with F - continuity [2];
(2) for $\varphi = \mathrm{cl} = \psi$, $F.\varphi\psi$ - continuity coincides with $F.\theta$ - continuity [6];
(3) for $\varphi = \mathrm{int} \circ \mathrm{cl} = \psi$, $F.\varphi\psi$ - continuity coincides with $F.\delta$ - continuity [4];
(4) for $\varphi = \imath$ and $\psi = \mathrm{cl}$, $F.\varphi\psi$ - continuity coincides with F.weak - continuity [1];
(5) for $\varphi = \mathrm{cl}$ and $\psi = \imath$, $F.\varphi\psi$ - continuity coincides with F.strong θ - continuity [7];
(6) for $\varphi = \imath$ and $\psi = \mathrm{int} \circ \mathrm{cl}$, $F.\varphi\psi$ - continuity coincides with F.almost continuity [1, 4];
(7) for $\varphi = \mathrm{cl}$ and $\psi = \mathrm{int} \circ \mathrm{cl}$, $F.\varphi\psi$ - continuous is called *F.almost strong θ - continuous*;
(8) for $\varphi = \mathrm{int} \circ \mathrm{cl}$, and $\psi = \imath$ $F.\varphi\psi$ - continuous is called *F.super - continuous*;
(9) for $\varphi = \mathrm{int} \circ \mathrm{cl}$, and $\psi = \mathrm{cl}$, $F.\varphi\psi$ - continuous is called *F.weakly θ - continuous.*

The next theorem characterizes fuzzy $\varphi\psi$ - continuous mappings in terms of the φ - interior (ψ - interior) and φ - closed (ψ - closed) of fuzzy sets.

Theorem 3.3. *For a mapping $f : (X, \tau, \varphi) \to (Y, \Delta, \psi)$ the following are equivalent:*

(1) f *is* $F.\varphi\psi$ *- continuous;*

(2) $(\forall \eta \in \Delta)(f^{-1}(\eta) \subseteq \operatorname{int}_{\varphi}(f^{-1}(\eta^{\psi})))$;

(3) $(\forall \mu \in I^X)(f(\operatorname{cl}_{\varphi}(\mu)) \subseteq \operatorname{cl}_{\psi}(f(\mu)))$;

(4) $(\forall \eta \in I^Y)(\operatorname{cl}_{\varphi}(f^{-1}(\eta)) \subseteq f^{-1}(\operatorname{cl}_{\psi}(\eta)))$;

(5) $(\forall \eta \in I^Y)(f^{-1}(\operatorname{int}_{\psi}(\eta)) \subseteq \operatorname{int}_{\varphi}(f^{-1}(\eta)))$.

Proof. (1) $\Longrightarrow$ (2): Let $\eta \in \Delta$ and $x_\varepsilon q f^{-1}(\eta)$. Then $f(x_\varepsilon) q \eta$. By (1), $(\exists \nu \in N_Q(x_\varepsilon))(f(\nu^{\varphi}) \subseteq \eta^{\psi})$ and hence $\nu^{\varphi} \subseteq f^{-1}(\eta^{\psi})$ which implies that $x_\varepsilon q \operatorname{int}_{\varphi}(f^{-1}(\eta^{\psi}))$. Thus by Prop. 2.1 (6), we have $f^{-1}(\eta) \subseteq$ $\subseteq \operatorname{int}_{\varphi}(f^{-1}(\eta^{\psi}))$.

(2) $\Longrightarrow$ (3): Let $\mu \in I^X$ and $f(x_\varepsilon) \not\subseteq \operatorname{cl}_{\psi}(f(\mu))$. Then $(\exists \eta \in$ $\in N_Q(f(x_\varepsilon)))(\eta^{\psi} \overline{q} f(\mu))$ and hence $f^{-1}(\eta^{\psi}) \overline{q} \mu$ which implies $\operatorname{int}_{\varphi}(f^{-1}(\eta^{\psi})) \overline{q} \mu$. From $x_\varepsilon q f^{-1}(\eta)$ and by (2) we obtain $(\exists \rho \in N_Q(x_\varepsilon))$ $(\rho^{\varphi} \subseteq f^{-1}(\eta^{\psi}))$. Hence $\rho^{\varphi} \overline{q} \mu$ and so $x_\varepsilon \not\subseteq \operatorname{cl}_{\varphi}(\mu)$ which implies that $f(x_\varepsilon) \not\subseteq f(\operatorname{cl}_{\varphi}(\mu))$. Thus $f(\operatorname{cl}_{\varphi}(\mu)) \subseteq \operatorname{cl}_{\psi}(f(\mu))$.

(3) $\Longrightarrow$ (4): Let $\eta \in I^Y$. From $ff^{-1}(\eta) \subseteq \eta$, we have $\operatorname{cl}_{\psi}(ff^{-1}(\eta)) \subseteq \operatorname{cl}_{\psi}(\eta)$. By (3), we have $f(\operatorname{cl}_{\varphi}(f^{-1}(\eta))) \subseteq$ $\subseteq \operatorname{cl}_{\psi}(ff^{-1}(\eta)) \subseteq \operatorname{cl}_{\psi}(\eta)$. Thus we have $\operatorname{cl}_{\varphi}(f^{-1}(\eta)) \subseteq f^{-1}(\operatorname{cl}_{\psi}(\eta))$.

(4) $\Longrightarrow$ (5): Let $\eta \in I^Y$ and $x_\varepsilon q f^{-1}(\operatorname{int}_{\psi}(\eta))$. Then $x_\varepsilon \not\subseteq$ $\not\subseteq \operatorname{co}(\operatorname{int}_{\psi}(\eta)) = f^{-1}(\operatorname{cl}_{\psi}(\operatorname{co}(\eta)))$. By (4), we have $x_\varepsilon \not\subseteq$ $\not\subseteq \operatorname{cl}_{\varphi}(f^{-1}(\operatorname{co}(\eta))) = \operatorname{co}(\operatorname{int}_{\varphi}(f^{-1}(\eta)))$ and hence $x_\varepsilon q \operatorname{int}_{\varphi}(f^{-1}(\eta))$. Thus, $f^{-1}(\operatorname{int}_{\psi}(\eta)) \subseteq \operatorname{int}_{\varphi}(f^{-1}(\eta))$.

(5) $\Longrightarrow$ (1): Let $x_\varepsilon \in S(X)$ and $\eta \in N_Q(x_\varepsilon)$. From $\eta^{\psi} \overline{q} \operatorname{co}(\eta^{\psi})$, we have $f(x_\varepsilon) \not\subseteq \operatorname{cl}_{\psi}(\operatorname{co}(\eta^{\psi})) = \operatorname{co}(\operatorname{int}_{\psi}(\eta^{\psi}))$ and hence $f(x_\varepsilon) q \operatorname{int}_{\psi}(\eta^{\psi})$ which implies that $x_\varepsilon q f^{-1}(\operatorname{int}_{\psi}(\eta^{\psi}))$. By (5), we have $x_\varepsilon q \operatorname{int}_{\varphi}(f^{-1}(\eta^{\psi}))$ and hence $(\exists \mu \in N_Q(x_\varepsilon))(\mu^{\varphi} \subseteq f^{-1}(\eta^{\psi}))$ and so $f(\mu^{\varphi}) \subseteq \eta^{\psi}$. $\Diamond$

Corollary 3.4. *Let* $f : (X, \tau, \varphi) \to (Y, \Delta, \psi)$ *be a mapping. If* $(\forall x_\varepsilon \in$ $\in S(X))(\forall \eta \in N_Q(f(x_\varepsilon)))(\exists \mu \in N_Q(x_\varepsilon) \cap \tau_{\varphi})(f(\mu) \subseteq \eta^{\psi})$, *then* f *is* $F.\varphi\psi$ *- continuous.*

Corollary 3.5. *Let* $f : (X, \tau, \varphi) \to (Y, \Delta, \psi)$ *is an* $F.\varphi\psi$ *- continuous mapping, then the inverse image of each* ψ *- closed (*ψ *- open) fuzzy set is* φ *- closed (*φ *- open).*

The converse need not be true as can be seen from the following example.

Example 3.6. Let $X = \{x, y\}$ and $\mu, \eta, \rho \in I^X$ defined by:

$$\mu = \underline{0.6} \qquad \rho = \underline{0.3}$$
$$\eta(x) = 0.6 \qquad \eta(y) = 0.7,$$

where $\underline{\alpha}$ denotes the constant mapping with value α. Let $\tau = \{X, \emptyset, \mu, \eta, \rho\}$ and $\Delta = \{X, \emptyset, \eta, \rho\}$. Then (X, τ) and (X, Δ) are fts's. Define $\varphi : \tau \to I^X$ and $\psi : \Delta \to I^X$ by:

$$\begin{array}{ll|ll} X^\varphi = X & \emptyset^\varphi = \emptyset & X^\psi = X & \emptyset^\psi = \emptyset \\ \mu^\varphi = \mu & \eta^\varphi = \eta & \eta^\psi = \eta & \rho^\psi = \underline{0.4}. \\ \rho^\varphi = \underline{0.5} & & & \end{array}$$

Clearly φ and ψ are regular operations. Moreover one easily finds: $\tau_\varphi = \{X, \emptyset, \mu, \eta\}$ and $\Delta_\psi = \{X, \emptyset, \eta\}$. Consider the identity mapping $f : (X, \tau, \varphi) \to (X, \Delta, \psi)$. Then the inverse image of each ψ-open is φ-open but f is not $F.\varphi\psi$-continuous. Indeed, for $x_\varepsilon, \varepsilon = 0.8$ and $\rho \in N_Q(f(x_\varepsilon))$ there is no $\nu \in N_Q(x_\varepsilon)$ such that $f(\nu^\varphi) \subseteq \rho^\psi$.

In the following theorem it is shown that ψ-regularity of the codomain space is a sufficient condition to obtain the converse of Cor. 3.5.

Theorem 3.7. *Let $f : (X, \tau, \varphi) \to (Y, \Delta, \psi)$ be a mapping. If the inverse image of each ψ-open is φ-open and (Y, Δ) is $\psi.FR_2$, then f is $F.\varphi\psi$-continuous.*

Proof. Let $x_\varepsilon \in S(X)$ and $\eta \in N_Q(f(x_\varepsilon))$. From (Y, Δ) is $\psi.FR_2$ and Th. 2.16, we infer $\eta \in \Delta_\psi$. By hypothesis $f^{-1}(\eta) \in \tau_\varphi$ and $x_\varepsilon q f^{-1}(\eta)$ and hence $(\exists \mu \in N_Q(x_\varepsilon))(\mu^\varphi \subseteq f^{-1}(\eta))$ which implies that $f(\mu^\varphi) \subseteq \subseteq \eta \subseteq \eta^\psi$. Thus f is $F.\varphi\psi$-continuous. $\Diamond$

Theorem 3.8. *A mapping $f : (X, \tau, \varphi) \to (Y, \Delta, \psi)$ is $F.\varphi\psi$-continuous iff $(\forall x_\varepsilon \in S(X))(\forall \lambda_1 \in \Delta'$ and $f(x_\varepsilon) \not\subseteq \lambda_1)(\exists \lambda_2 \in \tau')(x_\varepsilon \not\subseteq \lambda_2)$ and $f(\lambda_2^{\varphi^\sim}) \supseteq \lambda_1^{\psi^\sim}$, where $\varphi^\sim$, $\psi^\sim$ are the dual operations of φ and ψ respectively.*

Proof. Straightforward. $\Diamond$

Theorem 3.9. *The axioms $\varphi.FT_1$ and $\varphi.FT_2$ are inverse invariant under a $F.\varphi\psi$-continuous injective mapping.*

Proof. As example, we prove the $\varphi.FT_2$ inverse invariance. Let f be a $F.\varphi\psi$-continuous mapping from (X, τ, φ) into (Y, Δ, ψ), where (Y, Δ) is $\psi.FT_2$. Let $x_\varepsilon, y_\nu \in S(X)$ with $x \neq y$. Since f is injective, we have $f(x) \neq f(y)$. From (Y, Δ) is $\psi.FT_2$, we obtain $(\exists \eta_1 \in \in N_Q(f(x_\varepsilon)))(\exists \eta_2 \in N_Q(f(y_\nu)))(\eta_1^\psi \cap \eta_2^\psi = \emptyset)$. By $F.\varphi\psi$-continuity of f, $(\exists \mu_1 \in N_Q(x_\varepsilon))(\exists \mu_2 \in N_Q(y_\nu))(f(\mu_1^\varphi) \subseteq \eta_1^\psi$ and $f(\mu_2^\varphi) \subseteq \eta_2^\psi)$.

Hence $f(\mu_1^\varphi) \cap f(\mu_2^\varphi) = \emptyset$ and so $\mu_1^\varphi \cap \mu_2^\varphi = \emptyset$. Thus (X, τ) is $\varphi.FT_2$-fts. $\Diamond$

Theorem 3.10. *The axiom $\varphi.FR_2$ is inverse invariant under a $F.\varphi\psi$-continuous, F-open and injective mapping.*

Proof. Let f be a $F.\varphi\psi$-continuous, F-open and injective mapping from (X, τ, φ) into (Y, Δ, ψ), where (Y, Δ) is $\psi.FR_2$. Let $x_\varepsilon \in S(X)$ and $\mu \in N_Q(x_\varepsilon)$. From f is F-open, we have $f(\mu) \in N_Q(f(x_\varepsilon))$. Since (Y, Δ) is $\psi.FR_2$, we obtain $(\exists \eta \in N_Q(f(x_\varepsilon)))(\eta^\psi \subseteq f(\mu))$. By $F.\varphi\psi$-continuity of f, $(\exists \nu \in N_Q(x_\varepsilon))(f(\nu^\varphi) \subseteq \eta^\psi)$. Hence, $\nu^\varphi = f^{-1}f(\nu^\varphi) \subseteq \subseteq f^{-1}(\eta^\psi) \subseteq f^{-1}f(\mu) = \mu$ (f being injective). Thus, (X, τ) is $\varphi.FR_2$-fts. $\Diamond$

Theorem 3.11. *If $f, g : (X, \tau, \varphi) \to (Y, \Delta, \psi)$ are $F.\varphi\psi$-continuous mappings, φ is regular and (Y, Δ) is $\psi.FT_2$, then the set $\mu = \cup\{x_\varepsilon \mid x_\varepsilon \in \in I^X$ and $f(x_\varepsilon) = g(x_\varepsilon)\}$ is φ-closed in X and if $\mathrm{cl}_\varphi(\mu) = X$ and $(\forall x_\varepsilon \subseteq \mu)(f(x_\varepsilon) = g(x_\varepsilon))$, then $f = g$.*

Proof. For any $x \in X, f(x_\varepsilon) = g(x_\varepsilon)$ iff $f(x) = g(x)$. Hence, if $x_\varepsilon \not\subseteq \mu$, we have $f(x) \neq g(x)$. Since (Y, Δ) is $\psi.FT_2$, then $(\exists \eta_1 \in \in N_Q(f(x_\varepsilon)))(\exists \eta_2 \in N_Q(g(x_\varepsilon)))(\eta_1^\psi \cap \eta_2^\psi = \emptyset)$. By $F.\varphi\psi$-continuity of f and g, $(\exists \nu_1, \nu_2 \in N_Q(x_\varepsilon))(f(\nu_1^\varphi) \subseteq \eta_1^\beta$ and $g(\nu_2^\varphi) \subseteq \eta_2^\beta)$. Then $f(\nu_1^\varphi) \cap g(\nu_2^\varphi) = \emptyset$.

Now, since φ is regular then $(\exists \rho \in N_Q(x_\varepsilon))(\rho^\varphi \subseteq \nu_1^\varphi \cap \nu_2^\varphi)$. In the light of $\eta_1^\psi \cap \eta_2^\psi = \emptyset$, it is easily seen that $\rho^\varphi \cap \mu = \emptyset$ and hence $\rho^\varphi \bar{q} \mu$ which implies that $x_\varepsilon \not\subseteq \mathrm{cl}_\varphi(\mu)$. Thus μ is φ-closed. Finally, since $\mu = \mathrm{cl}_\varphi(\mu) = X$, we have $(\forall x \in X)(\exists x_\varepsilon \subseteq \mu)(f(x_\varepsilon) = g(x_\varepsilon))$ and consequently $(\forall x \in X)(f(x) = g(x))$. Thus $f = g$. $\Diamond$

4. Fuzzy $\varphi\psi$ – open and $\varphi\psi$ – closed mappings

Definition 4.1. A mapping $f : (X, \tau, \varphi) \to (Y, \Delta, \psi)$ is called:

(1) *$F.\varphi\psi$-open* iff for every $\mu \in I^X$, $f(\mathrm{int}_\varphi(\mu)) \subseteq \mathrm{int}_\psi(f(\mu))$;

(2) *$F.\varphi\psi$-closed* iff for every $\mu \in I^X$, $\mathrm{cl}_\psi(f(\mu)) \subseteq f(\mathrm{cl}_\varphi(\mu))$.

Examples 4.2.

(1) If $\varphi = \imath$ and $\psi = \imath$, then $F.\varphi\psi$-open ($F.\varphi\psi$-closed) mapping coincides with F-open (F-closed) [2];

(2) when $\varphi = \mathrm{cl}$ and $\psi = \mathrm{cl}$, then $F.\varphi\psi$-open ($F.\varphi\psi$-closed) mapping is called $F.\theta$-open ($F.\theta$-closed);

(3) if $\varphi = \mathrm{int} \circ \mathrm{cl}$ and $\psi = \mathrm{int} \circ \mathrm{cl}$, then $F.\varphi\psi$-open ($F.\varphi\psi$-closed)

mapping is called $F.\delta$ - open ($F.\delta$ - closed).

Theorem 4.3. *If a mapping $f : (X,\tau,\varphi) \to (Y,\Delta,\psi)$ is $F.\varphi\psi$ - open ($F.\varphi\psi$ - closed), then the image of every φ - open (φ - closed) fuzzy set is ψ - open (ψ - closed). The converse is true if (X,τ) is $\varphi.FR_2$.*

Proof. Let $\mu \in \tau_\varphi$. Then $\mu = \operatorname{int}_\varphi(\mu)$ and hence $f(\mu) = f(\operatorname{int}_\varphi(\mu))$. Since f is $F.\varphi\psi$-open, we have $f(\mu) \subseteq \operatorname{int}_\psi(f(\mu))$ and hence $f(\mu) \in \in \Delta_\psi$. Conversely, if (X,τ) is $\varphi.FR_2$, then by Th. 2.16, we have $(\forall \mu \in \in I^X)(\operatorname{int}_\varphi(\mu) \in \tau_\varphi)$ and hence $f(\operatorname{int}_\varphi(\mu)) \in \Delta_\psi$ which implies that $f(\operatorname{int}_\varphi(\mu)) \subseteq \operatorname{int}_\psi(f(\mu))$. Proof of other case can be given in similar way. $\Diamond$

The next example shows that $\varphi.FR_2$ is needed in the statement Th. 4.3.

Example 4.4. Let $X = \{x,y\}$, $\mu,\nu,\eta,\rho \in I^X$ defined by:

$$\begin{array}{llll} \mu(x) = 0.4 & \mu(y) = 0.3 & \eta(x) = 0.7 & \eta(y) = 0.6 \\ \nu(x) = 0.6 & \nu(y) = 0.7 & \rho = \underline{0.4} & \end{array}$$

Let $\tau = \{X,\emptyset,\mu,\nu\}$ and $\Delta = \{X,\emptyset,\eta,\rho\}$. Then (X,τ) and (X,Δ) are fts's. Define $\varphi : \tau \to I^X$ and $\psi : \Delta \to I^X$ by:

$$\begin{array}{ll|ll} X^\varphi = X & \emptyset^\varphi = \emptyset & X^\psi = X & \emptyset^\psi = \emptyset \\ \nu^\varphi = \nu & \mu^\varphi = \underline{0.4} & \eta^\psi = \eta & \rho^\psi = \underline{0.5}. \end{array}$$

Clearly φ and ψ are regular operations. Moreover one easily finds: $\tau_\varphi = = \{X,\emptyset,\nu\}$ and $\Delta_\psi = \{X,\emptyset,\eta\}$ and hence $\tau'_\varphi\{X,\emptyset,\mu\}$ and $\Delta'_\psi = = \{X,\emptyset,\operatorname{co}(\eta)\}$. Define $f : (X,\tau,\varphi) \to (X,\Delta,\psi)$ satisfying $f(x) = y$ and $f(y) = x$, then every image of φ - closed (φ - open) is ψ - closed (ψ - open), but f is not $F.\varphi\psi$ - closed. Indeed, for $\nu \in I^X$, we have $\operatorname{cl}_\varphi(\nu) = = \{(x,0.6),(y,0.9)\}$. So, $f(\operatorname{cl}_\varphi(\nu)) = \{(x,0.9),(y,0.6)\}$. Since $f(\nu) = \eta$, we have $\operatorname{cl}_\psi(f(\nu)) = \operatorname{cl}_\psi(\eta) = \underline{0.9}$. Hence $\operatorname{cl}_\psi(f(\nu)) \nsubseteq \nsubseteq f(\operatorname{cl}_\varphi(\nu))$.

Theorem 4.5. *Let $f : (X,\tau,\varphi) \to (Y,\Delta,\psi)$ be a mapping.*

(1) *If $(\forall \eta \in \tau)(f(\eta) \in \Delta$ and $f(\eta^\varphi) = (f(\eta))^\psi)$, then f is $F.\varphi\psi$ - open.*

(2) *If $(\forall \lambda \in \tau')(f(\lambda) \in \Delta'$ and $f(\lambda^\varphi) = (f(\lambda))^\psi)$, then f is $F.\varphi\psi$ - closed.*

Proof. (1) Let $\mu \in I^X$ and $y_\nu q f(\operatorname{int}_\varphi(\mu))$. Then $(\exists x_\varepsilon \subseteq f^{-1}(y_\nu))$ $(x_\varepsilon q \operatorname{int}_\varphi(\mu))$ and hence $(\exists \eta \in N_Q(x_\varepsilon))(\eta^\varphi \subseteq \mu)$. From hypothesis we obtain that $f(\eta) \in N_Q(y_\nu)$ and $(f(\eta))^\psi \subseteq f(\mu)$ and hence $y_\nu q \operatorname{int}_\psi(f(\mu))$. Thus $f(\operatorname{int}_\varphi(\mu)) \subseteq \operatorname{int}_\psi(f(\mu))$. The proof of (2) is similar. $\Diamond$

Corollary 4.6. *Let $f:(X,\tau,\varphi)\to(Y,\Delta,\psi)$ be a mapping.*

(1) *If $(\forall\eta\in\tau)(f(\eta)\in\Delta$ and $f(\eta^\varphi)=(f(\eta))^\psi)$, then the image of every φ - open fuzzy set is ψ - open.*

(2) *If $(\forall\lambda\in\tau')(f(\lambda)\in\Delta'$ and $f(\lambda^{\varphi^\sim})=(f(\lambda))^{\psi^\sim})$, then the image of every φ - closed fuzzy set is ψ - closed.*

The following example shows that the converse of Cor. 4.6 is not true in general.

Example 4.7. Let $X=\{x,y\}$, $\mu,\nu,\eta,\rho,\sigma\in I^X$ defined by:

$$\mu(x)=0.5 \qquad \mu(y)=0.6 \qquad \nu(x)=0.8 \qquad \nu(y)=0.9$$
$$\eta(x)=0.5 \qquad \eta(y)=0.4 \qquad \rho(x)=0.4 \qquad \rho(y)=0.6$$
$$\sigma=\underline{0.4}\,.$$

Let $\tau=\{X,\emptyset,\mu,\eta,\rho,\sigma\}$ and $\Delta=\{X,\emptyset,\mu,\nu,\rho,\sigma\}$. Then (X,τ) and (X,Δ) are fts's. Define $\varphi:\tau\to I^X$ and $\psi:\Delta\to I^X$ by:

$$\begin{array}{ll|ll} X^\varphi=X & \emptyset^\varphi=\emptyset & X^\psi=X & \emptyset^\psi=\emptyset \\ \mu^\varphi=\underline{0.7} & \eta^\varphi=\eta & \mu^\psi=\underline{0.6} & \nu^\psi=\nu \\ \rho^\varphi=\rho & \sigma^\varphi=\sigma & \rho^\psi=\rho & \sigma^\psi=\sigma\,. \end{array}$$

It is easy to see that φ and ψ are regular operations and $\tau_\varphi=\tau$ and $\Delta_\psi=\Delta$. Consider the identity mapping $f:(X,\tau,\varphi)\to(X,\Delta,\psi)$. It is easy to see that the image of every φ - open fuzzy set is ψ - open (and hence f is $F.\varphi\psi$ - open, since (X,τ) is $\varphi.FR_2$), but for $\mu\in\tau$ we have $f(\mu)\in\Delta$ and $f(\mu^\varphi)\neq(f(\mu))^\psi$.

Definition 4.8. A bijective mapping $f:(X,\tau,\varphi)\to(Y,\Delta,\psi)$ is called *$F.\varphi\psi$ - homeomorphism* iff both f and f^{-1} are $F.\varphi\psi$ - continuous.

Example 4.9.

(1) If $\varphi=\imath$ and $\psi=\imath$, then $F.\varphi\psi$ - homeomorphism coincides with F - homeomorphism [2].

(2) If $\varphi=$ cl and $\psi=$ cl, then $F.\varphi\psi$ - homeomorphism is called *$F.\theta$ - homeomorphism*.

(3) If $\varphi=$ int $\circ$ cl and $\psi=$ int $\circ$ cl, then $F.\varphi\psi$ - homeomorphism is called *$F.\delta$ - homeomorphism*.

Theorem 4.10. *If $f:(X,\tau,\varphi)\to(Y,\Delta,\psi)$ is bijective, then the following properties of f are equivalent:*

(1) *f is $F.\varphi\psi$ - homeomorphism;*

(2) *f is $F.\varphi\psi$ - continuous and $F.\varphi\psi$ - open;*

(3) *f is $F.\varphi\psi$ - continuous and $F.\varphi\psi$ - closed;*

(4) $(\forall \mu \in I^X)(f(\operatorname{cl}_\varphi(\mu)) = \operatorname{cl}_\psi(f(\mu)))$.

Proof. (1) $\Longrightarrow$ (2): Let $\mu \in I^X$. From f^{-1} is $F.\varphi\psi$ - continuous, we have $(f^{-1})^{-1}(\operatorname{int}_\varphi(\mu)) \subseteq \operatorname{int}_\psi((f^{-1})^{-1}(\mu))$ and hence $f(\operatorname{int}_\varphi(\mu)) \subseteq$ $\subseteq \operatorname{int}_\psi(f(\mu))$.

(2) $\Longrightarrow$ (3): Let $\mu \in I^X$. From f is $F.\varphi\psi$ - open and bijective, we obtain that $f(\operatorname{int}_\varphi(\operatorname{co}(\mu))) \subseteq \operatorname{int}_\psi(f(co(\mu)))$ and hence $\operatorname{co}(f(\operatorname{cl}_\varphi(\mu))) \subseteq \operatorname{co}(\operatorname{cl}_\psi(f(\mu)))$ which implies that $\operatorname{cl}_\psi(f(\mu)) \subseteq$ $\subseteq f(\operatorname{cl}_\varphi(\mu))$.

(3) $\Longrightarrow$ (4) and (4) $\Longrightarrow$ (1) can be easily proved. $\Diamond$

5. Good extensions

Definition 5.1 [13]. A property P_f of a fts is said to be a *good extension* of the property P in classical topology iff whenever the fts is topologically generated (induced) say by (X,T), then $(X,\omega(T))$ has property P_f iff (X,T) has property P.

Theorem 5.2 [8]. *Let (X,T) be a topological space and φ be an operation on T. Consider the induced fuzzy topological space $(X,\omega(T))$ and the operation $\varphi_\omega : \omega(T) \to I^X$ defined by: $(\forall \mu \in \omega(T))(\mu^{\varphi_\omega} =$ $= \bigcup_{0\le\alpha<h(\mu)} (\underline{\alpha} \cap 1_{(\mu_{\overline{\alpha}})^\varphi}))$, where $h(\mu) = \sup_{x\in X} \mu(x)$. Then:*

(1) $\omega(T_\varphi) = (\omega(T))_{\varphi_\omega}$;

(2) $\operatorname{cl}_{\varphi_\omega}(1_A) = 1_{\operatorname{cl}_\varphi(A)}$;

(3) $\operatorname{int}_{\varphi_\omega}(1_A) = 1_{\operatorname{int}_\varphi(A)}$;

(4) $\operatorname{cl}_{\varphi_\omega}(\mu) = \bigcup_{0\le\alpha<h(\mu)} (\underline{\alpha} \cap 1_{\operatorname{cl}_\varphi(\mu_{\overline{\alpha}})}),\ \forall \mu \in I^X$;

(5) $\operatorname{int}_{\varphi_\omega}(\mu) = \bigcup_{0\le\alpha<h(\mu)} (\underline{\alpha} \cap 1_{\operatorname{int}_\varphi(\mu_{\overline{\alpha}})}),\ \forall \mu \in I^X$.

Proposition 5.3. *Let $f : X \to Y$ be a mapping, $\mu \in I^X$, $A \subseteq X$ and $B \subseteq Y$. Then the following relations hold:*

(1) $f^{-1}(\mu_{\overline{\alpha}}) = (f^{-1}(\mu))_{\overline{\alpha}}$.

(2) $f(\mu_{\overline{\alpha}}) = (f(\mu))_{\overline{\alpha}}$.

(3) $f^{-1}(1_B) = 1_{f^{-1}(B)}$.

(4) $f(1_A) = 1_{f(A)}$.

Theorem 5.4. *A mapping $f : (X, T_1, \varphi) \to (Y, T_2, \psi)$ is $\varphi\psi$ - continuous iff $f : (X, \omega(T_1), \varphi_\omega) \to (T, \omega(T_2), \psi_\omega)$ is $F.\varphi_\omega\psi_\omega$ - continuous.*

Proof. Let $\mu \in (\omega(T_2))_{\psi_\omega}$. From Th. 5.2 (1), we have $\mu \in \omega((T_2)_\psi)$. Then $(\forall \alpha \in [0,1[\,)(\mu_{\overline{\alpha}} \in (T_2)_\psi)$. From f is $\varphi\psi$ - continuous and Prop.

5.3 (1), we have $(\forall\alpha\in[0,1[)((f^{-1}(\mu))_{\overline{\alpha}}\in(T_1)_\varphi)$ and hence $f^{-1}(\mu)\in\in\omega((T_1)_\varphi)=(\omega(T_1))_{\varphi_\omega}$. Thus f is $F.\varphi_\omega\psi_\omega$ - continuous. Conversely, let $B\in(T_2)_\psi$. Then by Th. 5.2 (1), $1_B\in(\omega(T_2))_{\psi_\omega}$. Since f is $F.\varphi_\omega\psi_\omega$ - continuous, we have $f^{-1}(1_B)=1_{f^{-1}(B)}\in\omega((T_1)_\varphi)$ and hence $f^{-1}(B)\in(T_1)_\varphi$. Thus is $\varphi\psi$ - continuous. ◊

Theorem 5.5. *A mapping $f:(X,T_1,\varphi)\to(Y,T_2,\psi)$ is $\varphi\psi$ - open iff $f:(X,\omega(T_1),\varphi_\omega)\to(Y,\omega(T_2),\psi_\omega)$ is $F.\varphi_\omega\psi_\omega$ - open.*

Proof. Let $\mu\in I^X$. Then $(\forall\alpha\in[0,1[)(\mu_{\overline{\alpha}}\subseteq X)$. From f is $\varphi\psi$ - open, it follows $f(\operatorname{int}_\varphi(\mu_{\overline{\alpha}}))\subseteq\operatorname{int}_\psi(f(\mu_{\overline{\alpha}}))$. Then we obtain successively:

$$1_{f(\operatorname{int}_\varphi)(\mu_{\overline{\alpha}})}\subseteq 1_{\operatorname{int}_\psi(f(\mu_{\overline{\alpha}}))},\quad \underline{\alpha}\cap 1_{f(\operatorname{int}_\varphi(\mu_{\overline{\alpha}}))}\subseteq\underline{\alpha}\cap 1_{\operatorname{int}_\psi(f(\mu_{\overline{\alpha}}))},$$

$$\bigcup_{0\le\alpha<h(\mu)}\left(\underline{\alpha}\cap 1_{f(\operatorname{int}_\varphi(\mu_{\overline{\alpha}}))}\right)\subseteq\bigcup_{0\le\alpha<h(\mu)}\left(\underline{\alpha}\cap 1_{\operatorname{int}_\psi(f(\mu_{\overline{\alpha}}))}\right),$$

$$f\Big(\bigcup_{0\le\alpha<h(\mu)}\left(\underline{\alpha}\cap 1_{\operatorname{int}_\varphi(\mu_{\overline{\alpha}})}\right)\Big)\subseteq\bigcup_{0\le\alpha<h(\mu)}\left(\underline{\alpha}\cap 1_{\operatorname{int}_\psi(f(\mu_{\overline{\alpha}}))}\right).$$

Then $f(\operatorname{int}_{\varphi_\omega}(\mu))\subseteq\operatorname{int}_{\psi_\omega}(f(\mu))$ and hence f is $F.\varphi_\omega\psi_\omega$ - open.

Conversely, let $A\subseteq X$. Then $1_A\in I^X$ and so $f(\operatorname{int}_{\varphi_\omega}(1_A))\subseteq\subseteq\operatorname{int}_{\psi_\omega}(f(1_A))$. Then we have successively:

$$f(1_{\operatorname{int}_\varphi(A)})\subseteq\operatorname{int}_{\psi_\omega}(1_{f(A)}),\quad 1_{f(\operatorname{int}_\varphi(A))}\subseteq 1_{\operatorname{int}_\psi(f(A))}.$$

Then $f(\operatorname{int}_\varphi(A))\subseteq\operatorname{int}_\psi(f(A))$ and hence f is $\varphi\psi$ - open. ◊

Theorem 5.6. *A mapping $f:(X,T_1,\varphi)\to(Y,T_2,\psi)$ is $\varphi\psi$ - closed iff $f:(X,\omega(T_1),\varphi_\omega)\to(Y,\omega(T_2),\psi_\omega)$ is $F.\varphi_\omega\psi_\omega$ - closed.*

Proof. It is similar to that of Th. 5.5. ◊

With the results seen above we conclude that:

Theorem 5.7. *$f:(X,T_1,\varphi)\to(Y,T_2,\psi)$ is $\varphi\psi$ - homeomorphism iff $f:(X,\omega(T_1),\varphi_\omega)\to(Y,\omega(T_2),\psi_\omega)$ is $F.\varphi_\omega\psi_\omega$ - homeomorphism.*

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A EGOROFF – TYPE THEOREM FOR SET – VALUED MEASURABLE FUNCTIONS

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Abstract: Results of the following type treated: If $\phi : \mathbf{S}^n \to C(\mathbf{S}^k)$, $k \geq 2$, is a Lebesgue measurable function it is shown that there exists a continuous function $f : \mathbf{B}_{n+1} \to \mathbf{S}^k \setminus \{\infty\}$ so that the radial cluster-set function f_R of f equals ϕ almost every where on $\mathbf{S}^n$.

In [6] and [7], question of interpolations by radial cluster set functions were addressed for functions of Baire class 1. The results of these papers can be used to prove Egoroff-type theorems and Lusin-type theorems for set-valued measurable functions. The present note illustrates one such Egoroff-type theorem. The construction found here can be used to establish other theorems of this type as well as Lusin-type theorems. Throughout the note, k and n are integers such that $k \geq 2$ and $n \geq 1$.

1. Statement of the theorem. We shall begin the statement of our Egoroff-type theorem. The notation used in its statement will be explained immediately after the statement and the proof of the theorem will be given in Sect. 3.

Theorem. *Let $\phi : \mathbf{S}^n \to C(\mathbf{S}^k)$ be a Lebesgue measurable function. Then, there is a continuous function $f : \mathbf{B}_{n+1} \to \mathbf{S}^k \setminus \{\infty\}$ and there is an increasing sequence of positive numbers $\{r_m\}$ converging to 1 such that*

(i) *the radial cluster-set function f_R of f is equal to ϕ Lebesgue almost everywhere on $\mathbf{S}^n$, and*

(ii) *for each positive number ε there is a Lebesgue measurable set E such that the continuous functions $\Gamma_m : \mathbf{S}^n \to C(\mathbf{S}^k)$, $m = 1, 2, \ldots$, defined by*

$$\Gamma_m(z) = \{f(rz) : r_m \leq r \leq r_{m+1}\} \quad z \in \mathbf{S}^n ,$$

converge uniformly to ϕ on E and $\mu(\mathbf{S}^n \setminus E) < \varepsilon$.

As usual, $\mathbb{R}^{n+1}$ is the $(n+1)$-dimensional Euclidean space. Its open unit ball and corresponding boundary are $\mathbf{B}_{n+1}$ and $\mathbf{S}^n$, respectively. The Lebesgue measure on $\mathbf{S}^n$ is denoted by μ. The point ∞ is the point $(0, \ldots, 1)$ on the k-sphere $\mathbf{S}^k$ of $\mathbb{R}^{k+1}$. By $C(\mathbf{S}^k)$ we mean the collection of all nonempty subcontinua of $\mathbf{S}^k$. When $C(\mathbf{S}^k)$ is endowed with the Hausdorff metric D, we have from a theorem of Curtis and Schori that $C(\mathbf{S}^k)$ is homeomorphic to the Hilbert cube I^{ω_0}, where $I = [0, 1]$ (see [10] and [11]).

Let us now turn to the radial cluster sets of a continuous function f defined on $\mathbf{B}_{n+1}$ into $\mathbf{S}^k$. Assign to each point z of the boundary $\mathbf{S}^n$ of $\mathbf{B}_{n+1}$ the set

$$f_R(z) = \cap\{\mathrm{Cl}\,(\{f(rz) : \delta \leq r < 1\}) : 0 < \delta < 1\}$$

called the *radial cluster set* of f at z, where Cl denotes the closure operator in $\mathbf{S}^k$. The set $f_R(z)$ is a nonempty subcontinuum of $\mathbf{S}^k$. The resulting function

$$f_R : \mathbf{S}^n \to C(\mathbf{S}^k)$$

is called the *radial cluster-set function* of f. It is proved in [6] that f_R is a Baire class 2 function and that there are continuous functions f for which f_R is not of Baire class 1. Of course, when $n = 1$ and $k = 2$, the classical complex analysis case results.

Finally, the Lebesgue measurability of $\phi : \mathbf{S}^n \to C(\mathbf{S}^k)$ is defined in the usual way, that is, $\phi^{-1}[F]$ is Lebesgue measurable for each closed set F of $C(\mathbf{S}^k)$.

2. Preliminary Lemmas. The proof of our theorem, which is given in Section 3, will rely heavily on the existence of certain homotopies. This section is devoted to these existence lemmas.

The first lemma is Lemma 5.7 of [6]. The statement of the lemma will require the use of the stereographic projection π in $\mathbb{R}^{k+1}$ of $\mathbf{S}^k \setminus \{\infty\}$ onto $\mathbb{R}^k$. Here, $\mathbb{R}^k$ is identified with the k-dimensional coordinate hyperplane of $\mathbb{R}^{k+1}$ formed by setting the last coordinate equal to 0. We shall denote the Lipschitz constant of π^{-1} by M.

Lemma 1. *Suppose that $\varepsilon > 0$. If $g : \mathbf{S}^n \to \mathbb{R}^k$ and $\phi : \mathbf{S}^n \to C(\mathbf{S}^k)$ are continuous, then there exists a homotopy $\alpha : \mathbf{S}^n \times I \to \mathbb{R}^k$ such that, for all z in $\mathbf{S}^n$,*

(i) $\alpha(z, 0) = \alpha(z, 1) = g(z)$, *and*

(ii) $D(\pi^{-1}[\alpha(z, I)], \phi(z)) < 2\operatorname{dist}(\pi^{-1}(g(z)), \phi(z)) + \varepsilon M$.

From [6, Lemma 5.2] we infer the next lemma.

Lemma 2. *Suppose that $\varepsilon > 0$, that E is a compact, totally disconnected subset of $\mathbf{S}^n$ and that h_0 and h_1 are continuous mappings of $\mathbf{S}^n$ into $\mathbf{S}^k \setminus \{\infty\}$. Let*

$$K = \{z \in \mathbf{S}^n : |h_0(z) - h_1(z)| \le 1\}.$$

Then, there exists a homotopy $\beta : \mathbf{S}^n \times I \to \mathbf{S}^k \setminus \{\infty\}$ such that

(i) $\beta(z, 0) = h_0(z)$ *and* $\beta(z, 1) = h_1(z)$ *for z in $\mathbf{S}^n$, and*

(ii) $|\operatorname{diam}(\beta(z, I)) - |h_0(z) - h_1(z)|| < \varepsilon$ *for z in $K \cap E$.*

3. Proof of the Theorem. Let us begin with the homeomorphism H of $C(\mathbf{S}^k)$ onto I^{ω_0} given by the theorem of Curtis and Schori. The p-th coordinate H_p of H is a continuous function of $C(\mathbf{S}^k)$ into I. Also, a function ϕ from a space X into $C(\mathbf{S}^k)$ is continuous if and only if $H_p \circ \phi$ are continuous for all p. Consequently, we can prove the following lemma.

Lemma 3. *Let $\phi : \mathbf{S}^n \to C(\mathbf{S}^k)$ be a Lebesgue measurable function and $\varepsilon > 0$. Then, there exists a closed, totally disconnected subset E of $\mathbf{S}^n$ such that the n-dimensional Lebesgue measure $\mu(\mathbf{S}^n \setminus E)$ does not exceed ε and ϕ restricted to E, is continuous.*

Proof. For each p, the function $H_p \circ \phi$ is real-valued. By classical real function theory, there exists a closed subset E_p of $\mathbf{S}^n$ such that $\mu(\mathbf{S}^n \setminus E_p) < 2^{-(p+1)}\varepsilon$ and $H_p \circ \phi$ restricted to E_p is continuous. Since μ is regular measure and $\mathbf{S}^n$ is locally Euclidean, we may further assume that E_p is totally disconnected. The set $E = \cap\{E_p : p \geq 1\}$ is the required set. $\Diamond$

Proof of the Theorem. By Lemma 3, there is a sequence $\{F_j\}$ of closed, totally disconnected subsets of $\mathbf{S}^n$ such that $\phi|F_j$, the restriction of ϕ to F_j, is continuous and $\mu(\mathbf{S}^n \setminus F) = 0$ where F is the union of $\{F_j\}$. Since the members of the collection $\{F_j\}$ are compact, totally disconnected sets, we may assume also that the collection is disjointed. By Michael's Theorem [14, Th. 2], the continuous set-valued function $\phi|F_j$ has a continuous selection $s_j : F_j \to \mathbf{S}^k$, that is, s_j is a continuous function such that $s_j(z) \in \phi|F_j(z)$ for $z \in F_j$. From [13, pp. 74–80], we infer for each j the existence of a sequence of continuous functions $s_{jm} : F_j \to \mathbf{S}^k \setminus \{\infty\}$, $m = 1, 2, \ldots$, such that $|s_{jm}(z) - s_j(z)| < 1/(2m)$ for all z in F_j.

For each m, let $G_m = \cup\{F_j : j \leq m\}$. We have already mentioned that the Curtis-Schori Theorem gives us the fact that $C(\mathbf{S}^k)$ is homeomorphic to the Hilbert cube. Consequently, the Tietze Extension Theorem can be applied to get a continuous extension $\phi_m : \mathbf{S}^n \to C(\mathbf{S}^k)$ of $\phi|G_m$ for each m. Next, for each m, let $h_m : G_m - \to \mathbf{S}^k \setminus \{\infty\}$ be the continuous function defined by $h_m(z) = s_{jm}(z)$ for z in F_j and $1 \leq$ $\leq j \leq m$. As the set G_m is compact, by the Tietze Extension Theorem, $h_m : G_m \to \mathbf{S}^k \setminus \{\infty\}$ also has a continuous extension to $\mathbf{S}^n$ which will be denoted again by h_m. Thus, for each m, there is a pair of continuous maps $\phi_m : \mathbf{S}^n \to C(\mathbf{S}^n)$ and $h_m : \mathbf{S}^n \to \mathbf{S}^k \setminus \{\infty\}$ with the properties:

$$\phi_m(z) = \phi(z) \quad \text{for} \quad z \quad \text{in} \quad G_m\,,$$
$$\operatorname{dist}(h_m(z), \phi(z)) < 1/m \quad \text{for} \quad z \quad \text{in} \quad G_m\,,$$
$$|h_m(z) - h_{m+1}(z)| < 1/m \quad \text{for} \quad z \quad \text{in} \quad G_m\,.$$

We apply Lemma 1 to ϕ_m and $g_m = \pi \circ h_m$ to get a homotopy $\alpha_m :$ $: \mathbf{S}^n \times I \to \mathbb{R}^k$ such that, for all z in $\mathbf{S}^n$,

$$\pi^{-1} \circ \alpha_m(z, 0) = \pi^{-1} \circ \alpha_m(z, 1) = h_m(z)$$

and

$$D(\pi^{-1} \circ \alpha_m(z, I), \phi_m(z)) < 2\operatorname{dist}(h_m(z), \phi_m(z)) + M/m\,.$$

Next, we apply Lemma 2 to h_m and h_{m+1} to get a homotopy β_m :

$\mathbf{S}^n \times I \to \mathbf{S}^k \setminus \{\infty\}$ such that, for all z in $\mathbf{S}^n$,

$$\beta_m(z,0) = h_m(z), \quad \beta_m(z,1) = h_{m+1}$$

and

$$|\operatorname{diam}(\beta_m(z,I)) - |h_m(z) - h_{m+1}(z)|| < 1/m .$$

Now we shall piece together the homotopies $\pi^{-1} \circ \alpha_m$ and β_m to get the desired function $f : \mathbf{B}_{n+1} \to \mathbf{S}^k \setminus \{\infty\}$. Let $\{r_m\}$ and $\{r'_m\}$ be increasing sequences of positive numbers converging to 1 with $r_m < r'_m < r_{m+1}$. On the closed set $\{x \in \mathbf{B}_{n+1} : r_m \le |x| \le r'_m\}$ we define f by rescaling the homotopy $\pi^{-1} \circ \alpha_m$ in the obvious manner, and on the closed set $\{x \in \mathbf{B}_{n+1} : r'_m \le |x| \le r_{m+1}\}$ we define f by rescaling the homotopy β_m in the obvious manner. This defines f on the relatively closed set $\{x \in \mathbf{B}_{n+1} : r_1 \le |x| < 1\}$ of $\mathbf{B}_{n+1}$. The Tietze Extension Theorem applied to the closed set $\{x \in \mathbf{B}_{n+1} : |x| \le r_1\}$ will complete the definition of the continuous function f on $\mathbf{B}_{n+1}$.

Let us verify that $\Gamma_m(z) = \{f(rz) : r_m \le r \le r_{m+1}\}$, $m = 1, 2, \ldots$, converges uniformly to $\phi(z)$ on G_j for each j. To this end, let $m > j$ and $z \in G_j$. Since G_j is contained in G_m, we obtain from the identity

$$\Gamma_m(z) = \pi^{-1} \circ \alpha_m(z, I) \cup \beta_m(z, I)$$

and the definition of the Hausdorff metric D the inequality

$$D(\Gamma_m(z), \pi^{-1} \circ \alpha_m(z, I)) \le \operatorname{diam}(\beta_m) .$$

Consequently,

$$\begin{aligned} D(\Gamma_m(z), \phi(z)) &\le D(\Gamma_m(z), \pi^{-1} \circ \alpha(z,I)) + D(\pi^{-1} \circ \alpha_m(z,I), \phi(z)) \le \\ &\le \operatorname{diam}(\beta_m(z,I)) + 2\operatorname{dist}(h_m(z), \phi(z)) + M/m \le \\ &\le |h_m(z) - h_{m+1}(z)| + 1/m + 2\operatorname{dist}(h_m(z), \phi(z)) + \\ &+ M/m < (4+M)/m . \end{aligned}$$

Thus, we have that $\Gamma_m(z)$ converges to $\phi(z)$ uniformly on G_j. Finally, let us show $f_R(z) = \phi(z)$ for each z in F. Each z in F is a member of G_j for some j. Clearly, for $p \ge m > j$, we have from the definition of the Hausdorff metric D that

$$D(\cup\{\Gamma_q(z) : m \le q \le p\}, \phi(z)) \le (4+M)/m .$$

Therefore,

$$D(\mathrm{Cl}(\{f(rz) : r_m \le r < 1\}),\, \phi(z)) \le (4+M)/m\,,$$

from which we conclude that $f_R(z) = \phi(z)$. Since $\mu(\mathbf{S}^n \setminus F) = 0$, we have that f_R is equal to ϕ Lebesgue almost everywhere on $\mathbf{S}^n$. $\Diamond$

Remark. The convergence of Γ_m to ϕ in the theorem is closely related to the concept of uniform convergence defined by Bagemihl and McMillan in [2]. Further investigations of this type of convergence can be found in [6] and [7]. The references [3], [4], [5], [8] and [9] contain discussions on radial limit behavior of continuous functions defined on an open ball.

Finally, consider the setting of classical complex variables. That is, $\mathbb{R}^2$ is identified with the set $\mathbb{C}$ of complex numbers and the unit disk and the unit circle are $\mathbf{B}_2$ and $\mathbf{S}^1$, respectively. Moreover, the set of extended complex numbers $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ becomes $\mathbf{S}^2$. By employing the Arakeljan Approximation Theorem [1] in the same manner as in [7], [8], [9] and [12], we can establish the following corollary. Since its proof is a straightforward modification of those in the above references, we shall not prove the corollary.

Corollary. *Let $\phi : \mathbf{S}^1 \to C(\hat{\mathbb{C}})$ be a Lebesgue measurable function. Then, there is an analytic function f from the unit disk $\{z \in \mathbb{C} : |z| < < 1\}$ into $\mathbb{C}$ and there is an increasing sequence of real numbers $\{r_m\}$ converging to 1 such that*

(i) *the radial cluster-set function f_R of f is equal to ϕ Lebesgue almost everywhere on $\mathbf{S}^1$, and*

(ii) *for each positive number ε there is a measurable set E such that the continuous functions $\Gamma_m : \mathbf{S}^1 \to C(\hat{\mathbb{C}})$, $m = 1, 2, \ldots$, defined by*

$$\Gamma_m(\xi) = \{f(r\xi) : r_m \le r \le r_{m+1}\}\,, \quad \xi \in \mathbf{S}^1\,,$$

converge uniformly to ϕ on E and $\mu(\mathbf{S}^1 \setminus E) < \varepsilon$.

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ON QUASI–CONTINUOUS FUNCTIONS HAVING DARBOUX PROPERTY

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Abstract: Some classes of quasi-continuous, Darboux like functions are studied. The maximal additive and multiplicative families for these classes are characterized. A necessary and sufficient condition for f to be the uniform limit of a sequence of quasi-continuous functions having the Darboux property is given.

1. Introduction. We shall consider the following families of real functions defined on some interval I:

$\mathcal{C}$onst – the class of all constant functions;

$\mathcal{C}$ – the class of all continuous functions;

$\mathcal{A}$ – the class of all almost continuous functions (in the sense of Stallings ([20]); $f : X \to Y$ is said to be almost continuous if for every open set $G \subset X \times Y$ containing f, there exists a continuous function $g : X \to Y$ lying entirely in G;

$\mathcal{C}$onn – the class of all connectivity functions; $f : X \to Y$ is a connectivity function if for every connected subset C of X, $f|C$ is a connected subset of $X \times Y$;

$\mathcal{D}$ – the class of all Darboux functions;

$\mathcal{B}_1$ – the family of all functions of the first class of Baire;

lsc(usc) – the class of all lower (upper) semicontinuous functions;

$\mathcal{M}$ – the class of Darboux functions f for which if x_0 is a right (left) hand sided point of discontinuity of f, then $f(x_0) = 0$ and there exists a sequence (x_n) such that $f(x_n) = 0$ and $x_n \searrow x_0$ ($x_n \nearrow x_0$) ([8] and [14]);

$\mathcal{Q}$ – the class of all quasi-continuous functions; a function $f : X \to Y$ is quasi-continuous at a point x_0 iff $x_0 \in \overline{\operatorname{int} f^{-1}(V)}$ for every neighbourhood V of $f(x_0)$ ([15]);

$\mathcal{U}_0(\mathcal{U})$ – the class of all functions defined on I such that for every subinterval $J \subset I$ (and for every set A of the cardinality less than the continuum) the set $f(J)$ (respectively $f(J \setminus A)$) is dense in the interval $[\inf f|J, \sup f|J]$ ([4]); it is remarked in [4] that in these definitions the interval $[\inf f|J, \sup f|J]$ can be replaced by the interval $[f(a), f(b)]$, where $J = (a, b)$;

$\mathcal{Y}$ – the family of all functions with the Young property, i.e. functions which are bilaterally dense in themselves ([21]); some authors call functions having this property peripherally continuous ([2], [9]). (We make no distinction between a function and its graph.)

The inclusions $\mathcal{A} \subsetneq \mathcal{C}\mathrm{onn} \subsetneq \mathcal{D}$ are noticed in [1], the inclusions $\mathcal{D} \subsetneq \mathcal{U} \subsetneq \mathcal{U}_0 \subsetneq \mathcal{Y}$ follow from [4]. The inclusion $\mathcal{M} \subset \mathcal{B}_1$ is remarked in [14]. Now we shall prove the inclusion $\mathcal{M} \subset \mathcal{Q}$.

Lemma 1. *If $f \in \mathcal{M}$ and x_0 is a point of right-hand (left-hand) sided discontinuity of f then there exists a sequence (x_n) of points at which f is right-hand sided or left-hand sided continuous with $f(x_n) = 0$ and $x_n \searrow x_0$ ($x_n \nearrow x_0$).*

Proof. Let us assume that f is right-hand sided discontinuous at some point x_0, $U = (x_0, x_0 + \varepsilon)$ for some $\varepsilon > 0$ and U contains no point of continuity of f at which f has the value zero. Observe that the set $B = \{x \in U : f(x) = 0\}$ is nowhere-dense and non-empty. Let (I_n) be a sequence of all components of the set $U \setminus \overline{B}$. Notice that $f(x) = 0$ for every $x \in \overline{B}$. Thus if $I_n = (a, b)$, then $f(a) = f(b) = 0$ and f is right-hand (left-hand) sided continuous at the point a (respectively, b). Hence there are points in $U \cap B$ at which f is right-hand or left-hand sided continuous. $\Diamond$

It follows easily from this lemma that for every point x_0 at which a function $f \in \mathcal{M}$ is discontinuous there exists a sequence (x_n) of continuity points of f such that $\lim_{n\to\infty} x_n = x_0$ and $\lim_{n\to\infty} f(x_n) = f(x_0)$, and this condition implies quasi-continuity of f at x_0 (see e.g. [10]).

Lemma 2. (a) *A function f is quasi-continuous and satisfies the Young condition iff for every $x_0 \in I$ there exist two sequences (x_n) and (z_n) of continuity points of f such that $x_n \nearrow x_0$, $z_n \searrow x_0$ and $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} f(z_n) = f(x_0)$ (this condition must be interpreted unilaterally for end-points of I).*

(b) *Let f be quasi-continuous. Then $f \in \mathcal{U}$ iff for each $x \in I$ the unilateral cluster sets of f at x are intervals and contain $f(x)$.*

Proof. (a) follows immediately from the fact that $f : I \to \mathbb{R}$ is quasi-continuous at some point x_0 iff there exists a sequence (x_n) of continuity points such that $\lim_{n\to\infty} x_n = x_0$ and $\lim_{n\to\infty} f(x_n) = f(x_0)$, i.e. $f|C(f)$ is $\mathfrak{c}$-dense in f, where we denote by $C(f)$ the set of all continuity points of f (see e.g. [10], Lemma 2). We can also write the following condition: $f \in \mathcal{QY}$ iff $f(x_0) \in C^-(f|C(f), x_0) \cap C^+(f|C(f), x_0)$ for each $x_0 \in I$. (By $C^-(f,x)$ and $C^+(f,x)$ we denote the left-hand and right-hand sided cluster sets of f at a point x.)

(b) follows from the fact that $f|C(f)$ is $\mathfrak{c}$-dense in f and the following characterization of the classes $\mathcal{U}_0$ and $\mathcal{U}$, which is proved in [4], theorems 3.1 and 3.2:

(i) $f \in \mathcal{U}_0$ iff for each $x \in I$ the unilateral cluster sets of f at x are intervals and contain $f(x)$;

(ii) $f \in \mathcal{U}$ iff $f \in \mathcal{U}_0$ and f is $\mathfrak{c}$-dense in itself. $\Diamond$

For the classes of real functions defined on an interval I we can state

$$\begin{array}{c} \mathcal{Q} \\ \cup\!\!\!| \\ \mathrm{Const} \subsetneq \mathcal{C} \subsetneq \mathcal{M} \subsetneq \mathcal{A} \subsetneq \mathrm{Conn} \subsetneq \mathcal{D} \subsetneq \mathcal{U} \subsetneq \mathcal{U}_0 \subsetneq \mathcal{Y}. \\ |\!\!\!\cap \\ \mathcal{B}_1 \end{array}$$

In the class $\mathcal{B}_1$ we have the following equalities:

$$\mathcal{AB}_1 = \mathrm{Conn}\,\mathcal{B}_1 = \mathcal{DB}_1 = \mathcal{U}_0\mathcal{B}_1 = \mathcal{UB}_1 = \mathcal{YB}_1 \quad \text{see [1] and [3].}$$

In the first part of the present paper we remark that in the class $\mathcal{Q}$ the following inclusions hold:

$$\mathcal{AQ} \subsetneq \mathrm{Conn}\,\mathcal{Q} \subsetneq \mathcal{DQ} \subsetneq \mathcal{U}_0\mathcal{Q} = \mathcal{UQ} \subsetneq \mathcal{YQ}.$$

Let $\mathcal{Z}$ be a class of real functions. We define the maximal additive (multiplicative, latticelike, respectively) class for $\mathcal{Z}$ as the class of all such functions $f \in \mathcal{Z}$, for which $f+g \in \mathcal{Z}$ ($fg \in \mathcal{Z}$ or $\max(f,g) \in \mathcal{Z}$ and $\min(f,g) \in \mathcal{Z}$, respectively) whenever $g \in \mathcal{Z}$. The adequate classes we denote by $\mathcal{M}_a(\mathcal{Z})$, $\mathcal{M}_m(\mathcal{Z})$ and $\mathcal{M}_\ell(\mathcal{Z})$. Moreover let $\mathcal{M}_{\min}(\mathcal{Z}) = \{f \in \mathcal{Z}\colon$ if $g \in \mathcal{Z}$ then $\min(f,g) \in \mathcal{Z}\}$ and $\mathcal{M}_{\max}(\mathcal{Z}) = \{f \in \mathcal{Z}\colon$ if $g \in \mathcal{Z}$ then $\max(f,g) \in \mathcal{Z}\}$. Note that $\mathcal{M}_\ell(\mathcal{Z}) = \mathcal{M}_{\min}(\mathcal{Z}) \cap \mathcal{M}_{\max}(\mathcal{Z})$.

The following equalities are known:

$\mathcal{K}$	$\mathcal{M}_a(\mathcal{K})$	$\mathcal{M}_m(\mathcal{K})$	$\mathcal{M}_{\max}(\mathcal{K})$	$\mathcal{M}_{\min}(\mathcal{K})$	$\mathcal{M}_\ell(\mathcal{K})$
$\mathcal{D}$	Const ([19])	Const ([19])	$\mathcal{D}$usc ([7])	$\mathcal{D}$lsc ([7])	$\mathcal{C}$
$\mathcal{DB}_1$	$\mathcal{C}$ ([3])	$\mathcal{M}$ ([8])	$\mathcal{D}$usc ([7])	$\mathcal{D}$lsc ([7])	$\mathcal{C}$
$\mathcal{A}$	$\mathcal{C}$ ([14])	$\mathcal{M}$ ([14])	?	?	$\mathcal{C}$ ([14])
$\mathcal{C}$onn	$\mathcal{C}$ ([14])	$\mathcal{M}$ ([14])	?	?	$\mathcal{C}$ ([14])

Recently D. Banaszewski and K. Banaszewski proved the following results:

$\mathcal{K}$	$\mathcal{M}_a(\mathcal{K})$	$\mathcal{M}_m(\mathcal{K})$	$\mathcal{M}_{\max}(\mathcal{K})$	$\mathcal{M}_{\min}(\mathcal{K})$	$\mathcal{M}_\ell(\mathcal{K})$
$\mathcal{Y}$	$\mathcal{C}$ ([23])	$\mathcal{M}$ ([23])	$\mathcal{C}$ ([23])	$\mathcal{C}$ ([23])	$\mathcal{C}$
$\mathcal{QDB}_1$	$\mathcal{C}$ ([22])	$\mathcal{M}$ ([22])	$\mathcal{QD}$usc ([22])	$\mathcal{QD}$lsc ([22])	$\mathcal{C}$

In the second part of the present paper we shall add next lines to this table, namely,

$\mathcal{QD}$	Const	Const	$\mathcal{QD}$usc	$\mathcal{QD}$lsc	$\mathcal{C}$
$\mathcal{QA}$	$\mathcal{C}$	$\mathcal{M}$	?	?	$\mathcal{C}$
$\mathcal{QC}$onn	$\mathcal{C}$	$\mathcal{M}$	?	?	$\mathcal{C}$

It is well-known that a uniform limit of Darboux functions can be a function without the Darboux property. It was proved in [4] that a function f is a uniform limit of Darboux functions iff $f \in \mathcal{U}$. Since the classes $\mathcal{B}_1$ and $\mathcal{U}$ are closed with respect to uniform limits and $\mathcal{DB}_1 = \mathcal{UB}_1$, the class $\mathcal{DB}_1$ is closed with respect to uniform limits too (see e.g, [3]). The class $\mathcal{Q}$ is closed with respect to this operation too, but the class $\mathcal{DQ}$ is not.

In the last part of this paper we shall prove that a function f is a uniform limit of quasi-continuous functions having Darboux property iff $f \in \mathcal{QU}$. Notice also that a real function defined on $\mathbb{R}$ is a pointwise limit of some sequence of functions from the class $\mathcal{QD}$ iff it is pointwise discontinuous ([12]).

2. We start with some universal construction of quasi - continuous functions having Darboux property. Let $A \subset \mathbb{R}$ be a set $\mathfrak{c}$ - dense in itself (where $\mathfrak{c}$ denotes the cardinality of the continuum) and let B be a subset of $\mathbb{R}$. Let $\mathcal{D}^*(A, B)$ denote the class of all functions $f : A \to B$ which take on every $y \in B$ in every non-empty interval of A (i.e. a set of the form $A \cap (a, b)$ for some $a, b \in \mathbb{R}$). It is well - known that the family $\mathcal{D}^*(A, B)$ is non - empty (see e.g. [5]).

Let $I = [0, 1]$, $C \subset I$ be the Cantor set and for each $n \in \mathbb{N}$ let $\mathcal{J}_n$ be the family of all components of the set $I \setminus C$ of the n - th order (i.e. such components of $I \setminus C$ which length is equal to 3^{-n}). Let $A = I \setminus \bigcup\{\overline{J} : J \in \bigcup_{n=1}^{\infty} \mathcal{J}_n\}$. Notice that this set is $\mathfrak{c}$ - dense in itself. Let (q_n) be a sequence of all rationals such that for every rational q the set $\{n : q_n = q\}$ is infinite. Then for a given function $\varphi \in \mathcal{D}^*(A, \mathbb{R})$ the function $f : I \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} \varphi(x) & \text{for } x \in A \\ q_n & \text{for } x \in \bigcup\{\overline{J}, J \in \mathcal{J}_n\},\ n \in \mathbb{N} \end{cases}$$

is quasi - continuous and has the Darboux property.

Now we shall employ this method to construct some example of a quasi - continuous function with the Darboux property but not connected. It is easy to find (by transfinite induction) a function $\varphi \in \mathcal{D}^*(A, \mathbb{R})$ such that $\varphi(x) \neq -x$ for each $x \in A$. We define a function $f : I \to \mathbb{R}$ in the following way:

$$f(x) = \begin{cases} \varphi(x) & \text{for } x \in A \\ q_n & \text{for } x \in \bigcup\{\overline{J} : J \in \mathcal{J}_n \text{ and } q_n \notin J\},\ n \in \mathbb{N} \\ x + 1 & \text{otherwise.} \end{cases}$$

Then $f \in \mathcal{QD}$ and $\operatorname{rng} f = \mathbb{R}$ but $f \cap \{(x, x) : x \in I\} = \emptyset$ and therefore f is not connected.

Notice also that the function f which was constructed by J. Jastrzębski in [13] is quasi - continuous and connected but not almost continuous. Moreover, the function $g : I \to \mathbb{R}$ defined by $g(x) = q_n$ for $x \in \bigcup\{\overline{J} : J \in \mathcal{J}_n\}$ and $g(x) = 0$ otherwise, belongs to the class $\mathcal{QU}$ but g does not have the Darboux property. Finally, the function $h : I \to \mathbb{R}$, $h(x) = \sin(1/x)$ for $x \in (0, 1]$ and $h(0) = 0$ is quasi - continuous and almost continuous but h is not continuous. Thus all inclusions $\mathcal{C} \subsetneq \mathcal{AQ} \subsetneq \mathcal{C}\mathrm{onn}\,\mathcal{Q} \subsetneq \mathcal{DQ} \subsetneq \mathcal{UQ}$ are proper. The equality

$\mathcal{QU}_0 = \mathcal{UQ}$ follows from Lemma 2 (b). Now let $(I_n)_n$ be a sequence of all components of the complement of the Cantor set such that the unions $\bigcup_{n=1}^{\infty} I_{2n+1}$ and $\bigcup_{n=1}^{\infty} I_{2n}$ are dense in C and let f be the characteristic function of the set $C \cup \bigcup_{n=1}^{\infty} I_{2n}$. Then $f \in \mathcal{YQ} \setminus \mathcal{U}_0\mathcal{Q}$.

3. Theorem 1. *Assume that $I = [0,1]$, $X, Y \subset \mathbb{R}$ are intervals, a, b, c are reals such that $a < b < c$ and $F : X \times Y \to \mathbb{R}$ satisfies the following conditions:*

(1) *$F_x : Y \to \mathbb{R}$, $F_x(y) = F(x,y)$ is continuous and $(F_x)^{-1}(b)$ is countable for each $x \in X$;*

(2) *$F^y : X \to \mathbb{R}$, $F^y(x) = F(x,y)$ is continuous and $(F^y)^{-1}(b)$ is countable for each $y \in Y$;*

(3) card$\{x \in X\colon \forall y \in Y\ F(x,y) \neq a\} < 2^\omega$;

(4) card$\{x \in X : \forall y \in Y\ F(x,y) \neq c\} < 2^\omega$.

Then for every non-constant, continuous function $f : I \to X$ there exists a Lebesgue measurable, quasi-continuous function $g : I \to Y$ with the Darboux property such that $F(f,g)$ does not have the Darboux property (compare with [24]).

Proof. Notice that the following condition follows from (1):

$$\text{(1')} \qquad \forall x \in X \quad \exists y(x) \in Y \quad F(x, y(x)) \neq b.$$

Let $f : I \to X$ be a non-constant, continuous function. Let D be the set of all points $x \in X$ for which the set $f^{-1}(x)$ has a positive measure. Then the set D is countable and it follows from (1) that the set $\{y \in Y\colon \exists x \in D\ F(x,y) = b\}$ is countable too. Thus there exists a countable, dense set $P \subset Y$ such that

$$\text{(5)} \qquad \forall x \in D \quad \forall p \in P \quad F(x,p) \neq b.$$

Moreover, we have also the following property

$$\text{(6)} \qquad \forall p \in P \quad m(\{z : F(f(z), p) = b\}) = 0,$$

where the symbol $m(A)$ denotes the Lebesgue measure of A. In fact, $\{z : F(f(z),p) = b\} = \bigcup\{f^{-1}(x) : F(x,p) = b\}$ and it follows from (2) and (5) that this union has a measure zero.

Let (p_n) be a sequence of all points of P such that for any $p \in P$ the set $\{n : p_n = p\}$ is infinite.

Now we shall modify the construction of quasi-continuous function having Darboux property from the second part of this paper. We choose (inductively) a sequence of finite families of open intervals $(\mathcal{J}_n)_{n=0}^{\infty}$ such that:

(7) $$\mathcal{J}_0 = \{\emptyset\};$$

(8) if L is a component of the set $I \setminus \bigcup\{J : J \in \mathcal{J}_k,\ k \leq n\}$ then there exists some $K \in \mathcal{J}_{n+1}$ such that $K \subset L$, and
$$m(L) > \sum_{K \in \mathcal{J}_{n+1}, K \subset L} m(K) \geq m(L)/3;$$

(9) $F(f(x), p_n) \neq b$ for each $x \in \bigcup\{\overline{J} : J \in \mathcal{J}_n\}$;

(10) if $J \in \mathcal{J}_n$ and K is an interval on which f is constant and $K \cap \overline{J} \neq \emptyset$, then $K \subset \overline{J}$;

(11) if d, e are the end-points of some interval $J \in \mathcal{J}_n$ then $f(e) \neq f(d)$.

Such a choice is possible. Indeed, let us assume that are have chosen a family $\mathcal{J}_n$. Let $L \in I \setminus \bigcup_{k \leq n} \bigcup \mathcal{J}_k$. Then the set $Z = L \cap \{z \in I : F(f(z), p_{n+1}) = b\}$ is closed and nowhere-dense. Moreover, it follows from (6) that Z has a measure zero. Let (L_m) be a finite sequence of components of $L \setminus Z$ such that $\sum_m m(L_m) \geq 2m(L)/3$. By (10), $f|L_m$ is constant on no neighbourhood of ends of L_m (for each m). Thus for each m we can choose a subinterval K_m of L_m which satisfies (9), (10) and (11) and with $m(K_m) \geq m(L_m)/2$. Finally we put $\mathcal{J}_{n+1} = \{K_m \subset L : L \in \mathcal{J}_n\}$ and observe that this family satisfies all conditions (8), (9), (10) and (11).

Now let $A = I \setminus \bigcup\{\overline{K} : K \in \mathcal{J}_n,\ n \in \mathbb{N}\}$. Evidently this set is $\mathfrak{c}$-dense in itself, nowhere-dense and has a measure zero. Additionally, it follows from (11) that f is not constant on any interval of A. Let $C = \overline{A}$. Then $C \setminus A$ is countable and f is constant on no interval of C. Hence we have the following property:

(12) for each subinterval J of I, if $J \cap A \neq \emptyset$ then the set $f(J \cap A)$ has the cardinality of the continuum.

Indeed, let us suppose that J is a closed subinterval of I such that $J \cap A \neq \emptyset$ and the set $f(J \cap A)$ has the cardinality less than the continuum. Because the set $C \setminus A$ is countable, the set $f(J \cap C)$ has the cardinality less than the continuum too. Since f is continuous and

$J \cap C$ is a compact set, the set $f(J \cap C)$ is closed and consequently, it is countable. Let (y_n) be a sequence of all points of $f(J \cap C)$ and for each $n \in \mathbb{N}$ let $C_n = J \cap C \cap f^{-1}(y_n)$. By (11) the sets C_n are nowhere-dense in $J \cap C$ and $\bigcup_{n=1}^{\infty} C_n = J \cap C$, which contradicts the Baire theorem. Therefore (12) holds.

Lemma 3. *If a set A is $\mathfrak{c}$-dense in itself and $f : A \to X$ is a continuous function which satisfies the condition* (12), *then there exists a function $\varphi \in \mathcal{D}^*(A, Y)$ such that $F(f(x), \varphi(x)) \neq b$ for each $x \in A$, $F(f(x_1), \varphi(x_1)) = a$ and $F(f(x_2), \varphi(x_2)) = c$ for some $x_1, x_2 \in A \cap J$ and each interval J for which $A \cap J \neq \emptyset$* (compare e.g. with [16]).

Proof (of Lemma 3). Let (I_n) be a sequence of all basis sets in A. We list all elements of the family $(I_n) \times Y$ in the sequence $(I_\gamma \times \{y_\gamma\})_{\gamma<2^\omega}$ and choose (by induction) sequences $s_\gamma, t_\gamma, w_\gamma \in I_\gamma$, $t'_\gamma, w'_\gamma \in Y$ such that:

(13) $s_\gamma \in I_\gamma \setminus \{s_\beta, t_\beta, w_\beta : \beta < \gamma\}$ and $F(f(s_\gamma), y_\gamma) \neq b$,

(14) $t_\gamma \in I_\gamma \setminus (\{s_\beta, t_\beta, w_\beta : \beta < \gamma\} \cup \{s_\gamma\})$ and $F(f(t_\gamma), t'_\gamma) = a$,

(15) $w_\gamma \in I_\gamma \setminus (\{s_\beta, t_\beta, w_\beta : \beta < \gamma\} \cup \{s_\gamma, t_\gamma\})$ and $F(f(w_\gamma), w'_\gamma) = c$.

Now we define a function $\varphi : A \to Y$ by

$$\varphi(x) = \begin{cases} y_\gamma & \text{for } x = s_\gamma, \\ t'_\gamma & \text{for } x = t_\gamma, \\ w'_\gamma & \text{for } x = w_\gamma, \\ y(x) & \text{otherwise,} \end{cases}$$

where $\gamma < 2^\omega$ and $y(x)$ is defined in (1′). It is easy to verify that the function φ has the required properties. The proof of Lemma 3 is completed.

Now we can finish the proof of Th. 1. We define a function $g : I \to Y$ by $g(x) = p_n$ for $x \in \cup\{\overline{J} : J \in \mathcal{J}_n\}$, $n = 1, 2, \ldots$, and $g(x) = \varphi(x)$ for $x \in A$. It is easy to see that the function g is quasi-continuous, measurable and has the Darboux property. Instead the function $F(f, g)$ takes the values a, c and does not take the value b, and consequently, $F(f, g)$ does not have the Darboux property. $\Diamond$

Corollary 1. (1) *If we put $X = Y = \mathbb{R}$, $F(x, y) = x + y$, $a = -1$, $b = 0$ and $c = 1$, then we obtain the following inclusion: $\mathcal{M}_a(\mathcal{QD}) \cap \mathcal{C} \subset \mathcal{C}$*onst. Since the opposite inclusion is clear, *we have the equality* $\mathcal{M}_a(\mathcal{QD}) \cap \mathcal{C} = \mathcal{C}$onst.

(2) *We have also the equality $\mathcal{M}_m(\mathcal{QD}) \cap \mathcal{C} = \mathcal{C}$onst. The inclu-*

sion "$\supset$" is trivial. The second inclusion follows from Th. 1, if we put $X = Y = \mathbb{R}$, $F(x,y) = x \cdot y$, $a = 0$, $b = 1$ and $c = 2$.

(3) Similarly we can conclude that

$$\{f : I \to \mathbb{R} : f \in \mathcal{C} \textit{ and } f/g \in \mathcal{D} \textit{ for each } g \in \mathcal{QD},\ g : I \to \mathbb{R}_+\} = \mathcal{C}\mathrm{onst}$$

and

$$\{f : I \to \mathbb{R}_+ : f \in \mathcal{C} \textit{ and } g/f \in \mathcal{D} \textit{ for each } g \in \mathcal{QD}\} = \{f : I \to \mathbb{R}_+ : f \in \mathcal{C}\mathrm{onst}\}.$$

Lemma 4. *Let us assume that* $f \in \mathcal{QY}\mathrm{usc}$ ($f \in \mathcal{QY}\mathrm{lsc}$) *and* $g \in \mathcal{QU}$. *Then* $\max(f,g) \in \mathcal{Q}$ ($\min(f,g) \in \mathcal{Q}$). (Notice that the assumption $f, g \in \mathcal{Y}$ is necessary; we have $\mathcal{M}_{\min}(\mathcal{Q}) = \mathcal{M}_{\max}(\mathcal{Q}) = \mathcal{C}$ ([17])).

Proof. Observe that for quasi-continuous functions f, g the set $C(f) \cap \cap C(g)$ is residual in I and $\max(f,g)$ is continuous at every point from this set. Thus it is enough to prove that for each $x \in I$ there exists a sequence (x_n) of points of the set $C(f) \cap C(g)$ such that $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} \max(f,g)(x_n) = \max(f,g)(x)$. Let $x_0 \in I$. We shall consider three cases.

(a) $f(x_0) \geq g(x_0)$ and there exists a sequence (x_n) of points of $C(f) \cap C(g)$ such that $\lim_{n\to\infty} x_n = x_0$, $\lim_{n\to\infty} f(x_n) = f(x_0)$ and $f(x_n) \geq \geq g(x_n)$ for each $n \in \mathbb{N}$. Then $\lim_{n\to\infty} \max(f,g)(x_n) = \lim_{n\to\infty} f(x_n) = = f(x_0) = \max(f,g)(x_0)$ and therefore $\max(f,g)$ is quasi-continuous at x_0.

(b) $f(x_0) \geq g(x_0)$ and $f(x_n) < g(x_n)$ (if n is sufficiently big) for every sequence (x_n) of points of $C(f) \cap C(g)$ such that $\lim_{n\to\infty} x_n = x_0$ and $\lim_{n\to\infty} f(x_n) = f(x_0)$. Since $f \in \mathcal{QY}$, there exists a sequence (x_n) such that $x_n \in C(f) \cap C(g)$, $\lim_{n\to\infty} x_n = x_0$ and $\lim_{n\to\infty} f(x_n) = f(x_0)$. We can assume that $\lim_{n\to\infty} g(x_n)$ exists (finite or infinite). Then $\lim_{n\to\infty} g(x_n) \geq \geq \lim_{n\to\infty} f(x_n) = f(x_0)$. Since $g \in \mathcal{U}$, $C(g, x_0)$ is an interval ([4]) and therefore there exists a sequence (x'_n) such that $x'_n \in C(f) \cap \cap C(g)$, $\lim_{n\to\infty} x'_n = x_0$ and $\lim_{n\to\infty} g(x'_n) = f(x_0)$. Since f is upper semi-continuous, $\overline{\lim_{n\to\infty}} f(x'_n) \leq f(x_0)$. Hence $\overline{\lim_{n\to\infty}} \max(f,g)(x'_n) = f(x_0)$ and there exists a subsequence (x'_{n_k}) of (x'_n) such that $\lim_{k\to\infty} \max(f,g)(x'_{n_k}) = f(x_0)$ and consequently, $\max(f,g)$ is quasi-continuous at the point x_0.

(c) $f(x_0) < g(x_0)$. Then there exists a sequence (x_n) of points such that $x_n \in C(f) \cap C(g)$, $\lim_{n\to\infty} x_n = x_0$, $\lim_{n\to\infty} g(x_n) = g(x_0) > f(x_0)$ and $g(x_n) > f(x_0)$ for each $n \in \mathbb{N}$. Since f is upper semicontinuous, $\overline{\lim_{n\to\infty}} f(x_n) \le f(x_0)$ and consequently, $\lim_{n\to\infty} \max(f,g)(x_n) = \lim_{n\to\infty} g(x_n) = g(x_0) = \max(f,g)(x_0)$. Hence $\max(f,g)$ is quasi-continuous at the point x_0. ◊

Lemma 5. *If $f \in \mathcal{M}$ and $g \in \mathcal{QY}$ then the product fg is quasi-continuous.*

Proof. Of course it is sufficient to prove that fg is quasi-continuous at every point x_0 at which f is not continuous. Then $f(x_0) = 0$ and, by Lemma 1, if f is not continuous at x_0 from the left (from the right) then there exists a sequence (x_n) of points at which f is unilaterally continuous such that $f(x_n) = 0$ for each n and $x_n \nearrow x_0$ ($x_n \searrow x_0$). For every $n \in \mathbb{N}$ we choose a unilateral neighbourhood U_n of x_n such that $|f(x)| < 1/(n \cdot |g(x_n)|)$ if $g(x_n) \neq 0$ and $|f(x)| < 1/n$ whenever $g(x_n) = 0$, for each $x \in U_n$. Since $g \in \mathcal{QY}$, Lemma 2 (a) implies that for every $n \in \mathbb{N}$ there exists $z_n \in U_n \cap (x_n - 1/n, x_n + 1/n) \cap C(f) \cap C(g)$ for which $|g(z_n) - g(x_n)| < \varepsilon_n$, where $\varepsilon_n = 1$ if $g(x_n) = 0$ and $\varepsilon_n = |g(x_n)|$ otherwise. Then fg is continuous at each z_n, $\lim_{n\to\infty} z_n = \lim_{n\to\infty} x_n = x_0$ and $\lim_{n\to\infty} (fg)(z_n) = 0 = (fg)(x_0)$. This implies the quasi-continuity of fg at the point x_0. ◊

We shall apply also the following two lemmata, which were proved in [14].

Lemma 6. *Let Φ be some property of functions, let $\mathcal{X}_1$ be the class of all functions $f : X \to \mathbb{R}$ (where X is a topological space) possessing the property Φ and let $\mathcal{X}_2$ be the class of all functions $g : X \to \mathbb{R} \times \mathbb{R}$ possessing the same property Φ. Let the classes $\mathcal{X}_1$ and $\mathcal{X}_2$ fulfil the following conditions:*

(i) *if $f \in \mathcal{X}_2$ and $g \in \mathcal{C}$ ($g : \mathbb{R}^2 \to \mathbb{R}$), then $g \circ f \in \mathcal{X}_1$;*

(ii) *if $f \in \mathcal{X}_1$ and $g \in \mathcal{C}$ ($g : X \to \mathbb{R}$), then $h = (f,g) \in \mathcal{X}_2$, where $h : z \mapsto (f(x), g(x))$ for $x \in X$.*

Then $\mathcal{C} \subseteq \mathcal{M}_a(\mathcal{X}_1) \cap \mathcal{M}_m(\mathcal{X}_1) \cap \mathcal{M}_\ell(\mathcal{X}_1)$.

Lemma 7. *Let $\mathcal{X}$ be a subfamily of $\mathcal{U}_0$ and let the following conditions hold:*

(iii) *if $f : I \to \mathbb{R}$, $f \in \mathcal{X}$ and J is a subinterval of an interval I, then $f|J \in \mathcal{X}$;*

(iv) *if $h : (a,b) \to \mathbb{R}$, $h \in \mathcal{X}$, $y \in C^+(h,a)$ and $z \in C^-(h,b)$, then the functions $h_1 : [a,b) \to \mathbb{R}$, $h_2 : (a,b] \to \mathbb{R}$ and $h_3 : [a,b] \to \mathbb{R}$ belong* *to* $\mathcal{X}$, *where* $h_1 = h \cup \{(a,y)\}$, $h_2 = h \cup \{(b,z)\}$, $h_3 = h_1 \cup h_2$;

(v) *if* $I \subset \mathbb{R}$ *is an interval,* $a \in I$ *and* $f|(I \cap (-\infty, a]) \in \mathcal{X}$, $f|(I \cap \cap[a,\infty)) \in \mathcal{X}$, *then* $f \in \mathcal{X}$;

(vi) $\mathcal{C}\mathrm{onst} \subseteq \mathcal{M}_a(\mathcal{X})$ *and* $-1 \in \mathcal{M}_m(\mathcal{X})$.

Then $\mathcal{M}_a(\mathcal{X}) \subseteq \mathcal{C}$, $\mathcal{M}_{\min}(\mathcal{X}) \subseteq \mathcal{X}\mathrm{lsc}$ *and* $\mathcal{M}_{\max}(\mathcal{X}) \subseteq \mathcal{X}\mathrm{usc}$ *(hence* $\mathcal{M}_\ell(\mathcal{X}) \subseteq \mathcal{C}$*).*

If moreover the class $\mathcal{X}$ *fulfils the additional condition*

(vii) *if* $f : I \to (0,\infty)$ *and* $f \in \mathcal{X}$ *then* $1/f \in \mathcal{X}$,

then also $\mathcal{M}_m(\mathcal{X}) \subseteq \mathcal{M}$.

Let us observe that the family $\mathcal{X} = \mathcal{QD}$ does not satisfy the assumptions of Lemma 6 but it satisfies all assumptions of Lemma 7. Thus

(a) $\mathcal{M}_a(\mathcal{QD}) \subseteq \mathcal{C}$,

(b) $\mathcal{M}_{\min}(\mathcal{QD}) \subseteq \mathcal{QD}\mathrm{lsc}$ and $\mathcal{M}_{\max}(\mathcal{QD}) \subseteq \mathcal{QD}\mathrm{usc}$,

(c) $\mathcal{M}_m(\mathcal{QD}) \subseteq \mathcal{M}$.

Now we can prove the following theorem.

Theorem 2. *We have the following equalities:*

(1) $\mathcal{M}_a(\mathcal{QD}) = \mathcal{C}\mathrm{onst}$,

(2) $\mathcal{M}_m(\mathcal{QD}) = \mathcal{C}\mathrm{onst}$,

(3) $\mathcal{M}_{\min}(\mathcal{QD}) = \mathcal{QD}\mathrm{lsc}$ *and* $\mathcal{M}_{\max}(\mathcal{QD}) = \mathcal{QD}\mathrm{usc}$.

Proof. Evidently, we have $\mathcal{C}\mathrm{onst} \subseteq \mathcal{M}_a(\mathcal{QD}) \cap \mathcal{M}_m(\mathcal{QD})$. The inclusion $\mathcal{M}_a(\mathcal{QD}) \subseteq \mathcal{C}\mathrm{onst}$ follows from Lemma 7 and from Cor. 1 (1). Hence $\mathcal{M}_a(\mathcal{QD}) = \mathcal{C}\mathrm{onst}$.

Now we shall prove that $\mathcal{M}_m(\mathcal{QD}) \subseteq \mathcal{C}\mathrm{onst}$. It is enough to prove that $\mathcal{M}_m(\mathcal{QD}) \subseteq \mathcal{C}$ and to use Cor. 1(2). Fix $f \in \mathcal{M}_m(\mathcal{QD})$ and suppose that f is not continuous, i.e. $I \setminus C(f) \neq \emptyset$. Since $f \in \mathcal{M}$, the set $A = I \setminus C(f)$ is nowhere-dense, $f(x) = 0$ for $x \in \overline{A}$ and f is continuous on every component of the set $I \setminus \overline{A}$. Since f is not continuous, f is not constant. Since $f \in \mathcal{D}$, $\mathrm{rng}(f)$ has the cardinality equals the continuum and consequently there exists a component J of $I \setminus \overline{A}$ such that $f|J$ is continuous and not constant. We apply Cor. 1(2) and obtain some quasi-continuous function $g : J \to \mathbb{R}$ having the Darboux property for which $f \cdot g \notin \mathcal{D}$. Thus there exists a function h defined on the interval I such that $h \in \mathcal{QD}$ and $f \cdot h \notin \mathcal{D}$, which contradicts to $f \in \mathcal{M}_m(\mathcal{QD})$.

Now we shall prove (3). By Lemma 7 it follows that we need to prove the following two inclusions: $\mathcal{QD}\mathrm{usc} \subseteq \mathcal{M}_{\max}(\mathcal{QD})$ and $\mathcal{QD}\mathrm{lsc} \subseteq \subseteq \mathcal{M}_{\min}(\mathcal{QD})$. To prove that $\mathcal{QD}\mathrm{usc} \subseteq \mathcal{M}_{\max}(\mathcal{QD})$ let $f \in \mathcal{QD}\mathrm{usc}$ and $g \in \mathcal{QD}$. Since $\mathcal{M}_{\max}(\mathcal{D}) = \mathcal{D}\mathrm{usc}$, $\max(f,g) \in \mathcal{D}$. By Lemma 4 it follows that $\max(f,g) \in \mathcal{Q}$ and therefore $\max(f,g) \in \mathcal{QD}$. The proof that $\mathcal{QD}\mathrm{lsc} \subseteq \mathcal{M}_{\min}(\mathcal{QD})$ is similar. $\Diamond$

Observe now observe that the family $\mathcal{Q}$ satisfies all assumptions of Lemma 6 (see [18]) and therefore,

$$\mathcal{C} \subseteq \mathcal{M}_a(\mathcal{Q}) \cap \mathcal{M}_m(\mathcal{Q}) \cap \mathcal{M}_{\max}(\mathcal{Q}) \cap \mathcal{M}_{\min}(\mathcal{Q}) \text{ ([11], [17]).}$$

We have also the inclusion

$$\mathcal{C} \subseteq \mathcal{M}_a(\mathcal{A}) \cap \mathcal{M}_m(\mathcal{A}) \cap \mathcal{M}_{\max}(\mathcal{A}) \cap \mathcal{M}_{\min}(\mathcal{A}) \text{ ([14])}$$

and consequently,

$$\mathcal{C} \subseteq \mathcal{M}_a(\mathcal{QA}) \cap \mathcal{M}_m(\mathcal{QA}) \cap \mathcal{M}_{\max}(\mathcal{QA}) \cap \mathcal{M}_{\min}(\mathcal{QA}).$$

Similarly,

$$\mathcal{C} \subseteq \mathcal{M}_a(\mathcal{QC}\mathrm{onn}) \cap \mathcal{M}_m(\mathcal{QC}\mathrm{onn}) \cap \mathcal{M}_{\max}(\mathcal{QC}\mathrm{onn}) \cap \mathcal{M}_{\min}(\mathcal{QC}\mathrm{onn}).$$

Moreover, the families $\mathcal{QA}$ and $\mathcal{QC}\mathrm{onn}$ satisfy all assumptions of Lemma 7. Thus we obtain the following theorem.

Theorem 3. *Let $\mathcal{K} = \mathcal{A}$ or $\mathcal{K} = \mathcal{C}\mathrm{onn}$. Then the following equalities hold:*

$$\mathcal{M}_a(\mathcal{QK}) = \mathcal{C}, \quad \mathcal{M}_\ell(\mathcal{QK}) = \mathcal{C} \text{ and } \mathcal{M}_m(\mathcal{QK}) = \mathcal{M}.$$

Proof. The first two equalities follow immediately from lemmata 6 and 7. In the third equality it is sufficient to prove the inclusion $\mathcal{M} \subseteq \mathcal{M}_m(\mathcal{QK})$. Fix $f \in \mathcal{M}$ and $g \in \mathcal{QK}$. Since $\mathcal{M}_m(\mathcal{K}) = \mathcal{M}$ ([14]), $f \cdot g \in \mathcal{K}$. By Lemma 5 we obtain that $f \cdot g \in \mathcal{Q}$. Hence $f \cdot g \in \mathcal{KQ}$ and consequently $\mathcal{M} \subseteq \mathcal{M}_m(\mathcal{QK})$. $\Diamond$

Problem. For $\mathcal{K} \in \{\mathcal{A}, \mathcal{C}\mathrm{onn}\}$ find $\mathcal{M}_{\max}(\mathcal{QK})$ and $\mathcal{M}_{\min}(\mathcal{QK})$.

4. In this section we shall prove that the family $\mathcal{QU}$ is the uniform closure of the class of all quasi-continuous functions having the Darboux property. Functions which we shall consider are defined on the unit interval $I = [0,1]$.

Lemma 8. *Assume that $f \in \mathcal{QU}$, $(J_n)_n$ is a sequence of pairwise disjoint open intervals and g is a function such that $g(x) = f(x)$ for $x \in \bigcup_n J_n$, $g|\bigcup_n J_n$ is continuous and $f(J_n) \subset C^+(g|J_n, a_n) \cap C^-(g|J_n, b_n)$, where $J_n = (a_n, b_n)$, $n \in \mathbb{N}$. Then $g \in \mathcal{QU}$.*

Proof. Note that the set $A = F(\bigcup_n J_n)$ is nowhere-dense and therefore $B = C(f) \setminus A$ is dense in I and $f|B$ is dense in f. Additionally g is continuous at each point $x \in B$. We shall verify that $g|B$ is dense in g. Let $U = U_1 \times U_2$ be a neighbourhood of $(x, g(x))$ (obviously it is sufficient to consider only $x \in A$). Then $g(x) = f(x)$ and since f is quasi-continuous, $(t, f(t)) \in U$ for some $t \in B$. If $t \notin \bigcup_n J_n$ then $g(t) = f(t)$ and $(t, g(t)) \in U$. Otherwise $t \in J_n$ for some n. Then $a_n \in U_1$ or $b_n \in U_1$. Let e.g. $a_n \in U_1$. Since $f(t) \in C^+(g|J_n, a_n)$, there exists $s \in U_1 \cap J_n \cap B$ such that $(s, g(s)) \in U$. Thus g is quasi-continuous.

Now we verify that $g \in \mathcal{U}$. By Lemma 2(b) it suffices to observe that for every $x \in I$ the sets $C^-(g, x)$ and $C^+(g, x)$ are intervals and $f(x) \in C^-(g, x) \cap C^+(g, x)$. Assume that g is not continuous at x e.g. from the right. Then $g(x) = f(x)$ and $C^+(f, x) \subset C^+(g, x)$. Moreover for $y \in C^+(g, x) \setminus C^+(f, x)$ there exists $t \in C^+(f, x)$ such that $[t, y] \subset \subset C^+(g, x)$. Indeed, since $y \notin C^+(f, x)$, there exist sequences $(k_n)_n$ of positive integers and $(y_n)_n$ such that $y_n \in J_{k_n}$, $\lim_{n \to \infty} y_n = y$ and the sequence $(g(a_{k_n}))_n$ converges to some limit $t \in \overline{R}$. Then $t \in C^+(f, x)$. Since $f|J_{k_n}$ is continuous, $(f(a_{k_n}), y_n) \subset g(J_{k_n})$. Therefore $[t, y] \subset \subset C^+(g, x)$. This proves that $C^+(g, x)$ is an interval and $g(x) \in \in C^+(g, x)$. $\Diamond$

Lemma 9. *For each $f \in \mathcal{QU}$ and positive ε there exists $g \in \mathcal{QU}$ which is constant on no interval and such that $||f - g|| \leq \varepsilon$. Moreover, if f is of the Baire class α or measurable, then g may be taken from the same class.*

Proof. Let $\{J_n \subset I \colon n \in \mathbb{N}\}$ be the family of all maximal open intervals on which f is constant. Let $J_n = (a_n, b_n)$ and let $f(J_n) = \{y_n\}$ for each $n \in \mathbb{N}$. Since $f \in \mathcal{U}$, we obtain $f(a_n) = f(b_n) = y_n$. For every n we define a continuous surjection $g_n : \overline{J_n} \to [y_n - \varepsilon, y_n + \varepsilon]$ such that $g_n(a_n) = g_n(b_n) = y_n$ and g_n is constant on no subinterval of J_n. Then the function $g : I \to \mathbb{R}$ defined by $g(x) = g_n(x)$ for $x \in J_n$, $n \in \mathbb{N}$ and $g(x) = f(x)$ otherwise has the desired properties. Evidently $||f - g|| \leq \varepsilon$ and g is constant on no subinterval of I. By Lemma 8, $g \in \mathcal{QU}$. Finally it is easy to verify that if f is of the Baire class α or measurable, then g is from the same class. $\Diamond$

Lemma 10. *For every $f \in \mathcal{QU}$ and $\varepsilon > 0$ there exists a function $g \in \mathcal{QD}$ such that $||f - g|| < \varepsilon$. Moreover, if f is of the Baire class α or measurable then g may be taken from the same class.*

Proof. By Lemma 9 we can assume that $f : I \to \mathbb{R}$ is constant on no subinterval of I. Fix $n \in \mathbb{N}$ with $1/n < \varepsilon$. Since $f \in \mathcal{U}$, $T = \overline{f(I)}$ is an interval. Assume that $T = (-\infty, \infty)$ (the proof is similar when $T = [a, b]$, $T = [a, \infty)$ or $T = (-\infty, a]$). Put $a_k = k/n$, $J_k = (a_k, a_{k+1})$, $A_k = f^{-1}(J_k)$ and $B_k = f^{-1}(a_k)$ for each integer k. Since f is quasi-continuous, $f|C(f)$ is bilaterally dense in f and therefore we obtain the following conditions (for each k):

(1) $A_k = G_k \cup K_k$, where G_k is a non-empty, open set, K_k is nowhere-dense, $G_k \cap K_k = \emptyset$ and $K_k \subset \overline{G_k \cap (x, \infty)} \cap \overline{G_k \cap (-\infty, x)}$,

(2) B_k is a nowhere-dense subset of $\overline{(G_{k-1} \cup G_k) \cap (x, \infty)} \cap \cap \overline{(G_{k-1} \cup G_k) \cap (-\infty, x)}$.

Fix an integer k. Let $(I_{k,m})_m$ be a sequence of all components of G_k. For every m we define a continuous surjection $g_{k,m} : I_{k,m} \to \overline{J_k}$ such that:

(3) the end-points of $I_{k,m}$ belong to $\overline{g_{k,m}^{-1}(y)}$ for each $y \in \overline{J_k}$.

Now we define the function $g : I \to \mathbb{R}$ by $g(x) = g_{k,m}(x)$ for $x \in I_{k,m}$ (for each k, m) and $g(x) = f(x)$ otherwise. Evidently $||f - g|| \leq \leq 1/n < \varepsilon$. By Lemma 8, $g \in \mathcal{QU}$. To show that g has the Darboux property fix $a < b$ with $g(a) \neq g(b)$ (e.g. $g(a) < g(b)$) and $y \in (g(a), g(b))$. Let $J = (a, b)$. Obviously it is sufficient to consider the case when J is included in no interval $I_{k,m}$. Because $g \in \mathcal{U}$, $[g(a), g(b)] \subset \overline{g(J)}$. Let k be an integer such that $y \in \overline{J_k}$. Then $(g(a), g(b)) \cap J_k \neq \emptyset$ and consequently, $J \cap g^{-1}(J_k) \neq \emptyset$. Since $g^{-1}(J_k) \subset \subset f^{-1}(J_k)$, the condition (1) implies $I_{k,m} \cap J \neq \emptyset$ for some $m \in \mathbb{N}$. Let $I_{k,m} = (c, d)$. Since J is not a subset of $I_{k,m}$, $c \in J$ or $d \in J$. Let e.g. $c \in J$. Then $g(x) = y$ for some $x \in (c, b)$.

Finally let us assume that f is of the Baire class α and let $G \subset R$ be an open set. Then $g^{-1}(G) = \bigcup_{k,m} g_{k,m}^{-1}(G) \cup (f^{-1}(G) \setminus \bigcup_{k,m} I_{k,m})$ is clearly a Borel set of the additive class α. Hence g is of the Baire class α. Similarly we can prove that g is measurable if so is f. $\Diamond$

Theorem 4. *A necessary and sufficient condition for f to belong to $\mathcal{QU}$ is that f be the uniform limit of a sequence of quasi-continuous functions having the Darboux property. Moreover, if f is of the Baire class α or measurable then the approximating functions may be taken to be Baire class α or measurable.*

Proof. Because the families of all quasi-continuous, of the Baire class α, measurable functions are closed with respect to uniform limits (see [6] and e.g. [3]) and the uniform limits of sequences of Darboux functions belong to the class $\mathcal{U}$ [4], we obtain the sufficiency. The necessity is proved by applying Lemma 10. ◊

Corollary 2. *The class $\mathcal{QU}$ is closed with respect to uniform limits.*

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w-JORDAN NEAR-RINGS I

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Abstract: Let N be a zero-symmetric near-ring with an invariant series whose factors are N-simple. We prove that the radical $J_2\ (N)$ is nilpotent and the factor $N/J_2\ (N)$ is a direct sum of a finite number of A-simple and strongly monogenic near-rings. Moreover we characterize nilpotent near-rings with invariant series whose factors are of prime order.

Introduction and general results

Many authors have studied near-rings containing particular chains of ideals (see [5,8,10]) and have often shown the existence of links between these chains of ideals and the structure of the near-rings under consideration. In this paper we begin a study of near-rings with an invariant series whose factors belong to certain given classes. In particular we study here the zero-symmetric case; the general case and the construction of finite near-rings satisfying these conditions will be covered in future papers.

For the zero-symmetric near-rings with an invariant series whose factors are N-simple, we obtain a result analogous to the Artin-Noether theorem. We prove that a zero-symmetric near-ring N with an invariant series whose factors are N-simple has the radical $J_2(N)$ nilpotent and the factor $N/J_2(N)$ is a direct sum of A-simple and strongly monogenic near-rings. Moreover we discuss the finite case and characterize the near-rings with an invariant series whose factors are of prime order. We prove a necessary and sufficient condition so that N is nilpotent and we establish a link between the nilpotence index and the length of the series. In the case in which index and length coincide, we prove that the order of N is a prime power.

In the following we will often refer to [12] without express recall.

Let N be a left near-ring. A finite system of subnear-rings of N contained in one another

$$N = N_1 \supset N_2 \supset \ldots \supset N_n = \{0\}$$

is called a *normal series* of N if every subnear-ring N_i, $i \in \{1,2,\ldots,n\}$, is a proper ideal in N_{i-1}, an *invariant series* of N if every subnear-ring N_i, $i \in \{1,2,\ldots,n\}$, is a proper ideal of N. The factor-near-rings N_i/N_{i+1} are called *principal factors* of the invariant series. For invariant series, in the following, we will indicate $N_i/N_{i+1}, N_i/N_{i+2},\ldots,N_i/N_{i+k}$ respectively with $N_i', N_i'' \ldots, N_i^k$ and with $f_i', f_i'',\ldots,f_i^k$ the corresponding canonical epimorphisms.

Let us consider now the following classes of near-rings:

S_0: class of simple near-rings ;

S_1: class of simple and strongly monogenic near-rings ;

S_2: class of N_0 -simple near-rings[(1)];

S_3: class of near-rings without proper subnear-rings;

S_4: class of near-rings of prime order.

Definition 1. A near-ring N is a *w-Jordan near-ring* (*wJ-near-ring*) if it has an invariant series whose factors belong to $S_w (w \in \{0,1,2,3,4\})$.

We can observe that in near-ring-theory the classes S_i ($i \in \{0,1,2,3,4\}$) never coincide without further conditions while in ring-theory, for instance, S_1 and S_2 coincide.

In order to establish relationships between the classes S_w, let us state some results that concern the near-rings belonging to S_2. We recall that: A near-ring N is N_0-simple if it is without proper additive subgroups S such that $SN_0 \subseteq S$.

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(1) We observe that if N is zero-symmetric, N_0-semplicity and N-semplicity coincide.

Definition 2. A zero-symmetric near-ring N is *A-simple* if it is without non-zero N-subgroups H such that $HN = \{0\}$.

Theorem 1. *A near-ring N belongs to S_2 iff N is a zero-ring of prime order, a constant near-ring of prime order or an A-simple and strongly monogenic near-ring .*

Proof. Let N be an N_0-simple near-ring. The constant and the zero-symmetric parts are both N_0-subgroups of N, hence N is constant or zero-symmetric . By [2] and Ex.3.9 p.78 of [12] a constant near-ring is N_0-simple iff it is cyclic of prime order. If N is zero-symmetric either $nN = \{0\}$ for every $n \in \mathbb{N}$, and thus N is a zero-ring of prime order, or N is strongly monogenic and obviously A-simple. Conversely, if N is a zero-ring of prime order or a constant near-ring of prime order, then N is N_0-simple. Let N be an A-simple and strongly monogenic near-ring. Let us suppose that M is a proper N_0-subgroup of N. Since N is an A-simple near-ring, then $MN \neq \{0\}$ and since N is a strongly monogenic near-ring there is an element $h \in M$ such that $hN = N$. Since M is an N_0-subgroup, hN is contained in M, a contradiction. Thus N is N_0-simple. $\diamondsuit$

We observe that a zero-symmetric near-ring which is A-simple and strongly monogenic is Blackett simple ([4]).

Definition 3. A near-ring N is *strongly N_0-simple* if its subnear-rings belong to S_2.

We will call S_2^s the class of the strongly N_0-simple near-rings .

Theorem 2. *If N is an N_0-simple near-ring and every subnear-ring M of N satisfies the d.c.c. on the M-subgroups, then N is strongly N_0-simple.*

Proof. By Th.1, if N is a zero-ring of prime order or a constant near-ring of prime order, then N is strongly N_0-simple. Let N be an A-simple and strongly monogenic near-ring and let M be a subnear-ring of N with d.c.c. on the M-subgroups. Our aim is to show that M does not contain additive subgroups S so proving that $SM \subseteq S$. Let us suppose S to be a proper M-subgroup of M. Since N is A-simple then $SN \neq \{0\}$, thus there is an element $s \in S$ such that $sN = N$, given that N is strongly monogenic. Firstly we observe that $r(s) = \{0\}$ (where $r(s)$ is the right annihilator of the element s). In fact $r(s) \neq \{0\}$ implies $r(s)N \neq \{0\}$, because $r(s)$ is an N-subgroup of N and N is A-simple; thus $r(s)N = N$ and $N = sN = s[r(s)N] = \{0\}N = \{0\}$ and this is absurd. Moreover, since S is a proper M-subgroup of M, sM

is strictly contained in M. We set $M_1 = sM$ and consider sM_1. It is an M-subgroup of M strictly contained in M_1, in fact if $sM_1 = M_1$, it would be $ssM = sM$, that is $s(sM - M) = \{0\}$. Since $r(s) = \{0\}$, then $sM = M$ and this was previously excluded. In this way we obtain a chain $M_1 \supset sM_1 \supset s^2M_1 \supset \ldots$ which becomes stationary, due to d.c.c. on the M-subgroups. Since this is excluded, M is M-simple. $\diamondsuit$

Proposition 1. *If N is a wJ-near-ring, then N is a $(w-1)J$-near-ring.*

Proof. We can easily prove that $S_4 \subset S_3 \subset S_2 \subset S_1 \subset S_0$ and consequently that a wJ-near-ring is a $(w-1)J$-near-ring. $\diamondsuit$

Proposition 2. *The classes S_w $(w \in \{0,1,2,3,4\})$ are closed under homomorphisms and the classes S_w $(w \in \{3,4\})$ are closed under substructures.*

Proof. The near-rings belonging to S_3 and S_4 are without substructures and simple, so they do not have proper homomorphic images. Moreover, if $N' = \varphi(N)$ is a homomorphic image of N, each proper N'_0-subgroup (ideal) of N' derives from some proper N_0-subgroup (ideal) of N, thus $N \in S_2$ implies $N' \in S_2$ ($N \in S_0$ implies $N' \in S_0$). Moreover, if N is strongly monogenic and simple, then N' is strongly monogenic and simple, therefore $N \in S_1$ implies $N' \in S_1$. $\diamondsuit$

Hence, by Prop.6 of [1]:

Proposition 3. *The classes of the $3J$-near-rings and of the $4J$-near-rings are closed under substructures, homomorphic images and N_0-subgroups.*

We should observe that the classes $S_w(w \in \{0,1,2\})$ are not closed under substructures. In fact for example $\mathbb{Q} \in S_2$ but $\mathbb{Z} \notin S_0$. Therefore we cannot apply Prop.6 of [1] and, in fact, even if we can prove that S_2 is closed under N_0- subgroups, the class of the $2J$-near-rings is not closed under N_0-subgroups.

2-Jordan near-rings

The following Th.3, which provides a necessary and sufficient condition so that the class S_2 is closed w.r.t. substructures, uses the Th.1.33 of [11].

Let I be an ideal of a near-ring N and S a subnear-ring of N. Then $I \cap S$ is an ideal of S, I is an ideal of $I+S$ and $I+S/I$ is isomorphic to $S/I \cap S$.

Theorem 3. *A near-ring N has all its subnear-rings as $2J$-near-rings iff it contains an invariant series $N = N_1 \supset N_2 \supset \dots \supset \supset N_n = \{0\}$ whose principal factors $N_i^!$ belong to S_2^s.*

Proof. Let N be a near-ring whose subnear-rings are $2J$-near-rings. So N is also a $2J$-near-ring. Hence let us consider an invariant series of N,

$$(\alpha) \qquad N = N_1 \supset N_2 \supset \dots \supset N_n = \{0\}$$

whose principal factors belong to S_2. In order to show that the principal factors of (α) belong to S_2^s, we will show that every subnear-ring M of $N_i^!$ has the d.c.c. on the M-subgroups. Let M be a subnear-ring of $N_i^!$. Since M is a homomorphic image of a subnear-ring of N_i and consequently of N, by Proposition 2, it is a $2J$-near-ring. Therefore M has an invariant series $M = M_1 \supseteq M_2 \dots \supseteq M_n = \{0\}$ whose factors belong to S_2. Hence these factors have the d.c.c. on the $(M_i^!)_0$-subgroups. By Th.1 and Ex a) of [1] we can deduce that M also has the d.c.c. on M-subgroups. Thus $N_i^!$ belong to S_2 and every subnear-ring M of $N_i^!$ has the d.c.c. M. We apply Th.2 and $N_i^! \in S_2^s$.

Conversely, let N be a near-ring with an invariant series $N = N_1 \supset \supset N_2 \supset \dots \supset N_n = \{0\}$ whose principal factors $N_i^!$ belong to S_2^s. We can prove that the subnear-rings of N are $2J$-near-rings. Let M be a subnear-ring of N. We set $M_i = M \cap N_i$ and we obtain an invariant series of M : $M = M_1 \supseteq M_2 \supseteq \dots \supseteq M_n = \{0\}$.

By the Theorem 1.33 of [11], $N_{i+1} + M_i/N_{i+1}$ is isomorphic to $M_i/N_{i+1} \cap M_i$ that coincides with M_i/M_{i+1}. Therefore $M_i^!$ is isomorphic to $N_{i+1} + M_i/N_{i+1}$ and the latter is a subnear-ring of $N_i^!$. Since $N_i^!$ belongs to $S_2^s, M_i^!$ belongs to S_2 and M is a $2J$-near-ring. $\diamondsuit$

Corollary 1. *The class of finite $2J$-near-rings is closed under substructures.*

Proof. It follows from Th.2 and 3, given that, in the finite case, the d.c.c. hold. $\diamondsuit$

In the following N will be a zero-symmetric near-ring.

Theorem 4. *If N is a near-ring with an A-simple and strongly monogenic ideal I such that N/I is a zero-ring of prime order, then $N = I \oplus J$ where $J = J_2(N)$.*[2]

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(2) $J_2(N)$ is the intersection of right annihilators of N_0-simple N-groups, see [12] p. 136.

Proof. Let I be a proper ideal of N, otherwise the thesis is trivial. Since N is zero-symmetric, I is an N-subgroup of N, therefore $IJ_2(N) = \{0\}$ and $J_2(N) \neq N$, $J_2(N) \neq I$ because I is A-simple. Moreover $J_2(N) \neq \{0\}$. In fact: if $J_2(N) = \{0\}$, then $J_2(I) = \{0\}$ and I is 2-semisimple with d.c.c. on the right annihilators. Hence I has a left identity e (see [2], [4], [12] p. 146) and by Pierce decomposition $N = r(e) + eN$. We observe that $r(e) \neq \{0\}$. In fact $r(e) = \{0\}$ implies $N = eN \subseteq I$ and this is excluded. Moreover N/I is a zero-ring, therefore $[r(e)]^2 \subseteq I$ and hence $[r(e)]^2 = \{0\}$. In this way $r(e)$ is a non trivial nilpotent N-subgroup of N and therefore $r(e) \subseteq J_2(N) = \{0\}$ (see [12] p. 153, [13]), a contradiction. Finally $I \cap J_2(N) = \{0\}$ because I is simple and $N = I + J_2(N)$ because N/I is of prime order. Hence $N = I \oplus J_2(N)$. $\diamondsuit$

The following theorem shows that, given a zero-symmetric near-ring with an invariant series whose factors are in S_2, it is possible to construct another invariant series whose factors are in S_2 such that the A-simple and strongly monogenic factors precede the zero-ring factors.

Theorem 5. *Let N be a $2J$-near-ring and $N = N_1 \supset N_2 \supset \ldots \supset N_n = \{0\}$ an invariant series whose principal factors are in S_2. If $N_i^!$ is a zero-ring and $N_{i+1}^!$ is an A-simple and strongly monogenic near-ring then there is an ideal M_{i+1} of N such that $N_i \supset M_{i+1} \supset N_{i+2}$, N_i/M_{i+1} is isomorphic to $N_{i+1}^!$ and M_{i+1}/N_{i+2} is isomorphic to $N_i^!$.*

Proof. Considering the near-ring N_i'', we set $I = f_i''(N_{i+1})$. Given that $N_i''/N_{i+1}^!$ is isomorphic to $N_i^!$ we have N_i''/I isomorphic to $N_i^!$. Therefore N_i''/I is a zero-ring of prime order and I is A-simple and strongly monogenic because it is isomorphic to $N_{i+1}^!$. Hence, by Th.4, $N_i'' = I \oplus J$ where $J \simeq N_i^!$ and therefore $N_{i+1}^! \simeq N_i''/J$. We set $M_{i+1} = (f_i'')^\circ(J)$, that is M_{i+1}/N_{i+2} is isomorphic to $N_i^!$. Obviously M_{i+1} is an ideal of N_i and $N_i/M_{i+1} \simeq (N_i/N_{i+2})/(M_{i+1}/N_{i+2}) \simeq N_i''/J \simeq I \simeq N_{i+1}/N_{i+2} = N_{i+1}^!$. Hence M_{i+1} is a maximal ideal of N_i.

Now we can show that M_{i+1} is an ideal of N: the near-ring N_{i+1} is an ideal of N, M_{i+1} is an ideal of N_i, hence $N_{i+1}M_{i+1} \subseteq N_{i+1} \cap M_{i+1}$. Moreover $N_{i+1} \cap M_{i+1} = N_{i+2}$. In fact if $x \in N_{i+1} \cap M_{i+1}$, then $x + N_{i+2} \in N_{i+1}^! \cap J = \{0\}$ and this implies that $x \in N_{i+2}$. Thus $N_{i+1} \cap M_{i+1} \subseteq N_{i+2}$. Obviously $N_{i+2} \subseteq N_{i+1} \cap M_{i+1}$, therefore $N_{i+1} \cap M_{i+1} = N_{i+2}$. We now set $(N_{i+2} : N_{i+1})_N = \{m \in N / N_{i+1}m \subseteq N_{i+2}\} = H$ which is an ideal of N (see [11]). We obtain $M_{i+1} \subseteq$

$\subseteq H \cap N_i$ and $H \cap N_i$ is strictly enclosed in N_i, otherwise it would be $N_{i+1}\, N_i \subseteq N_{i+2}$ and hence $N_{i+1}\, N_{i+1} \subseteq N_{i+2}$, but N'_{i+1} is A-simple and this is excluded. Hence $M_{i+1} = H \cap N_i$. Thus M_{i+1}, as intersection of two ideals of N, is an ideal of N. $\diamondsuit$

Theorem 6. *A non nilpotent $2J$-near-ring N, has the radical $J_2(N)$ nilpotent and the factor $N/J_2(N)$ is a direct sum of A-simple and strongly monogenic near-rings.*

Proof. By Th.5, if N is a zero-symmetric $2J$-near-ring, we can construct a new invariant series $N = N_1 \supset N_2 \supset \ldots \supset N_n = \{0\}$ whose factors are in S_2, such that, if N'_i is A-simple and strongly monogenic and N'_j is a zero-ring, then $i < j$. We set $h \in I_n$, the smallest index such that N'_h is a zero-ring. Obviously N_h is nilpotent. Therefore $N_h \subseteq J_2(N)$. Moreover, if $N_h \neq N$, the near-ring N/N_h contains an invariant series whose factors are N-simple and hence 2-semisimple. By Ex. f) of [1], N/N_h is 2-semisimple and therefore $J_2(N) \subseteq N_h$. Hence $J_2(N) = N_h$ and the radical $J_2(N)$ is nilpotent. In this way $N/J_2(N)$ has an invariant series satisfying the hypotheses of Th.4 of [1], thus $N/J_2(N)$ is the direct sum of A-simple and strongly monogenic near-rings. $\diamondsuit$

The analogous, in ring-theory, brings us to the famous theorem of Artin-Noether. In fact, rings with an invariant series whose factors are in S_2, are rings with an invariant series whose factors are without right ideals[(3)] and hence are either fields or zero-rings. Thus in a ring A satisfying the hypotheses of Th.6 the Jacobson radical $J(A)$ is nilpotent and the factor $A/J(A)$ is a direct sum of fields.

Corollary 2. *Let N be a $2J$-near-ring. Then* $\mathcal{P}(N) = \eta(N) = J_0(N) = J_1(N) = J_2(N)$.[(4)]

Proof. It can be easily demonstrated, since N has the d.c.c. on the N-subgroups and $J_2(N)$ is nilpotent (see 5.61 p. 162 of [12]). $\diamondsuit$

If N is a finite near-ring, we obtain:

Corollary 3. *Let N be a finite near-ring such that $N \neq J_2(N)$. Then:*

1. *If N is a $2J$-near-ring and the A-simple factors present in a principal series are planar, then the additive group $(N/J_2(N))^+$ is nilpotent;*
2. *If N is a $3J$-near-ring, the additive group $(N/J_2(N))^+$ is abelian.*

Proof. The group $(N/J_2(N))^+$ is a direct sum of finite groups sup-

(3) A ring having an invariant series whose factors are in S_2, is right artinian.

(4) For the definitions of $\mathcal{P}(N)$, $\eta(N)$ and $J_v(N)$ ($v \in \{0, 1, 2, \}$) see [9], [11], [12].

porting planar near-rings. Therefore, as shown in [3], $(N/J_2(N))^+$ is nilpotent.

If N is a $3J$-near-ring, the factors of the invariant series are without proper subnear-rings. Therefore, as proved in [6], (see also [7]) they are p-singular(5) and therefore their additive group is elementary abelian, because they are simple. Thus $(N/J_2(N))^+$, being a direct sum of elementary abelian groups, is abelian. $\diamondsuit$

4-Jordan near-rings

In this section we will study the $4J$-near-rings with particular reference to the nilpotent case. We recall that a near-ring N is *nilpotent* if there is an index $n \in \mathbb{N}$ such that $N^n = \{0\}$. We will call $g(N)$ the least $n \in \mathbb{N}$ such that $N^m = \{0\}$ and $\dim(N)$ the length of an invariant series whose factors are in S_4.

Theorem 7. *A near-ring N with an invariant series $N = N_1 \supset N_2 \supset \supset \ldots \supset N_n = \{0\}$ and whose factors are in S_4 is nilpotent iff $N^s \subseteq N_s$, for every $s \in I_n$.*

Proof. Let N be a nilpotent $4J$-near-ring. *We will show that, for every $i \in I_n$, $NN_i \subseteq N_{i+1}$.* If $NN_i \not\subseteq N_{i+1}$, there is an element $a \in N$ such that $aN_i \not\subseteq N_{i+1}$. Since aN_i is a subnear-ring of N_i and N_i/N_{i+1} is of prime order, $(aN_i + N_{i+1})/N_{i+1}$ is not a proper subnear-ring of N_i/N_{i+1}. Therefore, either $aN_i + N_{i+1} = N_{i+1}$ or $aN_i + N_{i+1} = N_i$. Given that $aN_i \not\subseteq N_{i+1}$, we have:

$$(\alpha) \qquad aN_i + N_{i+1} = N_i$$

and $a^hN_i = a^{h+1}N_i + a^hN_{i+1}$. Let h' be the smallest integer such that $a^{h'}N_i \subseteq N_{i+1}$. This h' exists and it is $h' > 1$ because otherwise, for every $t \in \mathrm{N}$, it would be $a^tN_i + N_{i+1} = N_i$ and since N is nilpotent, it would be $N_{i+1} = N_i$ and this is excluded. Therefore, by (α), we obtain $a^{h'}N_i + a^{h'-1}N_{i+1} = a^{h'-1}N_i$, hence $a^{h'-1}N_i \subseteq N_{i+1}$ in contrast to the hypothesis stating that h' is the smallest integer so that $a^{h'}N_i \subseteq N_{i+1}$. Thus $NN_i \subseteq N_{i+1}$ and consequently $N^s \subseteq N_s$ for every $s \in I_n$. The converse is trivial. $\diamondsuit$

(5) For the definition of p-singular near-ring see [6].

Corollary 4. *If N is a nilpotent $4J$-near-ring, $g(N) \leq \dim(N)$.*
Proof. It is a consequence of Th.7. $\diamondsuit$

We can characterize the case in which $g(N) = \dim(N)$.

Theorem 8. *Let N be a nilpotent $4J$-near-ring and let $N = N_1 \supset \supset N_2 \supset \ldots \supset N_n = \{0\}$ a series whose factors are in S_4. The length of the chain and the nilpotence index of N coincide iff $N_i = (N_{i+1} : N)_N$ for every $i \in I_{n-1}$.*

Proof. We set $M_i = (N_{i+1} : N)_N = \{n \in N/Nn \subseteq N_{i+1}\}$. Let $g(N) = dim\,(N) = n$. By Th.7, we have $NN_i \subseteq N_{i+1}$ and hence $N_i \subseteq \subseteq M_i$. If N_i is strictly contained in M_i, the series $N \supset M_i \supset N_i \supset \{0\}$ will be refinable (by Jordan-Hölder theorem) in a principal series where $M_i = \overline{N}_j$ with $j \leq i$. By Th.7, $N^j \subseteq \overline{N}_j$ and hence $N^j \subseteq M_i$. Therefore $N^{j+1} \subseteq NM_i \subseteq N_{i+1}$. Hence $N^{j+1+(n-i-1)} = N^{n-(i-j)} = \{0\}$. Given that $g(N) = n$, we obtain $i = j$, that is $M_i = \overline{N}_j = N_i$.

Conversely, let us suppose $N_i = M_i$ for every $i \in I_{n-1}$ and $g(N) = h$. Then $N^h = \{0\}$, therefore $N^{h-1} \subseteq (0 : N)_N = N_{n-1}$, in fact N_{n-1} is the right annihilator of N because $N_{n-1} = M_{n-1} = (N_n : N)_N$. Analogously $N^{h-2} \subseteq (N_{n-1} : N)_N = N_{n-2}$ and so on. After a finite number of steps we get $N \subseteq N_{n-h+1}$, thus $N = N_{n-h+1}$ and $n = h$. $\diamondsuit$

Finally:

Theorem 9. *If N is a nilpotent $4J$-near-ring such that $g(N) = \dim(N)$, then $|N| = p^\alpha$, (p prime).*

Proof. We can prove this theorem by induction on $g(N)$. If $g(N) = 2$, $N = N_1 \supset N_2 = \{0\}$ is the principal series required and hence $|N| = p$. Let us suppose the theorem proved for dim $(N) = n - 1$ and let $N = N_1 \supset N_2 \supset \ldots \supset N_n = \{0\}$ be a series of N whose factors are in S_4. Then $|N/N_{n-1}| = p$ and we can suppose $|N_{n-1}| = q$ (q prime). By Th.7, $N^{n-2}\,N = N^{n-1} \subseteq N_{n-1}$, therefore, for every $m \in \in N^{n-2}$, $mN \subseteq N_{n-1}$ and given that N_{n-1} is of prime order, either $mN = \{0\}$ or $mN = N_{n-1}$. If $mN = \{0\}$, for every $m \in N^{n-2}$, then $N^{n-1} = \{0\}$ and this is excluded, thus $mN = N_{n-1}$ for some $m \in N$. Considering now the left translation $\gamma_m : N \to mN$, we obtain an endomorphism of N^+ whose kernel is $r(m)$, the right annihilator of m and whose image is N_{n-1}. Therefore $|\text{im}\,\gamma_m| = |\,N/\ker\gamma_m|$ that is $q = |N/\ker\gamma_m|$. Given that $\ker\gamma_m = r(m) \supseteq r(N) = N_{n-1}$, either $|\ker\gamma_m| = q$ or $|\ker\gamma_m| = q^\beta$. Thus: $q = qp^\alpha/q^\beta$ and this implies $q^\beta = p^\alpha$, hence $p = q$. $\diamondsuit$

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THE CONGRUENCE LATTICE OF IMPLICATION ALGEBRAS*

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Abstract: The variety of implication algebras is a minimal quasivariety. It is 3-filtral but not 2-filtral. An implication algebra A is tolerance-trivial iff $(A, \leq)$ is a lattice, where the partial ordering $''\leq''$ is defined as follows: $a \leq b \Leftrightarrow \exists x \in A$ such that $b = x \cdot a$.

1. Introduction

Implication algebras are groupoids with a simple binary operation, which yields a partially order. This derived order structure can be considered as a generalization of Boolean lattices (see Prop.2).

Definition 1 ([1], [9]). A groupoid $(A, \cdot)$ is called an *implication algebra* if the operation " $\cdot$ " satisfies the following axioms:

$$(a \cdot b) \cdot a = a$$
$$(a \cdot b) \cdot b = (b \cdot a) \cdot a$$
$$a \cdot (b \cdot c) = b \cdot (a \cdot c).$$

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Example. If $(B, \vee, \wedge, 0, 1, ^-)$ is a Boolean algebra then $(B, \rightarrow)$ and $(B, /)$, where $a \rightarrow b = a^- \vee b$ and $a \,/\, b = a^- \wedge b$ for all $a, b \in B$, are both implication algebras.

Remark. If the algebra above is the Boolean algebra of propositional calculus then "$\rightarrow$" represents ordinary implication.

Implication algebras are examples of algebraic varieties which are *3-permutable, 3-congruence distributive* and *3-congruence modular* but are not either *congruence permutable* or *2-distributive* or *2-modular*: [9], [4].

In this paper we shall prove a new property of implication algebras, namely that they are *3-filtral* but not *2-filtral* (§2) and we shall characterize those implication algebras on which every compatible tolerance is a congruence (§3)

Let us first review a few concepts:

A variety V is *congruence permutable (congruence 3-permutable)* if $\Theta_1 \circ \Theta_2 = \Theta_2 \circ \Theta_1$ $(\Theta_1 \circ \Theta_2 \circ \Theta_1 = \Theta_2 \circ \Theta_1 \circ \Theta_2)$ for any two congruences $\Theta_1, \Theta_2 \in \operatorname{Con} A$ and for any $A \in V$ (where " $\circ$ " is the relational product of congruences); *3-congruence modularity* and *3-congruence distributivity* mean that the systems of equations of H.P. Gumm and B. Johnson respectively for congruence modularity and congruence distributivity consist of at least 3+1 terms.

For example *3-distributivity* means that the following system of equations (where $n, i \in \mathrm{N}$; $q_o, q_1, \ldots, q_n$ are 3-variable terms):

$$(1)\qquad \begin{aligned} &q_o(x,y,z) = x, \quad q_n(x,y,z) = z \\ &q_i(x,y,x) = x, \quad 0 \leq i \leq n \\ &q_i(x,x,y) = q_{i+1}(x,x,y), \ i \text{ even} \\ &q_i(x,y,y) = q_{i+1}(x,y,y), \ i \text{ odd} \end{aligned}$$

must contain at least 3+1 terms, i.e.: $n = 3$.

For implication algebras these terms are:

$$(2)\qquad \begin{aligned} &q_0(x,y,z) = x, && q_3(x,y,z) = z \\ &q_1(x,y,z) = [y \cdot (z \cdot x)] \cdot x, && q_2(x,y,z) = (x \cdot y) \cdot z \end{aligned}$$

for all $x, y, z \in A$.

Filtral varieties can be defined using the notion of *product congruence:*

Let A be the subdirect product of algebras A_i ($i \in I$) and let a_i denote the i-th component of $a \in A$ belonging to A_i. A congruence $\varphi \in \operatorname{Con} A$, is called the *product of the congruences* $\varphi_i \in \operatorname{Con} A_i$, $i \in I$ if $a \varphi b$ exactly when $a_i \varphi_i b_i$ for all $i \in I$. We write $\varphi = \prod_{i \in I} \varphi_i$.

Definition 2 ([7],[8]). A variety $\mathcal{V}$ is called an *ideal variety* iff for all $A \in \mathcal{V}$ every compact congruence on A is a product congruence.

Definition 3 ([7],[8]). A variety $\mathcal{V}$ is called *filtral* if it is an ideal variety and it is *semi-simple* i.e. all its subdirect irreducible algebras are congruence-simple.

We shall denote the class of subdirect irreducible algebras of a variety $\mathcal{V}$ by $\mathrm{SI}\mathcal{V}$, and the variety of implication algebras by $\mathcal{V}(\mathrm{I})$. E.Fried and E. Kiss [5] gave the following characterization of filtral varieties by term functions (see also [8]):

Theorem ([5],[8]).: *A variety $\mathcal{V}$ is filtral iff there is an $n \in \mathrm{N}$ and there are 3-variable terms $f_0, f_1, \ldots, f_n$ ($n > 1$) such that for any x, y, z in any algebra of $\mathcal{V}$ we have:*

$$
(3) \qquad
\begin{array}{ll}
\text{(a)} & f_0(x,y,z) = x, \quad f_n(x,y,z) = z, \\
\text{(b)} & f_i(x,y,x) = x, \quad (\textit{for all } \ i : 0 \le i \le n), \\
\text{(c)} & f_i(x,x,z) = f_{i+1}(x,x,z), \quad \textit{for } i \textit{ even}, \\
\text{(d)} & \textit{for all } A \in \mathrm{SI}\mathcal{V} \quad \textit{and} \quad x,y,z \in A,\ x \neq y : \\
 & f_i(x,y,z) = f_{i+1}(x,y,z), \quad \textit{for } \ i \textit{ odd}.
\end{array}
$$

Proceding in the same way as in characterization of congruence modular and congruence distributive varieties by a system of term equation, we can use the following concept:

Definition 4. According to the theorem above, if the system (3) of equations for $\mathcal{V}$ needs at least $n+1$ terms, then $\mathcal{V}$ is called *n-filtral.* Eg. $\mathcal{V}$ is *3-filtral* if $n = 3$ and f_0, f_1, f_2, f_3 satisfy conditions (3).

Let us now list some properties of implication algebras:

Property 1 ([1]). Let be A an implication algebra. We can define an partially ordering relation "$\le$" on A as follows:

$$a \le b \Leftrightarrow \exists x \in A : b = x \cdot a.$$

J.C.Abbott has shown [1] that this relation is isotone on the left and antitone on the right with respect to "$\cdot$" (i.e. $\forall c \in A$, if $a \le b : c \cdot a \le c \cdot b$ and $a \cdot c \ge b \cdot c$); furthermore $(A, \le)$ is a semilattice with identity, i.e. $\sup\{a, b\} = (a \cdot b) \cdot b$ exists for all $a, b \in A$ and there is an element

$1 \in A$ such that $x \leq 1$ for all $x \in A$. "$\leq$" can be defined using 1, since $a \leq b \Leftrightarrow a \cdot b = 1$.

Property 2 ([1]). If $(A, \leq)$ is the semilattice corresponding to the implication algebra $(A, \cdot)$, then every principal filter $(\{x | x \geq a\}, \leq)$ is a Boolean lattice. Vice versa in every semilattice with the above mentioned property one can define a binary operation "$\cdot$" for which $(A, \cdot)$ is an implication algebra in the following way:

$$a \cdot b = (a \vee b)_b^-$$

where $(a \vee b)_b^-$ denotes the complement of $a \vee b$ in the Boolean lattice $(\{x | x \geq b\}, \leq)$.

Property 3 ([1]). For a pair $a, b \in A$, $\inf\{a, b\}$ exists exactly when $\{a, b\}$ has a common lower bound $c \in A$. In that case $\inf\{a, b\} = [a \cdot (b \cdot c)] \cdot c$.

Remark ([1]). $(A, \leq)$ is a Boolean lattice iff it has a least element, denoted by 0 ($0 \leq x$, for all $x \in A$).

Definition 5 ([1]). If $(A, \cdot)$ is an implication algebra and if the derived partially ordered set $(A, \leq)$ is a lattice (i.e. for all $a, b \in A$ $\inf\{a, b\} = a \wedge b$ exists), then $(A, \leq)$ (and $(A, \cdot, \leq)$ as well) is called an *implication lattice*.

2. The variety and congruences of implication algebras

One of the most notable properties of implication algebras is that is a one-to-one correspondence between their congruences and their filters.

A subset $F \subseteq A$ of a partially ordered set $(A, \leq)$ is called a *filter* if for all $a \in F$ and $x \in A$, $x \geq a \Rightarrow x \in F$ and if $\inf\{x_1, x_2\} = x_1 \wedge x_2$ exists for $x_1, x_2 \in F$, then $x_1 \wedge x_2 \in F$. E.g. $[a] = \{x \in A | x \leq a\}$ is a filter, called the *principal filter* belonging to a. By Property 1 if $a \neq b$ then $[a] \neq [b]$.

One can easily show that the intersection of a given family $\{F_i\}_{i \in I}$, $I \neq \emptyset$ of filters of $(A, \leq)$ is also a filter; $\coprod_{i \in I} F_i$ can be defined as the intersection of all filters containing the set $\bigcup_{i \in I} F_i$. If $\mathcal{F}_A$ denotes the set of all filters of an implication algebra $(A, \cdot)$, then $(\mathcal{F}_A, \coprod, \bigcap, A, \{1\})$ is a distributive complete lattice with 1 and 0.

From now on let $\Theta[a]$ denote the congruence class of Θ belonging to $a \in A$, i.e.: $\Theta[a] = \{x \in A | x \, \Theta \, a\}$.

Property 4 ([1]). *The mapping* $i : \operatorname{Con} A \to \mathcal{F}_A$, $i(\Theta) = \Theta[1]$ *is an isomorphism between* $(\operatorname{Con} A, \wedge, \vee, 1_A, 0_A)$ *and* $(\mathcal{F}_A, \bigcap, \coprod, A, \{1\})$. *For any* $F \in \mathcal{F}_A$, $i^{-1}(F) = \Theta_F$, *where* $a \, \Theta_F b \Leftrightarrow a \cdot b,\ b \cdot a \in F$ (i^{-1} denotes the inverse of the mapping i).

Proposition 1. *The variety of implication algebras is a minimal quasivariety.*

Proof. We begin by showing that $\mathcal{V}(\mathrm{I})$ has only one subdirect irreducible algebra, namely the 2-element one.

Let $A \in \mathrm{SI}\mathcal{V}(\mathrm{I})$, γ its monolit, and F_γ the filter belonging to γ. Since $\gamma \leq \Theta$ for all $\Theta \in \operatorname{Con} A (\Theta \neq 0_A)$, therefore $F_\gamma \subseteq \bigcap_{x \in A} [x]$ and so there exists an $a \in F_\gamma$ such that $F_\gamma = [a] = \{1, a\}$ and

$$a \geq x \text{ for all } x \in A \setminus \{1\}. \tag{4}$$

Suppose now that there exists an $x \in A \setminus \{1\}$ such that $x \neq a$. Since $([x], \leq)$ is a Boolean lattice (see Prop.2) and $a \in [x]$, there exists an $a^- \in [x]$ such that $a^- \wedge a = x$, and $a^- \vee a = 1$.

Now (4) gives $a^- \leq a \neq 1$ - which is a contradiction. Thus $A = \{1, a\}$, i.e. A has two elements.

Two element implication algebras are isomorphic to each other and so $\mathrm{SI}\mathcal{V}(\mathrm{I})$ contains only one non-trivial algebra (and this one is congruence and subalgebra simple at the same time).

A locally finite variety $\mathcal{V}$ is a minimal quasivariety exactly when it has only one SI algebra and this can be embedded into every non-trivial $B \in \mathcal{V}$ (see [2], Cor.2).

By [1] the number of elements in any free implication algebra generated by n elements is at most 2^{2^n}. Therefore any finitely generated implication algebra is finite and so $\mathcal{V}(\mathrm{I})$ is locally finite.

On the other hand for every nontrivial $B \in \mathcal{V}(\mathrm{I})$ and $x \in B$, $x \neq$ $\neq 1$, $\{1, x\}$ is a two-element subalgebra of B and thus $\mathcal{V}(\mathrm{I})$ satisfies all previous conditions. $\diamondsuit$

Corollary 1. *Every implication algebra* $(A, \cdot)$ *is a subdirect power of two element implication algebra* $(\{1, a\}, \cdot)$.

Theorem 1. *The variety of implication algebras is 3-filtral but not 2-filtral.*

Proof. Assuming that $\mathcal{V}(\mathrm{I})$ is 2-filtral means there are three 3-variable terms f_0, f_1, f_2 sufficient for $\mathcal{V}(\mathrm{I})$ in the system (3) of equations. But in this case from (3) we get that $\mathcal{V}(\mathrm{I})$ is 2-distributive, contradicting [8].

To prove that $\mathcal{V}(\mathrm{I})$ is 3-filtral we shall use the terms q_0, q_1, q_2, q_3 from (2)-which were used first for distributivity. Let us check the identities of (3):

(a) is clear;

(b) $q_i(x,y,z) = x, \quad 0 \leq i \leq 2$ (by distributivity - (1));

(c) From (1) we have $q_0(x,x,z,) = q_1(x,x,z)$ and $q_2(x,x,z) = = q_3(x,x,z)$;

(d) Let x, y, z be elements of the subdirect irreducible algebra $(\{0,1\}, \cdot)$ and let $x \neq y$:
If $x = 0$ and $y = 1$ then $q_1(0,1,z) = [1 \cdot (z \cdot 0)] \cdot 0 = (z \cdot 0) \cdot 0 = = \sup\{z, 0\} = z$, $q_2(0,1,z) = (0 \cdot 1) \cdot z = z$;
If $x = 1$ and $y = 0$ then $q_1(1,0,z) = [0 \cdot (z \cdot 1)] \cdot 1 = 1$, $q_2(1,0,z) = = (1 \cdot 0) \cdot z = 0 \cdot z$. Since $0 \cdot 0 = 1$ and $0 \cdot 1 = 1$, we have $0 \cdot z = 1$.
To sum up: if $x \neq y$ then $q_1(x,y,z) = q_2(x,y,z)$ and so all the identities of (3) are satisfied. $\Diamond$

Corollary 2. *Every compact* $\Theta \in \mathrm{Con}\, A\, (A \in \mathcal{V}(\mathrm{I}))$ *has a complement.*

Proof. By [7] (and [8]) if $\mathcal{V}(\mathrm{I})$ is filtral then every compact congruence on $\mathcal{V}$ has a complement. $\Diamond$

Let $\mathrm{Con}^c A$ denote the lattice of compact congruences of A ; $\mathrm{Con}^{*c} A$ is the same lattice together with the element "1_A" and let $\mathcal{B}(\mathrm{Con}^{*c} A)$ be the Boolean lattice generated by $\mathrm{Con}^{*c} A$. (This one always exists, see [6]). Denoting the complement of $\Theta \in \mathrm{Con}\, A$ by Θ^-, let us define the operation " $*$ " on $\mathrm{Con}\, A$ as follows: $\Theta * \varphi = \Theta^- \vee \varphi$. (This way we obtain from $\mathcal{B}(\mathrm{Con}^{*c} A)$ an implication algebra in which, by [1], $(A, \cdot)$ can be dually embedded). Let Θ_a denote the congruence belonging to the principal filter [a] $(a \in A)$, (and at the same time to the element $a \in A$ as well).

Proposition 2. *Let* $(A, \cdot)$ *be an implication algebra and* $(A, \leq)$ *the derived partially ordered set. The following statements are equivalent:*

(i) $(A, \leq)$ *is a Boolean lattice*;

(ii) $(A, \leq)$ *and* $(\mathrm{Con}^{*c} A, \leq)$ *are dually order-isomorphic*;

(iii) $(A, \cdot)$ *and* $(\mathcal{B}(\mathrm{Con}^{*c} A), *)$ *are dually isomorphic implication algebras.*

Proof. (i) $\Rightarrow$ (ii) by [11]. (For a more general construction see [6]).

(ii) $\Rightarrow$ (i) and (iii) $\Rightarrow$ (ii): Since $\mathrm{Con}^{*c} A$ and $\mathcal{B}(\mathrm{Con}^{*c} A)$ both have a greatest element, $(A, \leq)$ has a least element and therefore by [1] it is a Boolean algebra.

(i) $\Rightarrow$ (iii): If $\Theta \in \mathrm{Con}^{*c} A$, then Θ can be written as a finite union of principal filters $[a_1], \dots, [a_n]$ $(a_1, \dots, a_n \in A,\ n \in \mathbb{N})$. Since $(A, \leq)$ is a lattice, $[a_1] \amalg \dots \amalg [a_n] = [a_1 \wedge \dots \wedge a_n]$ and therefore $\Theta[1]$ is a principal filter, i.e. there is an $a_\Theta \in A$ such that $[a_\Theta] = \Theta[1]$.

If $\overline{a}$ denotes the complement of a and $\Theta_{\overline{a}}$ the corresponding congruence then $[a] \cap [\overline{a}] = \{x | x \geq a \text{ and } x \geq \overline{a}\} = \{x | x \geq 1\} = \{1\}$, so $\Theta_a \wedge \Theta_{\overline{a}} = 0_A$ and $[a] \cup [\overline{a}] = [a \wedge \overline{a}] = [0] = A$, i.e.: $\Theta_a \vee \Theta_{\overline{a}} = 1_A$. Hence Θ_a and $\Theta_{\overline{a}}$ are complements of each other; furthermore since for all $\Theta \in \mathrm{Con}^c A$ there is an $a \in A$ such that $\Theta_a = \Theta$, $\overline{\Theta} \in \mathrm{Con}^c A$ holds as well (for all $\Theta \in \mathrm{Con}^c A$). However, this means that $\mathrm{Con}^{*c} A = \mathcal{B}(\mathrm{Con}^{*c} A)$ and by (i)$\Leftrightarrow$ (ii) $(A, \leq)$ and $(\mathcal{B}(\mathrm{Con}^{*c} A), \leq)$ are dually order isomorphic Boolean algebras. But in that case, by [1] again, they are dually isomorphic as implication algebras. $\diamondsuit$

3. Reflexive, compatible relations on implication algebras

A compatible relation $\rho \leq A \times A$ on $(A, \cdot)$ is called a *compatible tolerance* if ρ is reflexive and symmetric ([3]).

Definition 6 ([3]). An algebra $A \in \mathcal{V}$ is called *tolerance-trivial* (*T-trivial*) if every compatible tolerance on A is a congruence (i.e. transitive as well).

Theorem 2. *Let $(A, \cdot)$ be an implication algebra. Then the following statements are equivalent:*

(i) *Every reflexive compatible relation on $(A, \cdot)$ is a congruence;*

(ii) *$(A, \cdot)$ is tolerance-trivial;*

(iii) *$(A, \leq)$ is an implication lattice.*

Proof. (i) $\Rightarrow$ (ii) is clear.

(ii) $\Rightarrow$ (iii): Let us define a relation ρ as follows: $a \,\rho\, b \Leftrightarrow$ there is a $k \in A$ such that $a \geq k$ and $b \geq k$. By definition ρ is reflexive and symmetric. Let us show that ρ is compatible as well. Consider $c \,\rho\, d$ $(c, d \in A)$. This means that there is an $l \in A$ such that $c \geq l$ and $d \geq l$. Then $ca \geq a \geq k$ and $db \geq b \geq k$, while $ac \geq c \geq l$ and $bd \geq d \geq l$, thus $ca \,\rho\, db$ and $ac \,\rho\, bd$, ie.: ρ is compatible. By (ii) ρ is a congruence and $1 \,\rho\, a$ for any $a \in A$. Therefore $\rho = 1_A$. However, this means that for any $a, b \in A$, $\{a, b\}$ has a lower bound $m \in A$. By Prop.3 of [1] inf $\{a, b\}$ exists for all $a, b \in A$ and hence $(A, \leq)$ is an implication lattice.

(iii) $\Rightarrow$ (i): Let us now assume that $(A, \cdot)$ is an implication lattice. Using the idea of [4] (Th.8) first we show that if $(A, \leq)$ is a Boolean lattice then it satisfies (i). Indeed in that case there is a $0 \in A$ such that $0 \leq x$ for all $x \in A$ and by [1] again the complement of a, denoted by $\overline{a}$, can be obtained as $\overline{a} = a \cdot 0$. Since $a \vee b = (a \cdot b) \cdot b, a \wedge b = [a \cdot (b \cdot 0)] \cdot 0$, every compatible relation on $(A, \cdot)$ is also a compatible relation on $(A, \wedge, \vee, 1, 0, ^-)$. But since this algebra belongs to a Mal'cev variety all its reflexive compatible relations are congruences [3].

Now let $(A, \cdot)$ be an implication lattice and ρ a compatible reflexive relation on A. Let $a \rho b, b \rho c$ (for $a, b, c \in A$). Then $(a \wedge b) \wedge c = d$ exists and it is the greatest lower bound of $\{a, b, c\}$. The restriction of " $\cdot$ " to the principal filter $[d]$ is a Boolean algebra (with "0" element d) and $a, b, c \in [d]$.

On the other hand the restriction of ρ to $[d]$ is also compatible and reflexive and thus it is also a congruence on $([d], \cdot)$. But this means that $a \rho b \Rightarrow b \rho a$ and $a \rho b, b \rho c \Rightarrow a \rho c$. In conclusion ρ is a congruence on $(A, \cdot)$ as well. $\diamondsuit$

Corollary 3. *Let $(A, \cdot)$ be an implication algebra. If the derived structure $(A, \leq)$ is an implication lattice, then the congruences of $(A, \cdot)$ permute.*

Proof. In this case $(A, \cdot)$ is tolerance-trivial by Th.2. According to [10] every tolerance-trivial algebra has permutable congruences. $\diamondsuit$

Corollary 4. *For a finite implication algebra $(A, \cdot)$ the following statements are equivalent:*

(i) The derived partially ordered set $(A, \leq)$ is a Boolean lattice;

(ii) $(A, \cdot)$ is tolerance-trivial;

(iii) $(A, \cdot)$ *and* $(\mathrm{Con}\, A, *)$ *are dually isomorphic;*

(iv) $(A, \leq)$ *and* $(\mathrm{Con}\, A, \leq)$ *are dually order isomorphic.*

Proof. The proof is based on the fact that if A is finite then all its congruences are compact and so $\mathrm{Con}\, A = \mathrm{Con}^c A = \mathrm{Con}^{*c} A = \mathcal{B}(\mathrm{Con}^{*c} A)$. Applying Prop.2 we get Cor.4.

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