Mathematica Pannonica
3/1 (1992), 3 – 17

SEMI-HOMOMORPHISMS OF NEAR-RINGS

Y. **Fong**

Department of Mathematics, National Cheng Kung University, Tainan 70102, Taiwan, Republic of China.

L. **van Wyk**

Department of Mathematics, University of Stellenbosch, 7600 Stellenbosch, Republic of South Africa.

Received September 1990

AMS Subject Classification: 16 A 76, 16 A 72

Keywords: Semi-homomorphisms, semi-subgroups, finite abelian groups, near-rings, Dickson near-fields.

Abstract: The semi-subgroups of finite abelian groups are characterized and comparisons are made between semi-homomorphisms of rings and near-rings. This study leads to an alternative proof of a result by Zassenhaus in 1936, viz. that the automorphism group of the smallest Dickson non-field is isomorphic to the symmetric group of degree 3.

1. Introduction

Projectivity in classical projective geometry led to a study of semi-automorphisms of rings (see [1] and [3]). In [2] and [8] it was proven that every semi-automorphism of a division ring is either an automorphism or an anti-automorphism, and similarly for a matrix ring over a division ring.

Huq [9] presented a general study of semi-homomorphisms of rings, following the mentioned papers and, amongst others, Herstein's study of semi-homomorphisms of groups in [7]. In [4] the authors introduced semi-subgroups of groups and provided counterexamples to some of the assertions in [9]. The purpose of this paper is, on the one hand, to continue the investigation of the structure of semi-subgroups.

In Section 2 we characterize the semi-subgroups of finite abelian groups and the semi-subrings of $\mathbb{Z}_n$, the ring of integers modulo n, which shows that the notions semi-subgroup and semi-subring are not equivalent in $\mathbb{Z}_n$, unlike the case of subgroups and subrings of $\mathbb{Z}_n$.

In Section 3 we initiate a study of semi-homomorphisms of near-rings, although th first part applies to groups in general. Herstein [7] calls a mapping $\varphi : G \to H$ between groups (written additively) a *semi-homomorphism* if

$$\varphi(a+b+a) = \varphi(a) + \varphi(b) + \varphi(a) \tag{1}$$

for all $a, b \in G$. By taking $b = -a$, it follows that

$$\varphi(-a) = -\varphi(a) \tag{2}$$

for every $a \in G$; in particular we have $2\varphi(0) = 0$, where 0 denotes the neutral elements of G and H. Herstein showed that if the centralizer of $\varphi(G)$ in H is 0, then (2) can be generalized to

$$\varphi(na) = n\varphi(a) \tag{3}$$

for every integer n and every $a \in G$. We prove that the condition that the subset $\{\varphi(2a) - 2\varphi(a) : a \in G\}$ of H contains no elements of order 2, is also sufficient for (3). This result strengthens [4, Corollary 3.4] in which G and H are assumed to be abelian.

We use Huq's definition of a semi-homomorphism of rings as the definition of a semi-homomorphism of near-rings, i.e. a mapping $\varphi : R \to S$ between near-rings satisfying (1) and the condition

$$\varphi(aba) = \varphi(a)\varphi(b)\varphi(a) \tag{4}$$

for all $a, b \in R$. The left hand mapping convention is used for near-rings, since we shall be dealing with right near-rings. For details about near-rings we refer the reader to the books by Meldrum [11] or Pilz [12].

The image T of a semi-homomorphism φ of groups (rings, near-rings) is easily seen to be a *semi-subgroup* (*semi-subring, semi-subnear-ring*) of the codomain of φ, i.e.

$$a + b + a \in T \quad (\text{and } aba \in T) \tag{5}$$

for all $a, b \in T$. Note that a semi-subnear-ring in general does not concern a semi-near-ring (see e.g. Weinert [15]).

As far as semi-homomorphisms of near-rings in particular are concerned, we show that there are many similarities to the ring case when the near-rings under consideration happen to be abelian (e.g. in the case of near-fields), but that there are in general also some striking differences. (Recall that abelian near-rings can still be very much "non-ring-like".) During the investigation of the problem whether every semi-automorphism of a near-field is an automorphism, we obtained a surprising result, viz. that every automorphism φ of $(GF(3^2), +)$ satisfying $\varphi(1) = 1$ is an automorphism of $(GF(3^2), +, \circ)$, the smallest Dickson near-field which is not a field. Hence every semi-automorphism of $(GF(3^2), +, \circ)$ is an automorphism, and so the automorphism group of $(GF(3^2), +, \circ)$ comprises 6 elements (and is isomorphic to S_3, the symmetric group of degree 3). This provides an alternative proof of a special case of [16, Theorem 18].

Throughout the paper the symbol $\subset$ denotes strict inclusion and all near-rings are associative.

2. A characterization of the semi-subgroups of finite abelian groups

Let $(G, +)$ be a (not necessarily abelian of finite) group. Every subgroup of G is obviously a semi-subgroup, but the converse need not be true. The term *non-subgroup* will be used for a semi-subgroup which is not a subgroup. Our purpose in this section is to give a characterization of the semi-subgroups of finite abelian groups.

We denote the semi-subgroup of G generated by a subset $\{a_1, a_2, \ldots, a_n\}$, $n \geq 1$, of G by $(a_1, a_2, \ldots, a_n)_s$ and we stick to the usual notation $(a_1, a_2, \ldots, a_n)$ for the subgroup of G generated by $\{a_1, a_2, \ldots, a_n\}$. The order of an element a of G will be denoted by

$o(a)$, and $|X|$ will stand for the number of elements of a subset X of G.

The first two results describe the semi-subgroup of a group generated by a singleton.

Proposition 2.1. *If $o(a)$, $a \in G$, is even or infinite, then $(a)_s$ is a non-subgroup of G.*

Proof. Firstly, let $o(a) = 2k$ for some $k > 0$. We assert that $|(a)_s| = k$; to be more precise, $(a)_s$ comprises the following different elements: $a, 3a, 5a, \ldots, (2(k-1)+1)a$. For if $(2i+1)a = (2j+1)a$ for some i and j, $0 \leq i, j \leq k-1$, then $2(i-j)a = 0$. But $2(i-j) \leq 2k-2 < k$, which contradicts the assumption that $o(a) = 2k$. It is easily verified that these elements compose a semi-subgroup of G which is contained in $(a)_s$, and so our assertion is valid. Furthermore, $0 \notin (a)_s$, otherwise the assumption that $o(a) = 2k$ is contradicted again. Hence $(a)_s$ is not a subgroup of G. The case where $o(a)$ is infinite, is dealt with similarly. $\diamondsuit$

Proposition 2.2. *If $o(a)$, $a \in G$, is odd, then $(a)_s = (a)$.*

Proof. Let $o(a) = 2l+1$ for some $l > 0$. (If $l = 0$, then $a = 0$, and the result is trivial.) Consider the subset $T := \{a, 3a, 5a, \ldots, (2(l-1)+1)a, (2l+1)a, (2(l+1)+1)a, \ldots, (2(2l)+1)a\}$ of $(a)_s$. Clearly $T = \{a, 3a, 5a, \ldots, (2l-1)a, 0, 2a, 4a, \ldots, 2la\} = (a)$, and so $(a)_s = a(a)$. $\diamondsuit$

Henceforth in this section G will be a finite abelian group.

Theorem 2.3. *If $|G|$ is odd, then G has no non-subgroups.*

Proof. By Proposition 2.2 it clearly suffices to show that $(a,b)_s = (a,b)$ for all $a, b \in G$, since the order of every element of G is odd. Let $o(a) = 2k+1$ and $o(b) = 2l+1$ for some $k, l \geq 0$. Since $(2m+1)a + nb = (k+m+1)a + nb + (k+m+1)a$ and $2ma + nb = ma + nb + ma$ for every $m, n \geq 0$, it follows from Proposition 2.2 that $(a,b) \subseteq (a,b)_s$, and so $(a,b)_s = (a,b)$. $\diamondsuit$

As a result of Theorem 2.3 and the Fundamental Theorem on finite abelian groups, we study now the semi-subgroups of $\mathbb{Z}_{2^i}$, $i \geq 1$. The greatest common divisor of $m, n \in \mathbb{Z}$ will be denoted by $\gcd(m,n)$.

Proposition 2.4. *If $0 \neq a \in \mathbb{Z}_{2^i}$, $i \geq 1$, then $(a)_s = \{g, 3g, 5g, \ldots, (2^i/g - 1)g\}$, where $g = \gcd(a, 2^i)$.*

Proof. Since $a \neq 0$ and $o(a)$ divides 2^i, it follows that $o(a)$ is even, and so by Proposition 2.1 $(a)_s = \{a, 3a, 5a, \ldots, (2(2^{j-1}-1)+1)a\}$, where $o(a) = 2^j$ for some j, $1 \leq j \leq i$. But $o(a) = 2^i/g$, and so $g = 2^{i-j}$. Therefore $2^{j-1} = 2^i/2g$, which implies that the $2^i/2g$ elements of $(a)_s$

are all the odd multiples of $g(\text{mod } 2^i)$, because a is an odd multiple of g. Hence $(a)_s = \{g, 3g, 5g, \ldots, (2(2^i/2g - 1) + 1)g\}$. ◇

Theorem 2.5. *The semi-subgroups in Proposition 2.4 are precisely the non-subgroups of* $\mathbb{Z}_{2^i}$, $i \geq 1$.

Proof. We show that, for $a, b \in \mathbb{Z}_{2^i}$, either $(a, b)_s = (a)_s = (b)_s$ or $(a, b)_s = (g)$, where $g := \min(\gcd(a, 2^i), \gcd(b, 2^i))$. Firstly, if $\gcd(a, 2^i) = \gcd(b, 2^i)$, then either $a = 0 = b$, in which case $(a, b)_s = (0) = (a)_s = (b)_s$, or $a \neq 0$ and $b \neq 0$, in which case by Proposition 2.4 we have $(a)_s = (b)_s = \{g, 3g, 5g, \ldots, (2^i/g - 1)g\}$, and so $(a, b)_s = (a)_s = (b)_s$. Secondly, let $\gcd(a, 2^i) < \gcd(b, 2^i) =: h$. Then $h = 2^k g$ for some $k \geq 1$, since g and h are powers of 2. It follows from Proposition 2.4 that $(a)_s$ comprises all the odd multiples of g, and $(b)_s$ comprises all the odd multiples of h (mod 2^i). Hence $(a, b)_s$ contains $(a)_s$ as well as at least one even multiple of g. It can now be readily seen that $(a, b)_s = (g)$, since $x + y + x \in (a, b)_s$ for all $x \in (a)_s$, $y \in (b)_s$. ◇

As in the case of subgroups, it is easy to see that if K is a semi-subgroup of the direct sum of two (not necessarliy abelian of finite) groups G_1 and G_2, then $\pi_i K$ is semi-subgroup of G_i, $i = 1, 2$, where π_i denotes the i-th coordinate projection.

The foregoing results lead to a characterization of the semi-subgroups and non-subgroups of a finite abelian group:

Theorem 2.6. (a) *If* $H : \mathbb{Z}_{m_1} \oplus \mathbb{Z}_{m_2} \oplus \cdots \oplus \mathbb{Z}_{m_n}$ *is a finite abelian group, with* m_k *an odd prime for* $k = 1, 2, \ldots, n, n \geq 1$, *then* H *has no non-subgroups.*

(b) *If* $G := \mathbb{Z}_{n_1} \oplus \mathbb{Z}_{n_2} \oplus \cdots \oplus \mathbb{Z}_{n_s} \oplus H$ *is a finite abelian group, with* n_k *a power of 2 for* $k = 1, 2, \ldots, s$, $s \geq 1$, *and where* $H = 0$ *or* H *is as in (a), then:*

(i) If G_{j_q}, $q = 1, 2, \ldots, r$, $1 \leq r \leq s$, is a non-subgroup of $\mathbb{Z}_{n_{j_q}}$, where $j_q \in \{1, 2, \ldots, s\}$, G_t is a subgroup of $\mathbb{Z}_{n_t}$ for $t \in \{1, 2, \ldots, s\} \setminus \{j_1, j_2, \ldots, j_r\}$, and K is a subgroup of H, then the direct sum of these G_{j_q}'s, G_t's and K is a non-subgroup of G.

(ii) If A is a semi-subgroup of G, then $\pi_j A$ is a semi-subgroup of $\mathbb{Z}_{n_j}$ for $j = 1, 2, \ldots, s$, and $\pi_{s+1} A$ is a subgroup of H; furthermore, if A is a non-subgroup of G, then $\pi_t A$ is a non-subgroup of $\mathbb{Z}_{n_t}$ for some t, $1 \leq t \leq s$.

We have seen that $0 \in S$, S a semi-subgroup of a finite abelian group H, if and only if S is a subgroup of H. Also, every semi-subgroup of $\mathbb{Z}_n$ is "cyclic" in the sense that it is generated by a single element of $\mathbb{Z}_n$. The picture might be much more different in general, even in finitely generated abelian groups; e.g. $\{2k : k \geq 0\}$ and $(2,5)_s = = \{2,5,6,9,10\} \cup \{m : m \geq 12\}$ are non-subgroups of $\mathbb{Z}$.

Although every subgroup of $\mathbb{Z}_n$, $n > 1$, is a subring of $\mathbb{Z}_n$ (and vice versa), the same is not for semi-subgroups and semi-subrings, e.g. it follows from Proposition 2.4 that $\{2,6\}$ is a semi-subgroup of $\mathbb{Z}_8$, but is not a semi-subring of $\mathbb{Z}_8$, since $2^3 = 0 \notin \{2,6\}$. However, $\mathbb{Z}_n$ may contain semi-subrings which are not subrings, and which we call *non-subrings*. We make this scheme of affairs precise in the last part of this section, in which we show that the results about the semi-subring of $\mathbb{Z}_n$ generated by an element a of $\mathbb{Z}_n$, denoted by $(a)_{sr}$, are surprisingly different form those about the semi-subgroups as far as $o(a)$ is concerned, in the sense that if $o(a)$ is even, then it does not necessarily follow that $(a)_{sr}$ is a non-subring of $\mathbb{Z}_n$.

Lemma 2.7. *If n is odd and $n > 1$, then $\mathbb{Z}_n$ has no non-subrings.*

Proof. Let S be a semi-subring of $\mathbb{Z}_n$. Then $(S,+)$ is a semi-subgroup of $(\mathbb{Z}_n,+)$, and so the result follows from Theorem 2.3.

For the rest of this section n will be even.

Proposition 2.8. *If a is odd and $a < n$, then $(a)_{sr} = (a)_s$, a non-subring of $\mathbb{Z}_n$.*

Proof. First note that $o(a)$ is even, because $o(a) = n/g$ and g is odd, where $g := \gcd(a,n)$. Hence by Proposition 2.1 $(a)_s$ comprises the odd multiples of a (mod n). Furthermore, $(2i+1)a(2j+1)a(2i+1)a$ is an odd multiple of a for all i and j, because a is odd, and so $(a)_{sr} = (a)_s$, a non-subring of $\mathbb{Z}_n$, since it follows from Proposition 2.1 that $(a)_s$ is a non-subgroup of $\mathbb{Z}_n$. ♢

Proposition 2.9. *Let b be even, $b < n$. If*

(i) $o(b)$ is odd, then $(b)_{sr} = (b)_s$, a subring (subgroup) of $\mathbb{Z}_n$.

(ii) $o(b)$ is even, then $(b)_{sr}$ is a subring (subgroup) of $\mathbb{Z}_n$, and $(b)_s \subset (b)_{sr}$.

Proof. (i) By Proposition 2.2.

(ii) Since $(b)_s \subseteq (b)_{sr}$, it follows from Proposition 2.1 that $(b)_{sr}$ contains the odd multiples of b(mod n). But b^3 is a multiple of b and b is even, and so it can be seen, as in the last part of the proof of Theorem 2.5, that $(b)_{sr}$ contains all the even multiples of b(mod n) as well. Hence $(b)_s \subset (b)_{sr} = (b)$. ♢

The foregoing results lead to

Theorem 2.10. *The semi-subrings in Proposition 2.8 are precisely the*

non-subrings of $\mathbb{Z}_n$.
Proof. Very much similar to that of Theorem 2.5. ◇

3. Semi-homomorphisms of near-rings

Recall form Section 2 that we deal with right near-rings, i.e. the left distributive law is not required.

Proposition 2.4 provides a host of *non-subnear-rings* of near-rings, i.e. semi-subnear-rings which are not subnear-rings, viz. for any non-subgroup T of $\mathbb{Z}_{2^i}$, $i \geq 1$, as in Proposition 2.4, $\{f \in M(\mathbb{Z}_{2^i}) : f(T) \subseteq \subseteq T\}$ is a non-subnear-ring of the full near-ring $M(\mathbb{Z}_{2^i})$ of mappings on $\mathbb{Z}_{2^i}$.

The following two results, which explore properties of semi-homomorphisms of restricted classes of near-rings, can be proved exactly as in the ring case (see [4, Lemma 3.3] and [9, Proposition 8] respectively):
Lemma 3.1. *A semi-homomorphism* $\varphi : R \to S$ *of abelian near-rings is a homomorphism of the underlying additive groups if and only if the semi-subgroup* $\{\varphi(a+b) - \varphi(b) - \varphi(a) : a, b \in R\}$ *of* $(S, +)$ *contains no elements of order 2.*
Lemma 3.2. *Let* $\varphi : F \to F'$ *be a semi-homomorphism of near-fields. If* $\varphi(a) \neq 0$ *for some* $0 \neq a \in F$*, then* $\varphi(a^{-1}) = (\varphi(a))^{-1}$.

Herstein [7] showed that if the centralizer of $\varphi(G)$ in H is 0, where $\varphi : G \to H$ is a semi-homomorphism of groups, then (3) holds. The authors [4] showed that if G and H are abelian and the semi-subgroup $\{\varphi(2a) - 2\varphi(a) : a \in G\}$ of H contains no element of order 2, then (3) also holds. However, G and H need not be abelian, as will be shown shortly. We first need
Lemma 3.3. *Let* $\varphi : R \to S$ *be a semi-homomorphism of near-rings. Then* $\varphi(a+b) - \varphi(a) - \varphi(b) = \varphi(a) + \varphi(b) - \varphi(b+a)$ *for all* $a, b \in R$.
Proof. For $a, b \in R$ we have by (1) and (2):

$$\begin{aligned}\varphi(a+b) &= \varphi(a+b+(-(b+a))+b+a)\\ &= \varphi(a)+\varphi(b+(-(b+a))+b)+\varphi(a)\\ &= \varphi(a)+\varphi(b)+\varphi(-(b+a))+\varphi(b)+\varphi(a)\\ &= \varphi(a)+\varphi(b)-\varphi(b+a)+\varphi(b)+\varphi(a),\end{aligned}$$

from which the result follows. ◇

Proposition 3.4. *If $\varphi : R \to S$ is semi-homomorphism of near-rings such that the set $\{\varphi(2a)-2\varphi(a) : a \in R\}$ contains no elements of order 2, then $\varphi(na) = n\varphi(a)$ for every integer n and every $a \in R$.*
Proof. Firstly, since $2\varphi(0) = 0$, it follows that $\varphi(0) \in \{\varphi(2a)-2\varphi(a) : : a \in R\}$, and so the result holds for $n = 0$. The case $n = 1$ is trivial. Next, let $a = b$ in Lemma 3.3. Then $\varphi(2a) - 2\varphi(a) = 2\varphi(a) - \varphi(2a)$, and so $2(\varphi(2a) - 2\varphi(a)) = 0$, and so the result holds for $n = 2$. Using induction on n and assuming that $n > 2$, we get $\varphi(na) = \varphi(a + (n-2)a + a) = \varphi(a) + n(n-2)\varphi(a) + \varphi(a) = n\varphi(a)$, which establishes the result for every $n \geq 0$. Finally, by (2) and since $(-n)a = n(-a)$, the result also holds for $n < 0$. $\diamondsuit$

Note that the above two results hold merely in the presence of a semi-homomorphism of groups, since the multiplicative structure of the near-rings has not been invoked at all.

The multiplicative version of $2\varphi(0) = 0$ is, of course, $(\varphi(1))^2 = 1$, where 1 denotes the identities of the domain and codomain of φ. Huq [9] proved that if $\varphi : R \to S$ is a semi-homomorphism of rings with identities such that $1 \in \varphi(R)$ and S is a non-trivial ring without non-zero divisors of zero, then $\varphi(1) = 1$ of -1. We shall show in Example 3.7 that this result does not extend to near-rings in general, not even if φ is also a homomorphism of the underlying additive groups. However, we still have

Proposition 3.5. *If $\varphi : R \to S$ is a semi-homomorphism of near-rings with identities such that $\varphi(R)$ is a near-field and $1 \in \varphi(R)$, then $(\varphi(1))^2 = 1$ and $\varphi(1) = 1$ or -1.*
Proof. That $(\varphi(1))^2 = 1$, follows as in [9, Proposition 6]. A non-trivial near-ring-theoretic result states that if $r^2 = 1$ in a near-field, then $r = 1$ or -1 (see e.g. [12, Proposition 8.10]). $\diamondsuit$

The following example shows that it is possible that $\varphi(1) = -1$ under the conditions of Proposition 3.5. The reason for exhibiting this example must be seen against the background of the conjecture before Example 3.10.

Example 3.6. Let $(F, +, \circ)$ be the (infinite) Dickson near-field arising from $Q(x)$, the field of rational functions over the rationals, by defining multiplication as follows:

$$g(x)/h(x) \circ p(x)/q(x) = \begin{cases} 0, \text{ if } p(x)/q(x) = 0 \\ (g(x+d)/h(x+d)) \cdot (p(x)/qx)), \text{ otherwise,} \end{cases}$$

where $d := \deg(p(x)) - \deg(q(x))$ and $\cdot$ is the familiar multiplication in $\mathrm{Q}(x)$. (See [11, Example 8.29] for more details.) Then F is not a division ring, since the left distributive law does not hold. Define $\varphi : F \to F$ by $\varphi(g(x)/h(x)) = -g(x)/h(x)$. It can be verified that φ is a semi-homomorphism of near-fields (and an endomorphism of the underlying additive group). Furthermore, $\varphi(1) = -1$.

As in Heatherly and Olivier [5,6] we define a *near integral domain* as a (right) zerosymmetric near-ring having no non-zero divisors of zero and having at least one nonzero element which is not a right identity. (Note that some near-ringers call these nontrivial near integral domains "integral near-rings"). McQuarrie [10] originally devised the following (infinite) near integral domain which was later used by Heatherly and Olivier [6] to show that the additive group of a near integral domain may not be nilpotent. It is not only a near integral domain, but it is also a distributively generated (dg) near-ring with identity.

Let G_2 be the free (additive) group on two generators x and y, and define for every integer n the mapping $\Gamma_n : G_2 \to G_2$ by $\Gamma_n(h(x,y)) = h(nx, ny)$, where $h(x,y)$ is an arbitrary word in G_2. Every Γ_n is an element of the full near-ring $M(G_2)$ of mappings on G_2; in fact, the Γ_n's are distributive elements of $M(G_2)$. Let R be the subnear-ring of $M(G_2)$ generated by $\{\Gamma_n : n \in \mathbb{Z}\}$, i.e. $(R, \{\Gamma_n : n \in \mathbb{Z}\})$ is a dg near-ring. Then by [11, Lemma 9.11] $(R,+)$ is generated as a group by $\{\Gamma_n : n \in \mathbb{Z}\}$.

We use the above near integral domain in the following example, in which we show that Huq's result, which was mentioned just before Proposition 3.5, does not extend to near-rings.

Example 3.7. Define $\varphi : R \to R$ by

$$\varphi(\sum_{i=1}^{k} \varepsilon_{n_i}, \Gamma_{n_i}) = \sum_{i=1}^{k} \varepsilon_{n_i} \Gamma_{-n_i},$$

where $\varepsilon_{n_i} = \pm 1$ and $n_i \in \mathbb{Z}$ for $i = 1, 2, \ldots, k$. We show that φ is well-defined. First note that $\Gamma_n(h(-x,-y)) = h(n(-x), n(-y)) = h((-n)x, (-n)y) = \Gamma_{-n}(h(x,y))$ for every $n \in \mathbb{Z}$ and every word $h(x,y)$ in G_2. Suppose now that $\sum_{i=1}^{k} \varepsilon_{n_i} \Gamma_{n_i} = \sum_{j=1}^{l} \varepsilon_{m_j} \Gamma_{m_j}$. Then

$$\varepsilon_{n_1}\Gamma_{n_1} + \cdots + \varepsilon_{n_k}\Gamma_{n_k} - \varepsilon_{m_l}\Gamma_{m_l} - \cdots - \varepsilon_{m_1}\Gamma_{m_1} \equiv 0,$$

and so

$$(\varepsilon_{n_1}\Gamma_{n_1} + \cdots + \varepsilon_{n_k}\Gamma_{n_k} - \varepsilon_{m_l}\Gamma_{m_l} - \cdots - \varepsilon_{m_1}\Gamma_{m_1})(h(-x,-y)) = 0.$$

Hence, by the above remark

$$(\varepsilon_{n_1}\Gamma_{-n_1} + \cdots + \varepsilon_{n_k}\Gamma_{-n_k} - \varepsilon_{m_l}\Gamma_{-m_l} - \cdots - \varepsilon_{m_1}\Gamma_{-m_1})(h(x,y)) = 0,$$

i.e. $\sum_{i=1}^{k} \varepsilon_{n_i}\Gamma_{-n_i} = \sum_{j=1}^{l} \varepsilon_{m_j}\Gamma_{-m_j}$.

It is now obvious that φ is an endomorphism of $(R,+)$. Also, since $\varphi(\Gamma_n\Gamma_m\Gamma_n) = \varphi(\Gamma_{nmn}) = \Gamma_{-nmn} = \Gamma_{-n}\Gamma_{-m}\Gamma_{-n} = \varphi(\Gamma_n)\varphi(\Gamma_m)\varphi(\Gamma_n)$ for all $n,m \in \mathbb{Z}$, it follows easily that φ is a semi-homomorphism of near-rings.

The identity of R is Γ_1, and $\varphi(\Gamma_1) = \Gamma_{-1}$. Furthermore, $\Gamma_{-1} \neq -\Gamma_1$, because $\Gamma_{-1}(x+y) = -x + (-y) \neq (-y) + (-x) = -(x+y) = -\Gamma_1(x+y)$. Hence $\varphi(\Gamma_1) \neq -\Gamma_1$, and $\varphi(\Gamma_1) \neq \Gamma_1$.

It is well known that if F is a near-field, then either $F \cong M_c(\mathbb{Z}_2)$, the near-field of constant functions on $\mathbb{Z}_2$, or F is zero-symmetric (see e.g. [11, Proposition 8.1]). So if we exclude this "silly" near-field $M_c(\mathbb{Z}_2)$ (see Example 3.9), then the proof of [9, Proposition 9] serves to a great extent as the proof of the following proposition.

Let $Z(R)$ denote the center of a near-ring R.

Proposition 3.8. *If $\varphi : F \to F'$ is a semi-homomorphism of near-fields, then $\varphi(1) \in Z(\varphi(F))$.*

Proof. If φ is the zero map, then by the above remark $\varphi(1) = 0 \in Z(\varphi(F))$, since F' is zero-symmetric. If $\varphi(F) \neq 0$, then $\varphi(1) \neq 0$, otherwise $\varphi(a) = \varphi(1a1) = \varphi(1)\varphi(a)\varphi(1) = 0$ for all $a \in F$. Since $\varphi(a) = \varphi(1)\varphi(a)\varphi(1)$, the result follows from Lemma 3.3. ◇

Example 3.9. Let $\varphi : M_c(\mathbb{Z}_2) \to M_c(\mathbb{Z}_2)$ be the identity map. Then φ is an isomorphism of "near-fields" (in the sense of the remark preceding Proposition 3.8), but $\varphi(1) \notin Z(M_c(\mathbb{Z}_2))$, the empty set.

Hua [8] proved that every *semi-automorphism* of a division ring R, i.e. an automorphism φ of $(R,+)$ satisfying $\varphi(1) = 1$ and

$$\varphi(aba) = \varphi(a)\varphi(b)\varphi(a) \tag{6}$$

for all $a, b \in R$, is an automorphism of an anti-automorphism.

We have been unable to determine whether Hua's result can be "extended" to near-fields, i.e. whether every semi-automorphism of a near-field (where semi-automorphism is defined as above) is an automorphism. Notice that the lack of one distributive law should prevent a semi-automorphism from being an anti-automorphism, as is shown in [13] for finite simple near-rings with associated idempotents $e_1, e_2, \ldots, e_t$, where $t \geq 2$. (The case $t = 1$ produces the near-fields.)

After examining numerous examples, including the exceptional finite near-fields, i.e. the seven finite near-fields which are not Dickson near-fields (see e.g. [14, Chapter 4], where the structure of theses seven near-fields, which was originally determined by Zassenhaus [16], is made clear), we arrived at the following:

Conjecture. Every semi-automorphism of a near-field is an automorphism.

The examination of this problem yields a surprising result, the significance of which the authors do not understand fully at present and which perhaps has independent interest. It is well known that there are only two automorphisms of the Galois field $(GF(3^2), +, \cdot)$, viz. the (identity-preserving) automorphisms of $\mathbb{Z}_3[i]$ mapping i onto i and $2i$ respectively, where i is a root of the irreducible polynomial $x^2 + 1$ in $\mathbb{Z}_3[x]$. The term *non-field* is widely used in near-ring circles for a near-field which is not a field. The smallest (Dickson) non-field is given by $(GF(3^2), +, \circ)$, where $\circ$ is defined by

$$x \circ y = \begin{cases} x \cdot y, & \text{if } y \text{ is a square in the Galois field } (GF(3^2), +, \cdot) \\ x^3 \cdot y, & \text{otherwise.} \end{cases}$$

(See e.g. [12] for more details.) We show in the following example that every automorphism φ of $(GF(3^2), +)$ satisfying $\varphi(1) = 1$ is an automorphism of $(GF(3^2), +, \circ)$, and so every semi-automorphism of $(GF(3^2), +, \circ)$ is an automorphism. Hence there are precisely 6 automorphisms of $(GF(3^2), +, \circ)$ and the automorphism group of $(GF(3^2), +, \circ)$ is isomorphic to S_3. This provides an alternative proof of a special case of [16, Theorem 18].

Example 3.10. Let $(GF(3^2), +, \circ)$ be the smallest Dickson non-field as defined above, and let $a + bi$, $a, b \in \mathbb{Z}_3$, be the elements of $GF(3^2)$, with $i^2 = -1 = 2$. If φ is an automorphism of $(\mathbb{Z}_3[i], +)$ and $\varphi(1) = 1$, then $\varphi(a + bi) = a + b\varphi(i)$, and so $\varphi(i) \in \{i, 2i, 1+i, 1+2i, 2+i, 2+2i\}$. Since $k^3 = k$ and $3k = k$ and $3k = 0$ for every $k \in \mathbb{Z}_3$, we have, for all

$a+bi$ and $c+di$, the following:

$$(7)\qquad \begin{aligned}\varphi((a+bi)\circ(c+di)) &= \varphi((a+bi)\cdot(c+di))\\ &= \varphi(ac+2bd+(ad+bc)i)\\ &= ac+2bd+(ad+bc)\varphi(i)\end{aligned}$$

if $c+di$ is a square in $(\mathbb{Z}_3[i],+,\cdot)$, and

$$(8)\qquad \begin{aligned}\varphi((a+bi)\circ(c+di)) &= \varphi((a+bi)^3\cdot(c+di))\\ &= \varphi(ac+2bi)\cdot(c+di))\\ &= \varphi(ac+bd+(ad+2bc)i)\\ &= ac+bd+(ad+2bc)\varphi(i)\end{aligned}$$

if $c+di$ is not a square in $(\mathbb{Z}_3[i],+,\cdot)$. Also,

$$(9)\qquad \begin{aligned}\varphi(a+bi)\circ\varphi(c+di) &= (a+b\varphi(i))\circ(c+d\varphi(i))\\ &= ac+bd(\varphi(i))^2+(ad+bc)\varphi(i)\end{aligned}$$

if $c+d\varphi(i)$ is a square in $(\mathbb{Z}_3[i],+,\cdot)$, and

$$(10)\qquad \begin{aligned}\varphi(a+bi)\circ\varphi(c+di) &= (a+b\varphi(i))^3\cdot(c+d\varphi(i))\\ &= (a+b(\varphi(i))^3)\cdot(c+d\varphi(i))\\ &= ac+bd(\varphi(i))^4+ad\varphi(i)+bc(\varphi(i))^3\end{aligned}$$

if $c+d\varphi(i)$ is not a square in $(\mathbb{Z}_3[i],+,\cdot)$.
It can be verified that $c+di$ is a square in $(\mathbb{Z}_3[i],+,\cdot)$ if and only if $cd=0$, and so we consider the following cases:

(I) $c+di$ is a square in $(\mathbb{Z}_3[i],+,\cdot)$and $d=0$:

We have $c+di=c$, and so it follows from (7) and (9) that

$$\varphi((a+bi)\circ(c+di)) = \varphi(a+bi)\circ\varphi(c+di).$$

(II) $c+di$ is a square in $(\mathbb{Z}_3[i],+,\cdot)$and $d\neq 0$:

In this case $c=0$, but $c+d\varphi(i)$ may be or may not be a square in $(\mathbb{Z}_3[i],+,\cdot)$. The conditions on φ imply that $\varphi(k)=k$ for every $k\in\mathbb{Z}_3$, and so $\varphi(i)\notin\{0,1,2\}$. Therefore $\varphi(i)=k+li$ for some $k,l\in\mathbb{Z}_3$, $l\neq 0$.

If $k = 0$, then $c + d\varphi(i) = dli$, which is a square in $(\mathbb{Z}_3[i], +, \cdot)$, and so by (7) and (9) we have

$$\varphi((a + bi) \circ (c + di)) = \varphi(a + bi) \circ \varphi(c + di),$$

because $(\varphi(i))^2 = (li)^2 = 2l^2 = 2$. Suppose now that $k \neq 0$. Then $c + d\varphi(i) = d(k + li) = dk + dli$, which is not a square in $(\mathbb{Z}_3[i], +, \cdot)$. A direct calculation shows that $(k + li)^4 = 2$ in $(\mathbb{Z}_3[i], +, \cdot)$if $kl \neq 0$, and so the desired equality now follows from (7) and (10).

(III) $c + di$ is not a square in $(\mathbb{Z}_3[i], +, \cdot)$, and $\varphi(i) = li$ for some $l \in \mathbb{Z}_3$, $l \neq 0$:

Since $cd \neq 0$, it follows that $c + d\varphi(i)$ is not a square in $(\mathbb{Z}_3[i], +, \cdot)$. Furthermore, in this case $(\varphi(i))^2 = 2$ and $(\varphi(i))^4 = 1$ in $(\mathbb{Z}_3[i], +, \cdot)$, and so by (8) and (10) we have

$$\varphi((a + bi) \circ (c + di)) = \varphi(a + bi) \circ \varphi(c + di).$$

(IV) $c + di$ is not a square in $(\mathbb{Z}_3[i], +, \cdot)$, and $\varphi(i) = k + li$ for some $k, l \in \mathbb{Z}_3$, with $kl \neq 0$:

Now $c + d\varphi(i) = c + dk + dli$, which may be or may not be a square in $(\mathbb{Z}_3[i], +, \cdot)$. Firstly, suppose it is a square, i.e. $c + dk = 0$. Then $d + c\varphi(i) = d + c(k + li) = d + ck + cli$, and $d(\varphi(i))^2 = d(k^2 - l^2 + 2kli) = = 2dkli$, since $k^2 = l^2 = 1$. But $c = 2dk$ and $d + ck = dk^2 + ck = = (c + dk)k = 0$, and so $d + c\varphi(i) = d(\varphi(i))^2$. Hence by (8) and (9) we have

$$\begin{aligned}\varphi((a + bi) \circ (c + di)) &= ac + ad\varphi(i) + bc\varphi(i) + bd(\varphi(i))^2 \\ &= \varphi(a + bi) \circ \varphi(c + di),\end{aligned}$$

Secondly, suppose $c + d\varphi(i)$ is not a square in $(\mathbb{Z}_3[i], +, \cdot)$, i.e. $c + dk \neq 0$. Then $c = dk$, since $cdk \neq 0$ and $c, d, k \in \mathbb{Z}_3$. Therefore $2c\varphi(i) = = 2ck + 2cli = d + ck + 2cli = d + c(k + 2li) = d + c(\varphi(i))^3$. Also, $(\varphi(i))^4 = 2$, and so by (8) and (10) we have

$$\begin{aligned}\varphi((a + bi) \circ (c + di)) &= ac + bd + ad\varphi(i) + b(2c\varphi(i)) \\ &= ac + bd + ad\varphi(i) + b(d + c(\varphi(i))^3) \\ &= ac + 2bd + ad\varphi(i) + bc(\varphi(i))^3\end{aligned}$$

$$= ac + bd(\varphi(i))^4 + ad\varphi(i) + bc(\varphi(i))^3$$
$$= \varphi(a+bi) \circ \varphi(c+di).$$

This proves the assertion that every automorphism φ of $(GF(3^2),+)$ satisfying $\varphi(1) = 1$ is an automorphism of the smallest Dickson non-field $(GF(3^2),+,\circ)$.

Unfortunately(?) the above result does not hold for the Dickson non-field $(GF(5^2),+,\circ)$, where $\circ$ is defined by

$$x \circ y = \begin{cases} x \cdot y, & \text{if } y \text{ is a square in the Galois field } (GF(5^2),+,\cdot) \\ x^5 \cdot y, & \text{otherwise,} \end{cases}$$

as can be verified easily. However, as mentioned before, every semi-automorphism of this near-field and of all others investigated by us, is an automorphism.

Acknowledgements

1. Portions of this work were done while the second author was visiting the National Cheng Kung University in accordance with a bilateral exchange agreement between the NSC of the Republic of China and the CSIR of the Republic of South Africa. He wishes to express his appreciation for the financial assistance provided by the latter two bodies and for the hospitality bestowed upon him by the National Cheng Kung University.
2. Helpful discussions with Jim Clay during his visit to the National Cheng Kung University on the possibly significance of the result in Example 3.10, and on directions for future research in this regard, are gratefully acknowledged.

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Mathematica Pannonica
3/1 (1992), 19 – 29

ON FINITE SPHERE-PACKINGS

P.M. **Gandini**

Dipartimento di Matematica, Via Carlo Alberto 10, I-10123 Torino, Italy.

J.M. **Wills**

Math. Inst. Univ. Siegen, Hölderlinstraße 3, D-5900 Siegen, BRD.

Dedicated to Prof. Dr. Hans Sachs at the occasion of his 50th birthday

Received March 1990

AMS Subject Classification: 52 C 17, 52 C 07; 05 B 40

Keywords: Euclidean d-space, sphere-packing, sausage-catastrophe.

Abstract: Given k unit balls in Euclidean d-space E^d, what is the minimal volume of their convex hull? In E^2 hexagonal circle-packings, possibly degenerate, are best possible ([6], [8]). In E^d, $d \geq 5$ the linear arrangement of the k balls is conjectured to be optimal. L. Fejes Tóth's sausage conjecture [3], and several partial results (cf. [1],[4]) support this conjecture. In E^3 and E^4 no such general results can be expected, because the situation is more complicated. We consider $d = 3$: In the sausage-catastrophe (cf [9]) it is conjectured that for all $k < 56$ the linear arrangement is optimal, whereas for all but finitely many $k \geq 56$ clusters of spheres are best possible. Although this is supported by computer-aided calculation, a proof seems to be very hard. However, we can prove: For no $k \geq 56$ but 57,58,63 and 64 the sausage is optimal.

1. Introduction

Dense packings of finitely many spheres are good models for atom clusters. So in recent years there were several investigations about various aspects on finite circle- or sphere-packings (cf. e.g. [1], [3] – [6], [8], [9]). In this paper we define the density of finite sphere-packings via the minimal volume of the convex hull of the spheres. For simplicity we only consider unit spheres, i.e. $B^3 = \{x \in E^3 | \, ||x|| \leq 1\}$. Further $\mathcal{L}$ denotes the lattice of the densest lattice packing of unit spheres. Given k unit spheres $B_i^3 = B^3 + c_i$, $i = 1, \dots, k$ in E^3 with mutual disjoint interiors, the volume of their convex hull is given by the Steiner formula (cf. e.g. [7])

$$V(C_k + B^3) = V(C_k) + F(C_k) + M(C_k) + \frac{4}{3}\pi,$$

where $C_k = \mathrm{conv}(c_1, \dots, c_k)$ and V, F, M denote the volume, surface area and integral of mean curvature.

The problem is to minimize $V(C_k + B^3)$ for a given fixed k and all possible C_k i.e. with mutual distance ≥ 2 of any of the c_i.

The "icefern"-theorem ([1], Th. 2) says that if one restricts oneself to planar C_k, then the linear arrangement, i.e. $C_k = S_k$, where S_k is a segment of length $2(k-1)$, is minimal, i.e.

$$V(S_k + B^3) \leq V(C_k + B^3).$$

In other words, the sausage is better than any other planar arrangement of k unit balls. It is conjectured that for all $k < 56$ this inequality even holds for arbitrary C_k. Although computer-aided calculations support this conjecture, called sausage-catastrophe, an exact proof is still open for all $k \geq 4$. On the other hand simple considerations show that for all sufficiently large k there are lattice points $c_i \in \mathcal{L}$, $i = 1, \dots, k$ such that for the lattice-polyhedron $C_k = \mathrm{conv}(c_1, \dots, c_k)$ holds

$$(*) \qquad V(C_k + B^3) < V(S_k + B^3).$$

Obviously for sufficiently large k there are also C_k with $(*)$, which are no lattice-polyhedra. For k not too large, say $k < 100$, the difference in $(*)$ is so small that no general proof for $(*)$ and all possible k can be expected. However, the following result solves the problem for all but four $k \geq 56$.

Theorem. *For each $k \geq 56$, $k \neq 57, 58, 63, 64$ there is a C_k with $(*)$.*

Remarks. 1) For $k = 61, 67, 71, 77, 81, 83$ the C_k with $(*)$ are no lattice polyhedra. It remains open if for these k there are lattice polyhedra C_k with $(*)$.

2) We conjecture that for $k = 57, 58, 63$ and 64 the sausage is optimal. For the proof we need 11 lemmas. The theorem follows from Lemmas 5,6,7,8,9 and 11.

2. Definitions. The lattice polyhedra

Definition 1. Let T_1^n be the basic regular tetrahedron of $\mathcal{L}$ with edge-length 2, i.e. the convex hull of 4 lattice-points of $\mathcal{L}$. For $n \in \mathbb{N}$ let $T_n = nT_1$.

Definition 2. Let $P_{1,1,1}$ be the lattice parallelohedron of $\mathcal{L}$ with edges of length 2 parallel to those of T_1, i.e. the convex hull of 8 lattice-points of $\mathcal{L}$. For $0 \leq a \leq b \leq c$, $a, b, c \in \mathbb{N} \cup \{0\}$ let $P_{a,b,c}$ denote the lattice parallelohedron with edge-lengths $2a \leq 2b \leq 2c$, generated from $P_{1,1,1}$.

Remark. For $a = b = 0$, $c = k-1$ we get $P_{0,0,k-1} = S_k$ and $P_{0,0,k-1} + +B^3$ is the sausage with k balls. Besides this case we will only consider $2 \leq a \leq b \leq c$.

The T_n and $P_{a,b,c}$ are the basic lattice polyhedra and we obtain our general lattice polyhedra C_k for the theorem by omitting suitable lattice-points, or, in other words, by suitable truncations of the T_n and $P_{a,b,c}$. We will have two types of truncations: 1) by regular simplices, 2) by nonregular simplices. We start with the easier case:

1) From a vertex of T_n we cut off a copy of T_p, $p < n$. After compactifying the truncated or snub tetrahedron again we denote it by T_n^p. If we do so with each vertex of T_n we obtain

$$T_n^{p,q,r,s},\ 0 \leq p \leq q \leq r \leq s,$$

where 0 means no truncation; in particular $T_n^{0,0,0,0} = T_n$. Further we only consider n, p, q, r, s such that $T_n^{p,q,r,s} \neq \emptyset$. We can do the same truncation with $P_{a,b,c}$:

Each $P_{a,b,c}$ with $a \geq 1$ has exactly 2 acute vertices of the same type as T_n. So from these 2 vertices we cut off a copy of T_p and T_q with $0 \leq p \leq q \leq a$. After compactifying we obtain the truncated lattice parallelohedron $P_{a,b,c}^{p,q}$. For $P_{a,b,c}$ we write $P_{a,b,c}^{0,0}$.

2) The second type of truncation we describe via $T_2^{1,0,0,0}$ which has 3 vertices c_1, c_2, c_3 of the same type as a regular tetrahedron and 3 vertices v_1, v_2, v_3 of same type, which we call obtuse vertices. So

$T_2^{1,0,0,0} = \mathrm{conv}(c_3, v_1, v_2, v_3)$ with $||c_i - v_i|| = 2$ $(i = 1, 2, 3)$, $||v_i - v_j|| = 2$ and $||c_i - c_j|| = 4$ $(i \neq j)$. Now let $T = \mathrm{conv}(c_1, c_2, c_3, v_1, v_2, v_3)$ and $T^* = \mathrm{conv}(c_1, c_2, c_3, v_2, v_3)$. Then $T \cup T^* = T_2^{1,0,0,0}$ and $T \cap T^* = \mathrm{conv}(c_1, v_2, v_3) = T'$. Simple considerations show $||c_1 - v_2|| = ||c_1 - v_3|| = 2\sqrt{3}$. So T' is a triangle with two edges of length $2\sqrt{3}$ and one edge length 2. Further T is a tetrahedron with two edges of length $2\sqrt{3}$ and four edges of length 2.

The truncations of second type will be truncations of congruent copies of T. For this we consider $P_{a,b,c}^{p,q}$ with $a \geq 2$ and $0 < p \leq q < a$. Then easy considerations show that $P_{a,b,c}^{p,q}$ has at least two obtuse vertices as the vertices v_1, v_2, v_3 of $T_2^{1,0,0,0}$. At one or both of these vertices we cut off one or two copies of T and compactify. The new truncated polyhedron we denote by $P_{a,b,c}^{p,q,t}$ with $t \in \{1, 2\}$ and $0 < p \leq q < a$.

If we write $P_{a,b,c}^{p,q,o} = P_{a,b,c}^{p,q}$, we obtain the general truncated parallelohedron

$$P_{a,b,c}^{p,q,t} \; 0 \leq p \leq q \leq a \leq b \leq c, \quad t \in \{0, 1, 2\}, \tag{4}$$

which will solve $(*)$ for all but 14 values of k in the theorem.
This second type of truncation is only needed once for $T_n^{a,b,c,d}$ (namely for $k = 84$), so we do not introduce an extra notation for this special case.

3. Basic lemmas on lattice polyhedra

In this section we calculate V, F and M for the simplest polyhedra in our proof.

Lemma 1. $k(P_{a,b,c}) = (a+1)(b+1)(c+1)$, $V(P_{a,b,c}) = 4\sqrt{2}abc$, $F(P_{a,b,c}) = 4\sqrt{3}(ab + ac + bc)$, $M(P_{a,b,c}) = 2\pi(a + b + c)$.

Proof. Elementary calculation shows
$V(P_{1,1,1}) = 4\sqrt{2}$, $F(P_{1,1,1}) = 4\sqrt{3}(1+1+1)$, $M(P_{1,1,1}) = 2\pi(1+1+1)$.
From this one obtains the general case if one observes that $P_{a,b,c}$ can be dissected into abc copies of $P_{1,1,1}$. The calculation of $k(P_{a,b,c})$ is simple. $\diamondsuit$

Lemma 2. $k(T_n) = \binom{n+3}{3}$, $V(T_n) = \frac{2}{3}\sqrt{2}n^3$, $F(T_n) = 4\sqrt{3}n^2$,

$M(T_n) = 11,4638\dots n$ $k(T'_n) = \binom{n+2}{2}$, $F(T'_n) = 2\sqrt{3}n^2$, $M(T'_n) = 3\pi n$, *where T'_n is a facet of T_n.*

Proof. For $k(T_n)$ see [7], for $M(T_1)$ and hence for $M(T_n)$ see [2] or [7]. The other results are simple. ◇

In the following lemma we calculate V, F and M for the non-regular tetrahedron T (described in Sect. 2) and for its largest facet T'.

Lemma 3. $V(T) = V(T_1) = \frac{2}{3}\sqrt{2}$, $F(T) = 3\sqrt{3} + \sqrt{11}$, $M(T) = 14,3441\dots$, $F(T') = 2\sqrt{11}$, $M(T') = 14,0244\dots$.

Proof. For the calculation of $M(T)$ we introduce coordinates (only in this lemma). Again $T = \text{conv}(v_1, v_2, v_3, c_3)$.

Let $v_1 = \sqrt{2}(1,0,0)$, $v_2 = \sqrt{2}(0,1,0)$, $v_3 = \sqrt{2}(0,0,1)$, $c_3 = \sqrt{2}(-1,-1,1)$. Then $||v_1 - v_2|| = ||v_1 - v_3|| = ||v_2 - v_3|| = ||v_3 - c_3|| = 2$ and $||v_1 - c_3|| = ||v_2 - c_3|| = 2\sqrt{3}$ as required.

Elementary calculation shows $V(T) = V(T_1) = \frac{2}{3}\sqrt{2}$, $F(T) = 3\sqrt{3} + \sqrt{11}$ and $F(T') = 2\sqrt{11}$. (The surface area of T' is twice its 2-dimensional volume). Further $M(T')$ is the sum of the length of its three edges multiplied with $\frac{\pi}{2}$, hence $M(T') = \frac{\pi}{2}(2 + 2\sqrt{3}) = 14,0244\dots$.

It remains to calculate $M(T)$. For this we determine the affine hulls of the 4 facets of T:

$$
\begin{aligned}
E_1 &= \text{aff}(v_1, v_2, v_3) = \{(x,y,z) | x + y + z = \sqrt{2}\}\\
E_2 &= \text{aff}(v_1, v_3, c_3) = \{(x,y,z) | x - y + z = \sqrt{2}\}\\
E_3 &= \text{aff}(v_2, v_3, c_3) = \{(x,y,z) | -x + y + z = \sqrt{2}\}\\
E_4 &= \text{aff}(v_1, v_2, c_3) = \{(x,y,z) | -x - y - 3z = \sqrt{2}\}.
\end{aligned}
$$

From this one gets the angles of the outer normals of the E_i:

$$
\begin{aligned}
&\cos(E_1, E_2) = \cos(E_1, E_3) = \tfrac{1}{3} &&= \cos\alpha\\
&\cos(E_2, E_3) = -\tfrac{1}{3} &&= \cos\beta\\
&\cos(E_1, E_4) = -5/\sqrt{33} &&= \cos\gamma\\
&\cos(E_2, E_4) = \cos(E_3, E_4) = -3/\sqrt{33} &&= \cos\delta
\end{aligned}
$$

and hence (normalized to 2π):$\alpha = 0,5148\dots$,$\beta = 1,0213\dots$,$\gamma = 1,9106\dots$, $\delta = 1,2310\dots$. Now for M holds $M(T) = \sum_i \alpha_i l_i$ (cf. e.g. [7]), where the sum is taken over the 6 edges of T; l_i is the length of the i-th edge and α_i is the measure of the corresponding outer normals, normalized to π such that $\alpha_1 = \alpha_2 = \frac{1}{2}\alpha$, $\alpha_3 = \beta$, $\alpha_4 = \gamma$, $\alpha_5 = \alpha_6 = \frac{1}{2}\delta$. Then with $l_{1,2,3,4} = 2$, $l_5 = l_6 = 2\sqrt{3}$ one obtains $M(T) = 2\alpha + \beta + \gamma + 2\sqrt{3}\delta = 14,3441\dots$. ◇

4. The general case. Parallelohedra

Lemma 4. $k\left(P^{p,q,t}_{a,b,c}\right) = (a+1)(b+1)(c+1) - \binom{p+2}{3} - \binom{q+2}{3} - t.$

Proof. From the construction of $P^{p,q,t}_{a,b,c}$ and the additivity of the lattice point number follows with the Lemmas 1 and 2:

$$k\left(P^{p,q,t}_{a,b,c}\right) = (a+1)(b+1)(c+1) - \binom{p+3}{3} + \binom{p+2}{2} - \binom{q+3}{3} + \binom{q+2}{2} - t$$
$$= (a+1)(b+1)(c+1) - \binom{p+2}{3} + \binom{q+2}{3} - t.$$

Lemma 5. *Let* $k = (a+1)(b+1)(c+1) - \binom{p+2}{3} - \binom{q+2}{3}$ $(p, q \in \{0,1,2\})$ *and*

(a) $a \geq 2$, $b \geq 3$, $c \geq 8$ *or*
(b) $a \geq 2$, $b \geq 4$, $c \geq 5$ *or*
(c) $a \geq 3$, $b \geq 3$, $c \geq 4$.

Then $(*)$ *holds with* $C_k = P^{p,q}_{a,b,c}$.

Proof. Let k be given as above. Then by Lemma 4 we can choose $C_k = P^{p,q}_{a,b,c}$.

Further $V(S_k + B^3) = 2\pi(k - 1 + \frac{4}{3}\pi = 2\pi((a+1)(b+1)(c+1) - 1) - 2\pi\left(\binom{p+2}{3} + \binom{q+2}{3}\right) + \frac{4}{3}\pi$. From Lemmas 1 and 2 we have $V(C_k + B^3) = \{V(P_{a,b,c}) - V(T_p) - V(T_q)\} + \{F(P_{a,b,c}) - F(T_p) + F(T'_p) - F(T_q) + F(T'_q)\} + \{M(P_{a,b,c}) - M(T_p) + M(T'_p) - M(T_q) + M(T'_q)\} + \frac{4}{3}\pi = \{4\sqrt{2}abc - \frac{2}{3}\sqrt{2}(p^3 + q^3)\} + \{4\sqrt{3}(ab + ac + bc) - 2\sqrt{3}(p^2 + q^2)\} + \{2\pi(a+b+c) - (11,4638\ldots - 3\pi)(p+1)\} + \frac{4}{3}\pi$. So we get $V(C_k + B^3) - V(S_k + B^3) = abc(4\sqrt{2} - 2\pi) + (ab + ac + bc)(4\sqrt{3} - 2\pi) - \frac{1}{3}(2\sqrt{2} - \pi)(p^3 + q^3) - (2\sqrt{3} - \pi)(p^2 + q^2) + \delta(p+q) = \beta abc(a^{-1} + b^{-1} + c^{-1} - \gamma) + \{\frac{1}{6}\beta\gamma(p^3 + q^3) - \frac{1}{2}\beta(p^2 + q^2) + \delta(p+q)\} = A + B$, where $\beta = 2(2\sqrt{3} - \pi) = 0,64502\ldots$, $\gamma = (\pi - 2\sqrt{2}) : (2\sqrt{3} - \pi) = 0,9710\ldots$ and $\delta = 3\pi + \frac{2}{3}\pi - 11,4638\ldots = 0,0553\ldots$.

We show that $A + B < 0$. In all cases (a), (b), (c) we have

$$a^{-1} + b^{-1} + c^{-1} \leq \tfrac{23}{24} < \gamma,$$

hence $A < 0$. To show $B \leq 0$ it suffices to consider only p: $B_p = \frac{1}{6}\beta\gamma p^3 - \frac{1}{2}\beta p^2 + \delta p$. Now $B_0 = 0$, $B_1 = \frac{1}{2}\beta(\frac{1}{3}\gamma - 1) + \delta < 0$, $B_2 = \beta(\frac{4}{3}\gamma - 2) + 2\delta < 0$. So $B \leq 0$, i.e. $A + B < 0$ and $V(C_k + B^3) - V(S_k + B^3) < 0$.

Lemma 6. *Let* $k = 16(c+1) - \binom{p+2}{3} - \binom{q+2}{3} - t$, *and*
(a) $c \geq 4$, $t = 0$, $p, q \in \{0,1,2,3\}$ *or*
(b) $c \geq 5$, $t = 1$, $p, q \in \{1,2,\}$ *or*
(c) $c \geq 6$, $t = 1$, $p \in \{1,2\}$, $q = 3$ *or*
(d) $c \geq 7$, $t = 2$, $q \in \{2,3\}$.
Then $(*)$ *holds with* $C_k = P^{p,q,t}_{3,3,c}$.

Proof. Let k be given as above. Then by Lemma 4 we can choose $C_k = P^{p,q,t}_{3,3,c}$. As in the proof of Lemma 5 we get (now with $a = b = 3$) and with Lemma 3 for $(*)$:

$$V(C_k + B^3) - V(S_k + B^3) = A + B - t\{V(T) + F(T) - F(T') + M(T) - M(T') - 2\pi\} = A + B - t(\tfrac{2}{3}\sqrt{2} + 3\sqrt{3} - \sqrt{11} + 0{,}3197\ldots - 2\pi) =$$

$$= A + B - Ct = \Delta, \text{where} \tag{4.1}$$

$$A = 9\beta c(\tfrac{2}{3} + c^{-1} - \gamma) = 3\beta(2c + 3 - 3c\gamma)$$

$$B = \tfrac{1}{6}\beta\gamma(p^3 + q^3) - \tfrac{1}{2}\beta(p^2 + q^2) + \delta(p + q) = B_p + B_q,$$

$$C = 3{,}14\ldots.$$

It remains to prove $\Delta < 0$ in all cases. From the proof of Lemma 5 we have $B_0 = 0$, $B_1 = -0{,}162\ldots$, $B_2 = -0{,}344\ldots$, $B_3 = \frac{9}{2}\beta(\gamma - 1) + 3\delta = 0{,}082\ldots$, hence $B_2 < B_1 < B_0 = 0 < B_3$. To prove $\Delta < 0$ it suffices to prove $(*)$ for the worst cases in (a), (b), (c), (d):

(a) $c = 4$, $t = 0$, $p = q = 3$.
Then $\Delta = 3\beta(11 - 12\gamma) + 2B_3 < 0$.

(b) $c = 5$, $t = 1$, $p = q = 1$.
Then $\Delta = 3\beta(13 - 15\gamma) + 2B_1 + C < 0$.

(c) $c = 6$, $t = 1$, $p = 1$, $q = 3$.
Then $\Delta = 3\beta(15 - 18\gamma) + B_1 + B_3 + C < 0$.

(d) $c = 7$, $t = 2$, $p = 1$, $q = 3$.
Then $\Delta = 3\beta(17 - 21\gamma) + B_1 + B_3 + 2C < 0$.

These inequalities prove Lemma 6. ◇

Lemma 7. *The* k *in Lemmas 5 and 6 cover all* k *of the theorem except the fifteen cases* $k \in \{56, 59, 61, 62, 65, 67, 68, 71, 73, 74, 77, 81, 83, 84\}$.

Proof. We start with Lemma 6 which covers nearly all of these k. We write $k = 16c + 16 - R$, $R = \binom{p+2}{3} + \binom{q+2}{3} + t$ and calculate R for (a), (b), (c), (d):

(a) $t = 0$, $p, q \in \{0,1,2,3\}$ yield $R = 0, 1, 2, 4, 5, 8, 10, 11, 14$ and 20.

(b) $t = 1$, $p, q \in \{1,2\}$ yield $R = 3, 6, 9$.

(c) $t=1,\ p\in\{1,2\},\ q=3$ yield $R=12,15$.
(d) $t=2,\ p=1,\ q\in\{2,3\}$ yield $R=7,13$.
The special case $p=q=3$, i.e. $R=20$, is only needed for $c=4$ and yields $k=60$.
The other cases in (a),(b),(c),(d) cover all residue classes modulo 16, and from Lemma 6 follows with $c\geq 7$ that all $k\geq 112$ are covered.
For $c=6$ the only missing k are $k=112-R$, $R=7$ and 13, hence $k=105$ and 99.
For $c=5$ the only missing k are $k=96-R$, $R=7,12,13,15$, hence $k=81,83,84,89$.
For $c=4$ the only missing k are $k=80-R$, $R=3,6,7,9,12,13,15$, hence $k=65,67,68,71,73,74,77$.
Three of these k are covered by Lemma 5, namely $k=3\cdot 5\cdot 7=105$, $k=4\cdot 4\cdot 5-1=99$, and $k=3\cdot 5\cdot 6-1=89$.
This proves Lemma 7.

5. Truncated tetrahedra

In the preceeding section the theorem was proved for all but 14 k. In this section we prove it for eight of these k; seven in Lemma 8, one in Lemma 9.

Lemma 8. *Let $k\in\{56,59,62,65,68,73,74\}$. Then there are positive integers n,p,q,r,s with $p\leq q\leq r\leq s$, $r+s\leq n$ such that $(*)$ holds with $C_k=T_n^{p,q,r,s}$.*

Proof. From Lemma 2 and $r+s\leq n$ follows, if one observes that V is simply additive and that F,M and k are additive:

$$V(T_n^{p,q,r,s})=\tfrac{2}{3}\sqrt{2}(n^3-p^3-q^3-r^3-s^3)$$

$$F(T_n^{p,q,r,s})=2\sqrt{3}(2n^2-p^2-q^2-r^2-s^2)$$

$$M(T_n^{p,q,r,s})=11,4638\ldots(n-p-q-r-s)+3\pi(p+q+r+s)$$

$$(5.1)\qquad k(T_n^{p,q,r,s})=\binom{n+3}{3}-\binom{p+2}{3}-\binom{q+2}{3}-\binom{r+2}{3}-\binom{s+2}{3}.$$

So $k=\frac{1}{6}(n^3-p^3-q^3-r^3-s^3)+\frac{1}{2}(2n^2-p^2-q^2-r^2-s^2)+\frac{1}{3}(\frac{11}{2}n-$

$-p-q-r-s)+1$ and

$$V(C_k+B^3)-V(S_k+B^3)=$$

$$=\tfrac{1}{3}(2\sqrt{2}-\pi)(n^3-p^3-q^3-r^3-s^3)+(2\sqrt{3}-\pi)(2n^2-p^2-q^2-r^2-s^2)-$$

$$-(\tfrac{11}{3}\pi-11,4638\ldots)(n-p-q-r-s)=$$

$$=-0,10438\ldots(n^3-p^3-q^3-r^3-s^3)+0,3225\ldots(2n^2-p^2-q^2-r^2-s^2)$$

$$(5.2)\qquad -0,055\ldots(n-p-q-r-s)=\Delta.$$

We now consider the 7 cases separately by calculating k from (5.1) and Δ from (5.2). We omit the easy calculations for Δ.
(1) $k(T_6^{2,2,3,3})=56;\ \Delta=-0,183\ldots<0$
(2) $k(T_6^{1,2,3,3})=59;\ \Delta=-0,002\ldots<0$
(3) $k(T_6^{2,2,2,3})=62;\ \Delta=-0,610\ldots<0$
(4) $k(T_6^{1,2,2,3})=65;\ \Delta=-0,428\ldots<0$
(5) $k(T_6^{2,2,2,2})=68;\ \Delta=-1,036\ldots<0$
(6) $k(T_6^{1,1,2,2})=74;\ \Delta=-0,673\ldots<0$
(7) $k(T_7^{2,2,2,5})=73;\ \Delta=-0,356\ldots<0$
These seven inequalities prove Lemma 8. ◊
Lemma 9. *For* $k=84$ *holds* (*).
Proof. From (5.1) we get $k(T_7^{1,2,3,4})=85$.
With $C_{85}=T_7^{1,2,3,4}$ we get from (5.2) with some calculation $V(C_{85}+$ $+B^3)-V(S_{85}+B^3)=-3,27\ldots<0$.
Now $T_7^{1,2,3,4}$ obviously has at least one (in fact six) obtuse vertex as defined in Section 2. We cut off the irregular tetrahedron associated to this vertex as described for $P_{a,b,c}^{p,q,t}$, $t=1$ and obtain a truncated tetrahedron $\bar{T}_7^{1,2,3,4}$. Obviously $k(\bar{T}_7^{1,2,3,4})=84$, so we write $C_{84}=$ $=\bar{T}_7^{1,2,3,4}$. As in (4.1) we now get $C=3,14\ldots$: $V(C_{84}+B^3)-V(S_{84}-$ $-B^3)=V(C_{85}+B^3)-V(S_{85}+B^3)+C=$
$=-3,27\ldots+3,14\ldots<0$ which proves the lemma. ◊

6. Double tetrahedra

In this section we consider non-lattice packings for the last six k. If we fit two copies of T_n together at one facet, one obtains in an obvious way a double-tetrahedron (or bipyramide) D_n, endowed with the sphere-centres c_i of the two copies of T_n. D_n has exactly two acute vertices of same type as T_n. Hence we can truncate D_n by copies of T_p, T_q $(p, q < n)$ in the same way as we did to obtain $P^{p,q}_{a,b,c}$ and $T^{p,q,r,s}_n$. We denote this truncated and compactified D_n by $D^{p,q}_n$.

Lemma 10. *For $p \leq q < n$ we have*
$V(D^{p,q}_n) = \frac{2}{3}\sqrt{2}(2n^3 - p^3 - q^3)$
$F(D^{p,q}_n) = 2\sqrt{3}(3n^2 - p^2 - q^2)$
$M(D^{p,q}_n) = (2M(T_1) - 3\pi)n - (M(T_1) - 3\pi)(p + q)$
$k(D^{p,q}_n) = \binom{n+3}{3} + \binom{n+2}{3} - \binom{p+2}{3} - \binom{q+2}{3}$.

Proof. The results follow from Lemma 2, from the definition of $D^{p,q}_n$, and from the fact that V is simply additive and that F, M and k are additive. ◇

Lemma 11. *Let $k \in \{61, 67, 71, 77, 81, 83\}$. Then there are positive integers $p \leq q < n$, such that $(*)$ holds with $C_k = D^{p,q}_n$.*

Proof. From Lemma 10 we have

$$(6.1) \quad k(D^{p,q}_n) = \binom{n+3}{3} + \binom{n+2}{3} - \binom{p+2}{3} - \binom{q+2}{3} =$$
$$= \tfrac{1}{6}(2n^3 - p^3 - q^3) + \tfrac{1}{2}(3n^2 - p^2 - q^2) + \tfrac{13}{6}n - \tfrac{1}{3}(p+1) + 1.$$

So we get as in Lemma 8

$$V(C_k+B^3) - V(S_k+B^3) = V(D^{p,q}_n) + F(D^{p,q}_n) + M(D^{p,q}_n) - 2\pi(k-1) =$$
$$= \tfrac{1}{3}(2\sqrt{2} - \pi)(2n^3 - p^3 - q^3) + (2\sqrt{3} - \pi)(3n^2 - p^2 - q^2) + (2M(T_1) -$$
$$-3\pi - \tfrac{13}{3}\pi)n - (M(T_1) - 3\pi - \tfrac{2}{3}\pi)(p + q) = -0,104\ldots(2n^3 - p^3 - q^3) +$$
$$+0,3225\ldots(3n^2 - p^2 - q^2) - 0,11\ldots n - 0,055\ldots(p + q) = \Delta$$

We now consider the six cases separately by calculating k from (6.1) and Δ from the last equality. We omit the easy calculations for Δ.

(1) $k(D^{3,4}_5) = 61, \ \Delta = -1,40\ldots < 0$

(2) $k(D^{2,4}_5) = 67, \ \Delta = -1,72\ldots < 0$

(3) $k(D^{3,3}_5) = 71, \ \Delta = -2,95\ldots < 0$

(4) $k(D^{2,3}_5) = 77, \ \Delta = -3,27\ldots < 0$

(5) $k(D^{0,3}_5) = 81, \ \Delta = -2,70\ldots < 0$

(6) $k(D^{2,2}_5) = 83, \ \Delta = -3,58\ldots < 0$

These six inequalities prove Lemma 12. ◊

Acknowledgement. This paper was written while the second author was staying as visiting professor at the University of Turin.

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Mathematica Pannonica
3/1 (1992), 31 – 47

PROJECTIONS, SKEWNESS AND RELATED CONSTANTS IN REAL NORMED SPACES

Marco **Baronti***

Dipartimento di Matematica, Università, I-60100 Ancona, Via delle Brecce Bianche, Italia.

Pier Luigi **Papini***

Dipartimento di Matematica, Università, I-40127 Bologna, Piazza Porta S. Donato 5, Italia.

Received May 1990

AMS Subject Classification: 46 B 99; 41 A 65, 46 B 20, 47 A 30

Keywords: Projections, skewness, orthogonality.

Abstract: In real normed spaces, the notion of skewness was introduced by Fitzpatrick and Reznick. The radial projection constant had been already studied several years before and its relations with some projection constants had been pointed out. Here we introduce and study a modified version of skewness and we continue the study of the above notions. We compare all these constants and we establish several relations, some of them depending on properties of the underlying space.

1. Introduction

Let X be a normed space over the real field $\mathbb{R}$. We denote by S the unit sphere of $X : S = \{x \in X;\ \|x\| = 1\}$. Also, we set for x, y in

* While preparing this work, the authors were supported by G.N.A.F.A. of the C.N.R., and by the national research group ("40%") of Functional Analysis.

X:

$$\tau(x,y) = \lim_{t\to 0+} \frac{\|x+ty\|-\|x\|}{t} = \inf_{t>0} \frac{\|x+ty\|-\|x\|}{t}$$

and so

$$-\tau(x,-y) = \lim_{t\to 0-} \frac{\|x+ty\|-\|x\|}{t} = \sup_{t<0} \frac{\|x+ty\|-\|x\|}{t}.$$

Now define the multivalued map $J : X \to X^*$ in the following way. For $x \in X$ denote by $J(x)$ the nonempty set

$$J(x) = \{f \in X^*;\ \|f\| = \|x\|;\ f(x) = \|x\|^2\}$$

(X^* denoting the topological dual of X). For any x we have:

$$\|x\|\ \tau(x,y) = \sup\{f(y);\ f \in J(x)\}.$$

The space X is *smooth* if and only if $J(x)$ is a singleton for any $x \in X$, or equivalently, if and only if $\tau(x,y) = -\tau(x,-y)$ for any pair x,y; in this case $\tau(x,.)$ is linear in its second argument. We say that x is *orthogonal* to y, and we write $x\perp y$, when $\|x+ty\| \geq \|x\|$ for all $t \in \mathbb{R}$. Note that the following is true:

$$x\perp y \Leftrightarrow \pm x\perp \pm y \Leftrightarrow -\tau(x,-y) \leq 0 \leq \tau(x,y). \tag{1.1}$$

In particular, if X is smooth, then we have

$$x\perp y \Leftrightarrow \tau(x,y) = 0 \Leftrightarrow f(y) = 0 \text{ for } f \in J(x). \tag{1.2}$$

We shall write $x\perp M$ when $x\perp m$ for all $m \in M$. We denote by $[M]$ the linear span of M ($[y]$ = linear span of y).

We shall write: X is (H), when the norm of X derives from an inner product; in this case, $\tau(x,y)$ reduces to the inner product when $x,y \in S$. We recall that when $\dim(X) \geq 3$, then X is (H) if and only if orthogonality is symmetric (i.e., $x\perp y$ implies $y\perp x$).

The space X is said to be *uniformly nonsquare* (abbreviated: (UNS)) when $\sup\{\min(\|x+y\|,\|x-y\|);\ x,y \in S\} < 2$. Recall that (UNS) spaces are reflexive.

The notion of *skewness* for a normed space was introduced in [8] to describe the "asymmetry" of the norm:

$$s(X) = \sup\{s(x,y);\ x,y \in S\} \tag{1.3}$$

where

(1.4) $$s(x,y)=\tau(x,y)-\tau(y,x).$$

Note that

(1.5) $$s(x,y)=-s(y,x)=s(-x,-y) \text{ for any pair } x,y.$$

For any space X, $0\le s(X)\le 2$. Moreover, the extreme values 0 and 2 characterize - respectively – (H) spaces and spaces which are not (UNS) (see [8]).

We recall the definitions of some other constants that we shall compare with $s(X)$. The *radial map* T from X onto its unit ball, is the radial projection onto the unit ball defined by

$$T(x)=\begin{cases} x & \text{if} \quad \|x\|\le 1 \\ x/\|x\| & \text{if} \quad \|x\|>1. \end{cases}$$

The *radial constant* of X is the number

$$k(X)=\sup\{\frac{\|Tx-Ty\|}{\|x-y\|};\ x,y\in X;\ x\ne y\}\in[1,2].$$

Recall (see e.g. [9]) that

(1.6) $$k(X)=\sup\{\frac{1}{\|tx+y\|};\ x,y\in S;\ x\perp y, t\in\mathbb{R}\}$$

or also:

$$k(X)=\sup\{\frac{\|y\|}{\|x-y\|};\ y\ne 0;\ x\perp y\}=\sup\{\frac{1}{d(x,[y])};\ x\perp y;\ x,y\in S\}$$

where $d(x,A)=\inf\{\|x-a\|;\ a\in A\}$.
The extreme values of $k(X)$ (1 and 2) characterize – respectively – spaces where orthogonality is symmetric, and spaces which are not (UNS) (see [9] and the references there). For the sake of completeness, we recall that some results related to $k(X)$ were already given by Gurariĭ in two not too known papers (see [13] and [14]).
Remark. *Let* $x,y\in S, x\perp y, \lambda\ne 0$;*then* $\|y+\lambda x\|\ge|\lambda|\cdot\|x\|$. This shows that in (1.6), to obtain $\inf\|y+\lambda x\|$ it is enough to consider $\lambda\in[-1,1]$.

The radial constant is connected with the projection constants onto subspaces of X (see e.g. [2]). These and other similar relations will be studied in some detail here.

The present paper is organized in the following way. In Section 2 we indicate some general properties of the functional τ. In Section 3 we define new constants of skewness and we compare them with $s(X)$. In Section 4 we compare the radial constant with the constants of skewness. Section 5 deals with projection constants. Finally, in Section 6, we give some estimates for these constants in uniformly convex and uniformly smooth spaces.

Several results in sections 3,4 and 5 rely upon smoothness properties of X. Our modified measures of skewness are useful to obtain relations with constants which are related to orthogonal pairs. Relations among the constants we indicated and projections will be obtained by using what we shall call, according to [11], "polar" projections.

2. Some properties of the functional τ.

The following properties of τ are well known. For $x, y \in X, \mu \in \mathbb{R}$ and $\lambda > 0$ we have:

$$\tau(x, \mu x + \lambda y) = \mu\|x\| + \lambda\tau(x,y) \text{ (this is true also for } \lambda = 0) \tag{2.1}$$

$$\tau(\lambda x, y) = \tau(x,y) \tag{2.2}$$

$$|\tau(x,y)| \leq \|y\|. \tag{2.3}$$

We indicate a few more properties, which will be used later: though very simple, probably they are not so well known.

Lemma 2.1. *Let $x, y \in S$. Then $\tau(x,y) = 1$ implies $\tau(y,x) = 1$.*

Proof. Let $x, y \in S, 1 = \tau(x,y) = \inf_{t>0} \frac{\|x+ty\|-\|x\|}{t}$. Therefore for all $t > 0 : 1 \leq \frac{1+|t|-1}{t} = 1$, which implies $\|x + ty\| = 1 + t$. Thus, for $t > 0, \|y + tx\| = t\|\frac{y}{t} + x\| = 1 + t$, which implies $\tau(y,x) = = \lim_{t\to 0^+} \frac{\|y+tx\|-\|y\|}{t} = 1$. $\diamondsuit$

Lemma 2.2. *A space X is smooth if and only if the following condition holds:*

$$x \perp y \text{ if and only if } \tau(x,y) = 0. \tag{2.4}$$

If in addition orthogonality is symmetric, then smoothness is also equivalent to:

(2.5) *if* $x, y \in S$, *then* $x \perp y$ *implies* $\tau(x,y) = \tau(y,x)$.

Proof. Of course, if X is smooth, then $x \perp y$ implies $\tau(x,y) = 0$, and moreover, $\tau(x,y) = \tau(y,x) = 0$ if orthogonality is symmetric, so we have to prove the converse statements ("if" parts).

(2.4) $\Rightarrow$ X smooth: we prove the contrapositive. If X is not smooth, then there is a pair x, y with $x, y \in S$ and $-\tau(x,-y) = \lambda < \tau(x,y)$; then $-\tau(x, -y+\lambda x) = 0$, so $x \perp y - \lambda x$. But we have $\tau(x, y - \lambda x) > 0$, so (2.4) does not hold.

(2.5) $\Rightarrow$ X smooth (when orthogonality is symmetric): we prove the contrapositive. If $\dim(X) \geq 3$, then X must be (H) so we have nothing to prove; thus we assume $\dim(X) = 2$. Let us assume that X is not smooth; therefore (see (2.4)) there is a pair x, $y \in S, x \perp y$ (so $y \perp x$) such that $f_x(y) = \lambda \neq 0$ for some $f_x \in J(x)$; we can also assume $\lambda < 0$ (eventually, we change y into $-y$). Also, there exist $f \in J(x)$ and $g \in J(y)$ such that $f(y) = g(x) = 0$ (so $f \neq g$). Let $f_x = \alpha f + \beta g$, thus $\alpha = 1$ and $\beta = \lambda$. Take $z = \lambda x - y$ so $f_x(z) = 0$. Then also $z \perp x$, so we have: $1 = \|y\| \leq \|y - \lambda x\| \leq \|y - \lambda x + \lambda x\| = 1$. Therefore the value of the convex function of $t : F(t) = \|y + tx\|$ is 1 for $0 \leq t \leq -\lambda$, and so (set $a = 1/t$) $\|x + ay\| = a$ for $a \geq -1/\lambda$. By taking $\lambda_0 > -1/\lambda$ we obtain also, for $t \in R : \|x + \lambda_0 y\| = \lambda_0 \|y\| \leq \|\lambda_0 y + x + tx\|$, so $x + \lambda_0 y \perp x$ and then $x \perp x + \lambda_0 y$. Now we have: $\tau(x + \lambda_0 y, x) = \lim_{t \to 0^+} \frac{\|x+\lambda_0 y + tx\| - \lambda_0}{t} = \lim_{t \to 0^+} \frac{(1+t)\|x + (\lambda_0/(1+t))y\| - \lambda_0}{t}$; since $\lambda_0/(1+t) > -1/\lambda$ for t small enough, we obtain $\tau(x + \lambda_0 y, x) = \lim_{t \to 0^+} \frac{(1+t)(\lambda_0/(1+t)) - \lambda_0}{t} = 0$. But then, since $\frac{x+\lambda_0 y}{\lambda_0} \in S$ and $\frac{x+\lambda_0 y}{\lambda_0} \perp x$, (2.5) would imply also $\tau(x, \frac{x+\lambda_0 y}{\lambda_0}) = \tau(\frac{x+\lambda_0 y}{\lambda_0}, x) = 0$, against $f(\frac{x+\lambda_0 y}{\lambda_0}) = \frac{1}{\lambda_0} > 0$; this contradiction proves that (2.5) cannot hold when X is not smooth, which concludes the proof. $\Diamond$

For a result similar to the second part of Lemma 2.2, see Lemma 2 in [15].

3. Types of skewness

We introduce the following new constant:

$$s_1(X) = \sup\{s(x,y);\ x, y \in S;\ x \perp y\}. \tag{3.1}$$

We give first some indications about the range of $s_1(X)$.

Proposition 3.1. *We always have* $0 \leq s_1(X) \leq s(X) \leq 2$, *and these estimates are sharp.*

Proof. The estimates $s_1(X) \leq s(X) \leq 2$ are trivial. Now take a pair $x_0, y_0 \in S$ such that $x_0 \perp y_0$ and $y_0 \perp x_0$ (this is always possible: see [3]). But of course for any pair x, y we have either $\tau(x,y) - \tau(y,x) \geq 0$ or $\tau(y,x) - \tau(x,y) \geq 0$, so $s_1(X) \geq \max\{s(x_0 y_0), s(y_0, x_0)\} \geq 0$. If X is (H) then clearly $s_1(X) = 0$. If $X = \mathbb{R}^2$ with the norm given by: $\|(x,y)\| = \max\{|x|, |y|\}$, then it is easy to prove that $s_1(X) = 2$. In fact, it is enough to consider the following elements: $x = (1,1)$ and $y = (-1, a)$, with $0 < a < 1$, to prove that: $x, y \in S, x \perp y, \tau(x,y) = a, \tau(y,x) = -1$ and so $s_1(X) \geq 1 + a$; thus we have $s_1(X) = 2$. ◇

Now consider $\mathbb{R}^2$ with the norm given by

$$\|(x,y)\| = \begin{cases} \max\{|x|, |y|\} & \text{if} \quad xy \geq 0 \\ |x| + |y| & \text{if} \quad xy < 0 \end{cases}$$

The unit ball of X is a hexagon. Now take $y = (1,0)$ and $x = (a,1)$ with $0 < a < 1$. We have $x \perp y$ and $y \perp x$ (in fact orthogonality is symmetric in this space). Moreover, $\tau(y,x) = a$ and $\tau(x,y) = 0$, so $s_1(X) \geq s(y,x) = a$: this shows that $s_1(X) \geq 1$. Moreover, symmetry of orthogonality implies $\tau(y,x) \geq 0$, so $s(x,y) \leq 1$ for any pair x, y on S with $x \perp y$; thus $s_1(X) = 1$ for this space.

Remark 3.2. Note that Lemma 2.1 says that we have $s(x,y) < 2$ for every pair $x, y \in S$. This implies that $s(X)$ must be smaller than 2 when the norm has some property implying the continuity of the map J and the unit sphere has some kind of compactness. So, if X is smooth and $x, y \in S$, then $\tau(y,x) = s(y,x) < 1$ for $x \perp y$; thus $s_1(X) < 1$ in these spaces, under some assumptions of that type.

Lemma 3.3. *If* $s_1(X) = 0$, *then orthogonality is symmetric.*

Proof. Assume $s_1(X) = 0$. Let $x, y \in S$ and $x \perp y$, thus $0 \geq \tau(x,y) - \tau(y,x)$. Also, from $x \perp -y$, $\tau(x,-y) - \tau(-y,x) \leq 0$. But also, $\tau(x,y) \geq 0$ and $\tau(x,-y) \geq 0$, and so $\tau(y,x) \geq 0, 0 \leq \tau(-y,x) = \tau(y,-x)$. This implies $y \perp x$, which concludes the proof. ◇

Proposition 3.4. *Let* $\dim(X) \geq 3$; *then* $s_1(X) = 0$ *if and only if* X *is* (H). *If* $\dim(X) = 2$, *then* $s_1(X) = 0$ *if and only if orthogonality is symmetric and* X *is smooth.*

Proof. "If" part: The first statement is trivial; the second one is a consequence of Lemma 2.2.

"Only if" part: For $\dim(X) \geq 3$, the result follows from the above lemma. If $\dim(X) = 2$ and $s_1(X) = 0$, then orthogonality is symmetric (see Lemma 3.3); moreover for x, y in $S, x \perp y$, we have $s(x,y) \leq 0$ and $s(y,x) \leq 0$, so $\tau(x,y) - \tau(y,x) = 0$ and then smoothness of X follows from Lemma 2.2. ◇

We can also define the following constant.

$$s_2(X) = \sup\{s(y,x);\ x, y \in S;\ x \perp y\} = \tag{3.2}$$
$$= -\inf\{s(x,y);\ x, y \in S;\ x \perp y\}.$$

We have he following estimates.

Proposition 3.5. *For any space X we have $0 \leq s_2(X) \leq 1$ and these estimates are sharp.*

Proof. We obtain $0 \leq s_2(X)$ by considering a biorthogonal pair x, y in S. The inequality $s_2(X) \leq 1$ follows from the definition since $s(y,x) \leq$ $\leq \tau(y,x)$ when $x \perp y$. Moreover, we have $s_2(X) = 0$ if X is (H) and $s_2(X) = 1$ in the example of the hexagon (see after Proposition 3.1). ◇

Proposition 3.6. *We have $s_1(X) = s_2(X)$ (thus $s_1(X) \leq 1$) in the following cases:*

(i) *X is smooth*;

(ii) *orthogonality is symmetric.*

Proof. Let X be smooth, then $s(x,y) = -s(-x,y) = s(y,-x)$. Since $x \perp y$ is equivalent to $-x \perp y$, we easily obtain from this $s_1(X) = s_2(X)$. When orthogonality is symmetric, equality follows immediately from the definitions of $s_1(X)$ and $s_2(X)$. ◇

Remark 3.7. As we recalled in the introduction, the extreme values of $s(X)$, 0 and 2, characterize two important classes of spaces. Our Prop. 3.4 indicates the situations for which we have $s_1(X) = 0$. It is possible to have X smooth, thus $s_1(X) \leq 1$ (see Propositions 3.5 and 3.6) and X not (UNS); compare this result with Cor. 4.2.

We could raise the following questions:

Questions 3.8. For what spaces we have $s_1(X) = 1$? Note that the condition X (UNS) does not imply $s_1(X) < 1$ or $s_2(X) < 1$ (see again the hexagon; see also Cor. 6.2). Moreover, X (UNS) implies $s_1(X) \leq s(X) < 2$. We do not know if X (UNS) implies $s_1(X) \leq 1$. Note that, for Propositions 3.4 and 3.6, when $s_1(X) = 0$ then also $s_2(X) = 0$. Is the converse true? Or for what spaces we have $s_2(X) =$ $= 0$? Also: is the inequality $s_2(X) \leq s_1(X)$ always true?

We do not know if $s_2(X) < 1$ implies X (UNS), but we can prove a partial result in this direction.

Lemma 3.9. *Let there exist in S a pair x, y such that $x \perp y$ and $\|x \pm y\| = 2$. Then $s_2(X) = 1$.*

Proof. Let be x, y as in the assumptions. Consider the convex functions of $t \in \mathbb{R}$: $g(t) = \|x + y + t(x - y)\|$ and $f(t) = \|x - y + t(x + y)\|$. We have $g(-1) = g(0) = g(1) = 2 = f(1) = f(0) = f(-1)$, so $g(t) \geq 2$ and $f(t) \geq 2$ for all $t \in \mathbb{R}$; moreover $f(t) = g(t) = 2$ for $-1 \leq t \leq 1$. Now take any $a \in (0, 1)$ and set $u = \frac{x-y+a(x+y)}{2}$, $v = \frac{x+y}{2}$. We have $g(0)/2 = \|v\| = 1 = \|u\| = f(a)/2$. Let $0 < t < 1$, so $\frac{t}{1+ta} < 1$. We obtain $2\|v+tu\| = \|x+y+t(x-y+a(x+y))\| = \|(1+ta)(x+y)+t(x-y)\| =$ $= (1 + ta)\|x + y + \frac{t}{1+ta}(x - y)\| = (1 + ta) \cdot g(\frac{t}{1+ta}) = 2(1 + ta)$. Therefore $\tau(v, u) = \lim_{t\to 0^+} \frac{\|v+tu\|-\|v\|}{t} = \lim_{t\to 0^+} \frac{1+ta-1}{t} = a$. Also, $u \perp v$: in fact, for all $t > 0$ and small enough $(a < a + t < 1)$ we have $2\|u + tv\| = f(a + t) = 2 = 2\|u\|$, which implies $\tau(u, v) = 0$. Thus $s_2(X) \geq \tau(v, u) - \tau(u, v) = a$, and this implies the thesis. ◇

From the above lemma we obtain the following result.

Proposition 3.10. *If* $\dim(X) < \infty$ *and* $s_2(X) < 1$, *then* X *is* (UNS).

Proof. Assume X not (UNS). Then we have (see [2, Th. 6]) $\sup\{\|x + +y\| + \|x - y\|;\ x, y \in S; x \perp y\} = 4$. Now, by using the compactness of S and the fact that orthogonality is preserved when passing to the limit, we see that there exists in S a pair x, y with $x \perp y$ and such that $\|x + y\| + \|x - y\| = 4$. An application of Lemma 3.9 implies the thesis. ◇

We indicate another simple fact concerning $s_1(X)$.

Lemma 3.11. *Let X be smooth. Then*

$$s_1(X) \leq \sup\{\|x + y\|;\ x, y \in S;\ x \perp y\} - 1. \tag{3.3}$$

Proof. Let X be smooth, so $x \perp y$ is equivalent to $\tau(x, y) = 0$. Moreover, by using Prop. 3.6, for any $t > 0$ we have $s_1(X) = s_2(X) =$ $= \sup\{\tau(y, x);\ x, y \in S, x \perp y\} \leq \sup\{\frac{\|y+tx\|-1}{t};\ x, y \in S, x \perp y\}$. By setting $t = 1$ we obtain the thesis. ◇

We conclude this section with the following lemma.

Lemma 3.12. *For any space X and any $\lambda \in \mathbb{R}$ we have*

$$\lambda s(X) + 2 \leq \sup\{\|x + \lambda y\| + \|y - \lambda x\|;\ x, y \in S\}, \tag{3.4}$$

$$\lambda s_1(X) + 2 \leq \sup\{\|x + \lambda y\| + \|y - \lambda x\|;\ x, y \in S, x \perp y\}, \tag{3.5}$$

$$\lambda s_2(X) + 2 \leq \sup\{\|x + \lambda y\| + \|y - \lambda x\|;\ x, y \in S;\ y \perp x\}. \tag{3.6}$$

Proof. Since all these constants are non negative and $x \perp y$ implies $x \perp -y$, it is enough to reason for $\lambda \geq 0$. For $\lambda = 0$ there is nothing to prove. Now fix $\lambda > 0$; if $x, y \in S$, then we have $\|x + \lambda y\| + \|y - -\lambda x\| \geq \tau(x, x + \lambda y) + \tau(y, y - \lambda x) = 1 + \lambda\tau(x, y) + 1 + \lambda\tau(y, -x) \geq \geq 2 + \lambda(\tau(x, y) - \tau(y, x))$. This implies (3.4). A similar reasoning, applied to orthogonal pairs, implies (3.5) or (3.6). $\diamondsuit$

Remark 3.13. By the above lemma we reobtain easily that *if X is* (UNS), *then* $s_1(X) \leq s(X) < 2$.

4. Radial constant and skewness

We recalled in the introduction the definition of the radial constant $k(X)$ and its main properties. Now we shall indicate some relations between this constant and those dealing with skewness.

Proposition 4.1. *For any space X we have:*

$$k(X) \leq 1 + s_1(X). \tag{4.1}$$

Proof. Let $s_1 = s_1(X)$; let $x, y \in S, x \perp y$, thus $\tau(x, y) \geq 0$ and then we have $\tau(y, x) \geq \tau(x, y) - s_1 \geq -s_1$. But also $-x \perp y$ so $\tau(y, -x) \geq -s_1$. Let $\alpha = \|y + \lambda x\|$, so $\alpha \geq \tau(y, y + \lambda x)$. If $\lambda \geq 0$, then $\alpha \geq 1 + +\lambda\tau(y, x) \geq 1 - \lambda s_1$. Also, if $\lambda < 0$, then $\alpha \geq 1 + (-\lambda)\tau(y, -x) \geq 1 + +(-\lambda)(-s_1) = 1 + \lambda s_1$. Therefore $\alpha \geq 1 - |\lambda| s_1$; but also (from $x \perp y$) $\alpha \geq |\lambda|$. This implies $\alpha \geq \max\{|\lambda|, 1 - |\lambda| s_1\}$. Since $\min_{\lambda \in \mathbb{R}}(\max\{|\lambda|, 1 - |\lambda| s_1\}) = \frac{1}{1+s_1}$, we obtain $\alpha \geq \frac{1}{1+s_1}$. Therefore, by (1.6), we have $k(X) = \sup\{\frac{1}{\|y+\lambda x\|};\ x, y \in S;\ x \perp y;\ \lambda \in \mathbb{R}\} \leq 1 + s_1$, so (4.1). $\diamondsuit$

Proposition 4.1 has the following consequences, which contain Lemma 3.3:

Corollary 4.2. *If* $s_1(X) = 0$, *then orthogonality is symmetric. Also: if* $s_1(X) < 1$, *then X is* (UNS).

Proof. From $s_1(X) = 0$ we obtain $k(X) = 1$, so the first statement. Concerning the second statement, the contrapositive follows immediately: in fact, if X is not (UNS), then $k(X) = 2$, so (by (4.1)) $s_1(X) \geq 1$. $\diamondsuit$

Recall that, for any space X, we have

$$k(X) = k(X^*) \tag{4.2}$$

and

$$s(X) = s(X^*). \tag{4.3}$$

Prop. 3.4 shows that in general $s_1(X) \neq s_1(X^*)$, and also (see Prop. 3.6) $s_2(X) \neq s_2(X^*)$. But this cannot happen in "good" spaces. In fact we have the following result.

Proposition 4.3. *If both X and X^* are smooth, then*

$$s_1(X) = s_1(X^*). \tag{4.4}$$

Proof. If X is smooth but not reflexive, then it is not (UNS), so (by Cor. 4.2) $s_1(X) \geq 1$; therefore, by Prop. 3.6, $s_1(X) = 1$. For the same reasons, $s_1(X^*) = 1$, so (4.4) is proved in this case.

Now assume X reflexive, so our assumptions imply that it is also smooth and strictly convex. Under these assumptions J is a one-to-one isometry between X and X^*; moreover, $x \perp y$ if and only if $J(y) \perp J(x)$. Therefore, by setting $J(x) = f_x$ and $J(y) = f_y$, we obtain ($\hat{x}$, $\hat{y}$ denoting the point functionals in X^{**}) $s_1(X) = \sup\{\tau(x,y) - \tau(y,x);\ x, y \in \in S;\ x \perp y\} = \sup\{f_x(y) - f_y(x);\ x, y \in S;\ x \perp y\} = \sup\{\hat{y}(f_x) - - \hat{x}(f_y);\ f_x, f_y \in S^*;\ f_y \perp f_x\} = \sup\{\tau(f_y, f_x) - \tau(f_x, f_y);\ f_x, f_y \in \in S^*;\ f_y \perp f_x\} = s_1(X^*)$, which concludes the proof of Proposition 4.3. ◇

The above proposition, together with Prop. 3.6, shows that in those spaces $s_2(X) = s_2(X^*)$. We can still ask whether, in any space X, we have $s_2(X) \leq s_2(X^*)$, or $s_2(X) \geq s_2(X^*)$.

We prove a result which will be used in the next section.

Lemma 4.4. *Let X be smooth; let $x, y \in S, x \perp y$. If we set $-\tau(y,x) = \beta$ and $\|x + \beta y\| = \alpha$, then $\alpha \leq 1 + s_1^2(X)$.*

Proof. Our assumptions imply $\tau(x,y) = 0$; $-s(x,y) = \tau(y,x)$, thus $\beta = s(x,y)$. Also $\tau(y, \frac{x+\beta y}{\alpha}) = 0, \|\frac{x+\beta y}{\alpha}\| = 1$, so $\tau(\frac{x+\beta y}{\alpha}, y) = = s(\frac{x+\beta y}{\alpha}, y)$. So we obtain: $\alpha = \tau(x + \beta y, x + \beta y) = \tau(x + \beta y, x) + + \beta\tau(x + \beta y, y) \leq 1 + \beta s(\frac{x+\beta y}{\alpha}, y) \leq 1 + |s(x,y)| \cdot |s(\frac{x+\beta y}{\alpha}, y)|$. Since $x \perp \pm y, y \perp \pm \frac{x+\beta y}{\alpha}$ and $\|x\| = \|y\| = \|\frac{x+\beta y}{\alpha}\| = 1$, we have $\max\{|s(x,y)|, |s(\frac{x+\beta y}{\alpha}, y)|\} \leq s_1(X)$, so the thesis. ◇

Remark 4.5. In Prop. 4.1, in general we do not have equality (consider again the exagon); also, we do not known if Lemma 4.4 is true without smoothness. For a related result see (5.17).

We want to recall that in [4], Desbiens introduced the following constant:

$$\beta(X) = \sup\{\beta \in \mathbb{R};\ x + \beta y \perp y;\quad x, y \in S\}.$$

In fact, as the same author noticed later in [5] (and as it is not difficult to see by using (1.6)), $\beta(X) = k(X)$ for any X. Some properties of $\beta(X) = k(X)$ were indicated in [4]; we shall indicate them in the last section.

5. Projections

Let M be a linear subspace of X. Recall that M is said to be *proximinal* if for every $x \in X$ the set

$$\Pi_M(x) = \{x_0 \in M;\ \|x_0 - x\| \le \|m - x\| \text{ for every } m \in M\} =$$

$$= \{x_0 \in M;\ x - x_0 \perp M\}$$

is non-empty. Given a proximinal subspace M of X, set

$$\text{(5.1)}\qquad |||\,\Pi_M\,||| = \sup\{\|y\|;\ y \in \Pi_M(x);\ \|x\| = 1\}.$$

Also, set:

$$\text{(5.2)}\quad MPB\ (X) = \sup\{|||\,\Pi_M|||;\ M \text{ is a proximinal subspace of } X\}$$

and

$$\text{(5.3)}\quad \overline{MPB}(X) = \sup\{|||\Pi_M|||; M \text{ is a proximinal hyperplane of } X\}.$$

If $M = f^{-1}(0)$ for some norm-one functional $f \in X^*$, then f assumes its norm on S if and only if M is proximinal. Moreover, $f(y) = 1$ for $y \in S$ is equivalent to $y \perp M$. Also, there is exactly one $y \in S$ with that property if X is *strictly convex.*

A linear, continuous, idempotent operator $P : X \to M$ is called a *projection.* In case there exists some projection from X onto M, we set

$$\text{(5.4)}\qquad \lambda(M, X) = \inf\{\|P\|;\quad P \text{ is a projection onto } M\}.$$

Moreover, we set

$$\text{(5.5)}\qquad F(X) = \sup\{\lambda(M, X);\ M \text{ is a hyperplane of } X\}.$$

For any space X we have (see [2])

$$k(X) = MPB(X) \tag{5.6}$$

and (see [10])

$$F(X) \leq \overline{MPB}(X). \tag{5.7}$$

Moreover, if $\dim(X) \geq 3$, then (see [1]) $F(X) = 1$ if and only if X is (H).
But we can also prove the following result:
Proposition 5.1. *For any space* X

$$\overline{MPB}(X) = MPB(X) = k(X). \tag{5.8}$$

Proof. Given $\varepsilon > 0$, there exist $x, y \in X, y \neq 0$, such that $x \perp y$ and $\frac{\|y\|}{\|x-y\|} > k(X) - \varepsilon$. Take a functional $f_x \in J(x)$ such that $f_x(y) = 0$; let M be the kernel of f_x. Note that M is a proximinal hyperplane and that $x \perp M$, therefore $-y \in \Pi_M(x-y)$. Thus $\overline{MPB}(X) \geq |||\Pi_M||| \geq \geq \frac{\|y\|}{\|x-y\|} > k(X) - \varepsilon$, which shows that (use (5.6)): $\overline{MPB}(X) \geq k(X) = = MPB(X) \geq \overline{MPB}(X)$, and then all these are equalities. ♢

Recall that given a hyperplane $M = f^{-1}(0)$, a projection $P : X \to \to M$ has a specified form, namely:

$$P(x) = P_{y,M}(x) = x - f(x)y, \text{ where } f(y) = 1. \tag{5.9}$$

If in (5.9) y is chosen so that $\|y\| = 1$, then we say that $P_{y,M}$ is a *polar projection* over M. In this case $\|I - P_{y,M}\| = 1$ (since $y \perp M$) and $x - f(x)y \in \Pi_M(x)$. If in addition there is a unique y as above, then we set

$$P'_M = P_{y,M}, \text{ i.e., } P'_M(x) = x - f(x)y \quad (f(y) = 1). \tag{5.10}$$

In this case, if X is also reflexive, then $\Pi_M(x) = \{x - f(x)y\}$, thus $|||\Pi_M||| = \|P'_M\|$. Polar projections have been used in [11], where it was shown that in some classical Banach spaces they coincide with the projections of minimal norm.

We can state the following result, which slightly improves Lemma 8 in [11].
Proposition 5.2. *For any space* X

$$k(X) = \sup\{\|P_{y,M}\|; M \text{ is a proximinal hyperplane of } X;\ y \perp M\}. \tag{5.11}$$

Moreover, if X is reflexive and strictly convex, then

$$(5.12) \qquad k(X) = \sup\{\|P'_M\|;\ M \text{ is a hyperplane of } X\}.$$

Proof. By using Proposition 5.1 we have (see the discussion above): $k(X) = \overline{MPB}(X) = \sup\{|||\Pi_M|||;\ M$ is a proximinal hyperplane of $X\} = \sup\{\|P_{y,M}\|;\ M$ is a proximinal hyperplane of $X; y\perp M\}$, so (5.11). Moreover, if X is reflexive and strictly convex, then every hyperplane is proximinal, so we can write $P_{y,M} = P'_M$, and then we obtain (5.12). ◇

Now let X be smooth. If $M = f^{-1}(0), \|f\| = 1$, and $y\perp M$, then $\tau(y,x) = f(x)$ for every $x \in X$ and we can write:

$$(5.13) \qquad \|P_{y,M}\| = \sup\{\|x - \tau(y,x)y\|;\ \|x\| = 1\}.$$

Also, by smoothness we have $\|-x-\tau(y,-x)y\| = \|x-\tau(y,x)y\|$, while $\tau(y,-x) = -\tau(y,x) > 0$ if $\tau(y,x) < 0$; thus

$$\|P_{y,M}\| = \sup\{\|x - \tau(y,x)y\|;\ \|x\| = 1, \tau(y,x) \geq 0\} =$$
$$= \sup\{\|x - \tau(y,x)y\|;\ \|x\| = 1, \tau(y,x) \leq 0\}.$$

Now we recall the following result from [11, Lemma 7]:
Lemma 5.3. *If $M = f^{-1}(0)$ for some $f \in X^*, \|f\| = 1$, then*

$$(5.14) \qquad \|P_{y,M}\| = \sup\{\|P(x)\|, x \in S, x\perp y\}.$$

Proposition 5.4. *Let X be smooth. Then we have*

$$(5.15) \qquad k(X) \leq 1 + s_1^2(X).$$

Proof. If X is smooth but not reflexive, so not (UNS), then we have (see Corollary 4.2) $s_1(X) = 1$, thus (5.15) is trivial. If X is smooth and reflexive, then every hyperplane is proximinal; moreover, (5.14), (5.13) and Lemma 4.4 together imply ($y \in S$):

$$(5.16) \qquad \begin{aligned} \|P_{y,M}\| = \sup\{\|P(x)\|, x \in S, x\perp y\} = \sup\{\|x - \tau(y,x)y\|,\\ x \in S, x\perp y\} \leq 1 + s_1^2(X). \end{aligned}$$

An application of (5.11) implies the thesis. ◇
Remark 5.5. By using (5.7), (5.8) and (5.15), *we also have (in any smooth space X)*

$$(5.17) \qquad F(X) \leq 1 + s_1^2(X).$$

A direct proof of (5.17) can be achieved in this way. If $M = f^{-1}(0)$ is proximinal ($f \in X^*$; $\|f\| = 1$) and there exists $y \in S$ such that $f(y) = 1$, then we have (see (5.16))

$$\lambda(M,X) \leq \|P_{y,M}\| \leq 1 + s_1^2(X). \tag{5.18}$$

Moreover, it is not difficult to see that a constant $k \in \mathbb{R}$ exists such that if $M_1 = f^{-1}(0), M_2 = g^{-1}(0)$ and $\|f - g\| < \varepsilon$ ($f, g \in X^*$; $\|f\| = = \|g\| = 1$), then $|\lambda(M_1, X) - \lambda(M_2, X)| < k\varepsilon$. By combining this fact with the Bishop-Phelps theorem, we see that, in a smooth space, (5.18) is true for every M, so we obtain again (5.17).

6. Uniformly convex and uniformly smooth spaces

Recall that X is said to be *uniformly convex* when the function of $\varepsilon \in [0,2], \delta(\varepsilon) = \inf\{1 - \frac{\|x+y\|}{2};\ x, y \in S;\ \|x - y\| \geq \varepsilon\}$ is positive for all $\varepsilon > 0$.

By using the function δ-called the *modulus of convexity* of X - we can give some rough estimates concerning some of the constants considered in the paper.

Proposition 6.1. *We always have*

$$s_1(X) \leq 2 - 2\delta(1) \tag{6.1}$$

Moreover, if X is smooth, then

$$s_1(X) \leq 1 - 2\delta(1). \tag{6.2}$$

Proof. Let $x, y \in S$; $x \perp y$, so $\|x - y\| \geq 1$. This implies $\|x + y\| \leq \leq 2 - 2\delta(1)$. By using (3.5) with $\lambda = 1$, we obtain: $s_1(X) + 2 \leq \leq 2 - 2\delta(1) + 2$, so we have (6.1). If X is smooth, then we obtain (6.2) in a similar way, by using (3.3). ◇

We have immediately the following

Corollary 6.2. *The condition $\delta(1) > 0$ implies $s_1(X) < 2$, and also $s_1(X) < 1$ if X is smooth.*

Of course, if we assume X^* to be uniformly convex, then Prop. 4.3 can be used again to obtain estimates for $s_1(X)$; note that in this case X is smooth. But we can also indicate some direct estimates for uniformly smooth spaces. Recall that the *modulus of smoothness* is defined, for $\lambda \in \mathbb{R}$, in this way:

$$\rho(\lambda) = \sup\{\frac{\|x+\lambda y\| + \|x-\lambda y\|}{2} - 1;\ x, y \in S\}.$$

The space is *uniformly smooth* if $\lim_{t\to 0} \frac{\rho(t)}{t} = 0$; this happens if and only if X^* is uniformly convex. Also, if we denote by ρ^* and δ^* respectively the moduli of smoothness and rotundity of X^*, then we have (see [6,pp.63 – 64]):

$$2\rho^*(1) = \sup\{\varepsilon - 2\delta(\varepsilon);\ 0 \le \varepsilon \le 2\} \tag{6.3}$$

and

$$2\rho(1) = \sup\{\varepsilon - 2\delta^*(\varepsilon);\ 0 \le \varepsilon \le 2\}. \tag{6.4}$$

We can also define (see [7,p.129])

$$\rho_1(\lambda) = \sup\{\frac{\|x+\lambda y\| + \|x-\lambda y\|}{2} - 1;\ x, y \in S, x\perp y\}.$$

Proposition 6.3. *For any space* X

$$s(X) \le 2\rho(1);\ s_1(X) \le 2\rho_1(1). \tag{6.5}$$

Proof. Formulas (6.5) are trivial (see (3.4) and (3.5)). Moreover, $s(X) = s(X^*) \le 2\rho^*(1) = \sup\{\varepsilon - 2\delta(\varepsilon);\ 0 \le \varepsilon \le 2\}$. ◇

Concerning the radial constant, (6.5) and (4.1) together imply

$$k(X) \le 1 + 2\rho_1(1). \tag{6.6}$$

By using the modulus of convexity, we have

Proposition 6.4. *Let* X *be* (UNS). *Then*

$$k(X) + \delta(k(X)) \le 2. \tag{6.7}$$

Also:

$$k(X) \le 2 - \frac{\delta(\varepsilon_0)}{2},\ \textit{where } \varepsilon_0 = \sup\{\varepsilon > 0;\ \varepsilon + \delta(\varepsilon) \le 2\}. \tag{6.8}$$

Proof. Our assumptions imply $k(X) \in [1, 2)$. If $k(X) = 1$ there is nothing to prove. Now let $k(X) \in (1, 2)$; set, for $\varepsilon \in (0, k(X) - 1)$, $k_\varepsilon = k(X) - \varepsilon$. By using (1.6), we can find $x, y \in S, x\perp y$ and $t_\varepsilon \in \mathbb{R}$, so that $\|t_\varepsilon x - y\| \le \frac{1}{k_\varepsilon}$; from $|k_\varepsilon| - |k_\varepsilon t_\varepsilon| \le \|k_\varepsilon y - k_\varepsilon t_\varepsilon x\| \le 1$ we obtain $2(k_\varepsilon - 1) \leq 2k_\varepsilon|t_\varepsilon| = \|2k_\varepsilon t_\varepsilon x\| \leq \|k_\varepsilon t_\varepsilon x + k_\varepsilon t_\varepsilon x - k_\varepsilon y\|$. Now we observe that $\|k_\varepsilon t_\varepsilon x\| \leq \|k_\varepsilon y - k_\varepsilon t_\varepsilon x\| \leq 1$ by construction, while $\|k_\varepsilon t_\varepsilon x - (k_\varepsilon t_\varepsilon x - k_\varepsilon y)\| = k_\varepsilon$; this implies, by definition of δ, $\|k_\varepsilon t_\varepsilon x + {} +k_\varepsilon t_\varepsilon x - k_\varepsilon y\| \leq 2(1-\delta(k_\varepsilon))$, thus $2(k_\varepsilon - 1) \leq 2(1-\delta(k_\varepsilon))$, and then $k(X) + \delta(k(X)) \leq 2$.

For the second part of the thesis, recall that the following parameter was used in [12]:

$$\mu(X) = \sup\{\frac{\|x\| + \|ty\|}{\|x+ty\|};\ x \perp y;\ t \in \mathbb{R}\}.$$

It was proved there that $\mu(X) \leq 3 - \delta(\varepsilon_0)$ (ε_0 defined above). Then it was proved in [2] that, in any space:

$$2k(X) - 1 \leq \mu(X) \leq k(X) + 1. \tag{6.9}$$

This implies $k(X) \leq \frac{1+\mu(X)}{2} \leq 2 - \frac{\delta(\varepsilon_0)}{2}$. ◊

Better (but more complicated) relations similar to (6.9) were given in [5], where also the estimate (6.7) was proved, in the form $k(X) \leq \varepsilon_0$. By using those results, the second part of Proposition 6.4 could be slightly improved.

Remark 6.5. The function δ is non decreasing and continuous for $\varepsilon < 2$; therefore we have $\varepsilon_0 < 2$ when X is (UNS) (and also the converse is true). Moreover, for any space X (see e.g. [6, p.60]) $\delta(\varepsilon) \leq 1 - (1 - {} -\frac{\varepsilon^2}{4})^{\frac{1}{2}}$, so $k(X) + \delta(k(X)) \leq k(X) + 1 - \sqrt{1 - \frac{k(X)^2}{4}}$; therefore, $\varepsilon_0 \geq \frac{8}{5}$. Thus, the estimate given by (6.7), at most, can say that $k(X) \leq \alpha$ for some $\alpha \geq \frac{8}{5}$. Concerning the estimate (6.8), note that we always have $2 - \frac{\delta(\varepsilon_0)}{2} \geq \frac{9}{5}$. Also, note that the second part of (6.9), together with (4.1), implies $\mu(X) \leq 2 + s_1(X)$. Again, slightly better estimates can be given by using the results of [5].

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Mathematica Pannonica
3/1 (1992), 49 – 56

LOCALIZING FAMILIES FOR REAL FUNCTION ALGEBRAS

Maciej **Grzesiak**

Institute of Mathematics, Technical University, PL-60965 Poznań, ul. Piotrowo 3a, Poland.

Received August 1990

AMS Subject Classification: 46 J 10; 46 J 35

Keywords: $C(X,\sigma)$, real function algebra, localizing cover, basic subclass of algebras, antisymmetric algebra, (weakly) analytic algebra, antisymmetric measure, annihilating measure, (real part) representing measure.

Abstract: Let A be a real function algebra on (X,σ). A cover $\mathcal{R}$ of X by closed sets localizes A if from $f \in C(X,\sigma)$ and $f|_R \in \overline{A|_R}$ for each $R \in \mathcal{R}$, it follows $f \in A$. Examples of such covers and some relations between them are given.

For a compact Hausdorff space X and a homeomorphism $\sigma : X \to \to X$, $\sigma \circ \sigma = \text{id}$, $C(X,\sigma)$ is a real space of all complex continuous functions on X fulfilling $f(\sigma x) = \overline{f(x)}$ ([5]).

Let A be a real function algebra on (X,σ), i.e. a subalgebra of $C(X,\sigma)$ which is uniformly closed, separates points of X and contains real constants ([5]).

The well known Bishop theorem states that every uniform algebra A can be obtained by "gluing together" a family of antisymmetric algebras. In a sense, the class of antisymmetric algebras determines (forms a basis for) the class of uniform algebras. This idea of forming an algebra from more elementary "bricks" was precised by Arenson [1]. Following him we will define analogous notions for real function algebras.

Let $\mathcal{A}$ denote the class of all real function algebras (over all pairs

(X,σ)). For $A \in \mathcal{A}$, say $A \subseteq C(X,\sigma), R$ a closed subset of X, let $A|_R = \{ g|_R : g \in A\}$ and $A|_R^- :=$ the uniform closure of $A|_R$.

Definition 1. Let $\mathcal{R}$ be a cover of X by closed sets, $A \in \mathcal{A}, A \subseteq$ $\subseteq C(X,\sigma)$. We say that $\mathcal{R}$ *localizes* A if the conditions:

$$f \in C(X,\sigma) \text{ and } f|_R \in A|_R^- \text{ for each } R \in \mathcal{R}, \text{ imply } f \in A.$$

Definition 2. A subclass $\mathcal{B} \subseteq \mathcal{A}$ is called *basic* if for every $A \in \mathcal{A}$, say $A \subseteq C(X,\sigma)$, there exists a cover $\mathcal{R}$ of X such that

(i) $\mathcal{R}$ localizes A;

(ii) for $R \in \mathcal{R}, A|_R^- \in \mathcal{B}$.

More picturesquely, every $A \in \mathcal{A}$ can be obtained from algebras belonging to $\mathcal{B}$ by "gluing" them together in a specified way.

The Bishop theorem states then that the family of all maximal antisymmetric sets localizes A. Following [1], we will denote this family $\mathcal{R}_1$.

We will remind the definitions for real function algebras.

Definition 3. [6] Let A be a real function algebra on (X,σ). A nonempty subset R of X is called a *set of antisymmetry* if:

(i) $f \in A$ and $f|_R$ is real implies $f|_R$ is constant, and

(ii) $f \in A$ and $f|_R$ is purely imaginary implies $f|_R$ is constant.

Definition 4. [2] Let A be a real function algebra on (X,σ). A nonempty subset R of X is called a *set of r-antisymmetry* if:

(i) $f \in A$ and $f|_R$ is real implies $f|_R$ is constant, and

(ii) R is σ-invariant, i.e. $\sigma(R) = R$.

Note that if a set is σ-invariant then a function with nonzero imaginary part cannot be constant on it. It follows for example, that if $A = C(X,\sigma)$ then the only sets which are both antisymmetric and r-antisymmetric are the singleton fixpoints. So in general the notions of antisymmetric and r-antisymmetric sets are different.

In [2], Cor. 2.5. it was proved that if $A|_R$ is an algebra of real type (see [3]) then R is r-antisymmetric iff R is a set of antisymmetry for the complex algebra $A + iA$. From this fact and from [6], Lemma 2.12 and Th. 2.15 it follows that:

If $A|_R$ is an algebra of real type and $\sigma(R) = R$, then R is antisymmetric iff R is r-antisymmetric.

In general, if R is antisymmetric set for A, then $R \cup \sigma(R)$ is r-antisymmetric. We will soon use this fact.

From the analogue of Bishop theorem for real function algebras (see [6], Cor. 3.4. and Th. 3.6.), the cover $\mathcal{R}_1$ of X by maximal antisymmetric sets localizes A. The problem is, which other covers localize A, or, equivalently, which subclasses $\mathcal{B} \subseteq \mathcal{A}$ are basic.

It is not difficult to prove that the cover $\mathcal{R}_1{}'$ by maximal r-antisymmetric sets localizes A. To this end let us show first:

Proposition 5. *Let A be a real function algebra on (X, σ) and let R be a maximal r-antisymmetric set for A. Then $A|_R$ is closed in $C(R, \sigma|_R)$.*

Proof. Consider two cases. First, if A is of complex type then R is maximal antisymmetric for a complex algebra A', where A' means A with the multiplication extended to complex scalars. Second, if A is of real type then R is maximal antisymmetric for a complexification $B = A +$ $+iA$. In both cases the restriction algebras $A'|_R$ and $B|_R$ are closed in $C(R)$. Taking into account suitable inclusions it is easy to see that $A|_R$ is closed in $C(R, \sigma|_R)$. $\diamondsuit$

Now, Th. 3.3 from [6] (Machado theorem for real function algebras) states that for any $f \in C(X, \sigma)$ its distance from A is realized on some closed antisymmetric subset Y of X. Hence this distance is realized also on a r-antisymmetric set $Y \cup \sigma(Y)$ and repeating the proof of Cor. 3.4 in [6] we can show that the cover $\mathcal{R}_1{}'$ localizes A.

Let us consider other natural covers.

Definition 6. A closed set $F \subset X$ is a *peak set* for real function algebra $A \subset C(X, \sigma)$ if there exists $f \in A$ with $f = 1$ on F and $|f| < 1$ off of F. A closed set $E \subset X$ is a *weak peak set (p - set)* for A if E is an intersection of peak sets. If a function f equals 1 on a set (not necessarily closed) F and $|f| \leq 1$ off of F then we will say that f *peaks* on F.

Note that for any peak set, $F = \sigma(F)$ and that the countable intersection of peak sets is a peak set.

Definition 7. A real function algebra A on (X, σ) is called an *analytic (a weakly analytic) algebra* if from the fact that $f \in A$ and f is constant (f peaks) on an open subset of X it follows that f is constant on X.

It is clear that if A is analytic then it is weakly analytic.

We will call a closed set $R \subseteq X$ *(weakly) analytic* if the uniform closure $\overline{A|_R}$ of the algebra $A|_R$ is (weakly) analytic. This means that any subset of R which is also a peak set (in weakly analytic case), or a set of constancy (in analytic case) for some $f \in A|_R$ is nowhere dense in R or coincides with R.

This definition is the same as for uniform algebras (see [1]). In the case of uniform algebras it is known that these types of algebras: antisymmetric, analytic and weakly analytic are all different. In [1] it is also proved that $\mathcal{R}_2$ = the family of all weakly analytic sets, localizes A, while the family of all analytic sets does not.

Lemma 8. *If a set $F \subset X$ is weakly analytic, then it is antisymmetric.*

Proof. Let $f \in A$ be such a function that $f|_F$ is real. Suppose that $f|_F$ is not constant. Then the set $P(f)$ defined as the closure of the set of all polynomials of $f|_F$ contains a function g (defined on F) such that $\|g\| = 1, g \neq 1$ and $g^{-1}(1)$ contains a set which is open in F. This is impossible because F is weakly analytic. $\diamondsuit$

An easy consequence of this lemma is

Theorem 9. *If A is an analytic (weakly analytic) algebra, then it is also antisymmetric.*

From the lemma above, $\mathcal{R}_2 \subset \mathcal{R}_1$. We are going to show that $\mathcal{R}_2$ localizes A. First we define two smaller then $\mathcal{R}_2$ families of sets.

Given a probability measure ν on X we will consider A as a subspace in $L^p(\nu), 1 \leq p < \infty$ and denote its closure as $H^p(\nu)$. Also we define $H^\infty(\nu) = H^1(\nu) \cap L^\infty(\nu)$.

Definition 10. A probability measure ν is called an *antisymmetric measure* if every function in $H^\infty(\nu)$ that is real valued a.e. is constant a.e.

Let $\mathcal{R}_3$ denote the family of supports of antisymmetric measures.

Lemma 11. *The support of any antisymmetric measure is weakly analytic set.*

Proof. Let F be the support of any antisymmetric measure ν, let $f \in (A|_F)^{-}, \|f\| = 1$. Then $f \in H^\infty(\nu)$. If $G \subseteq f^{-1}(1)$ is open in F, then $\nu(G) > 0$ (because $F = \operatorname{supp} \nu$). It is easy to show that the sequence $((1+f)/2)^n$ converges a.e. to the characteristic function χ_H for some $H \supseteq G$. Since the measure is antisymmetric χ_H must be constant. $\diamondsuit$

It follows that $\mathcal{R}_3 \subset \mathcal{R}_2$.

Before defining the next cover we will remind some known facts.

Let $M(X, \sigma)$ be the set of all Radon self – conjugate measures on X, i.e.:

$$M(X, \sigma) = \{\mu \in M(X) : \mu = \overline{\mu} \circ \sigma\},$$

where $M(X)$ is the set of all Radon (= regular Borel) measures on X. We have:

Theorem 12. (Riesz type, [2]). *The mapping L defined by*

$$(L\mu)(f) = \int f d\mu \text{ for } \mu \in M(X,\sigma), f \in C(X,\sigma),$$

is a linear isometry from $M(X,\sigma)$ onto $C(X,\sigma)^*$.

Definition 13. ([2]) Let E be a subspace of $C(X,\sigma)$. A measure $\mu \in M(X,\sigma)$ is said to *annihilate (be orthogonal to) the subspace E* (in symbols $\mu \perp E$) if the functional F_μ represented by this measure fulfills the condition $F_\mu(f) = 0$ for every $f \in E$. The *annihilator* of $E, E^\perp$, is defined as the set of all measures orthogonal to E.

Definition 14. A Radon self-conjugate measure μ is an *extreme annihilating measure* for E if μ is an extreme point of the unit ball of $E^\perp, \mu \in \text{ext}B(E^\perp)$.

It is easy to prove ([2]) that if μ is an extreme annihilating measure then supp μ is an antisymmetric set.

Let $\mathcal{R}_4$ be the family of all supports of extreme annihilating measures along with all singleton subsets of X. The family $\mathcal{R}_4$ localizes A. (Let $f \in C(X,\sigma)$ be such that $f|_R$ belongs to $(A|_R)^-$ for any $R \in \mathcal{R}_4$. From the Krein – Milman theorem, any $\mu \in B(A^\perp)$ annihilates f. Suppose that $f \notin A$. Then from the Hahn – Banach theorem there exists $\mu \in B(A^\perp), \mu(f) = 1$, a contradiction.)

We are going to show that $\mathcal{R}_4 \subseteq \mathcal{R}_3$. Let $\mu \in \text{ext}B(A^\perp)$. It suffices to show that $|\mu|$ is antisymmetric. Let $f \in H^\infty(|\mu|)$ be a real valued function. If $\varepsilon > 0$ is sufficiently small then $h = (1/2) + \varepsilon f$ fulfills $0 < h < 1$ and obviously $h\mu \in A^\perp$. We have

$$\mu = \|h\mu\| \frac{h\mu}{\|h\mu\|} + \|(1-h)\mu\| \frac{(1-h)\mu}{\|(1-h)\mu\|}.$$

But $\mu \in \text{ext}B(A^\perp)$, hence $h\mu = \|h\mu\|\mu$. It follows that $h = \|h\mu\|$ a.e., so f is constant.

Since $\mathcal{R}_4 \subseteq \mathcal{R}_3 \subseteq \mathcal{R}_2 \subseteq \mathcal{R}_1$ and $\mathcal{R}_4$ localizes A, hence each of $\mathcal{R}_i, i = 1,2,3,4$ does so.

We are now going to investigate the problem whether the natural cover of X consisting of supports of real part representing measures localizes A.

Lemma 15. *Let μ be a probability measure. Then μ is antisymmetric iff for every Borel set F such that $\chi_F \in H^\infty(\mu), \mu(F) = 0$ or $\mu(F) = 1$.*

Proof. The necessity is obvious. To prove sufficiency, let $f \in H^\infty(\mu)$ be a real function, $a = \text{ess inf} f(x), b = \text{ess sup} f(x)$. We are going to show that $a = b$. Take (P_n), a sequence of polynomials with real coefficients which is point convergent on $[a, b]$ to the characteristic function of $[a, (a+b)/2]$, and such that $\max_{t\in[a,b]} |P_n(t)| \leq 1$. Then $P_n(f)$ is a sequence of functions from $H^\infty(\mu), \|P_n(f)\| \leq 1, \lim P_n(f) = 1$ on a set $F := f^{-1}([a, (a+b)/2])$. This sequence has a subsequence which is w^*-convergent to some $g \in H^\infty(\mu)$. But $g \equiv 1$ on a set F and $\|g\| \leq 1$, so it is easy to see that $((1+g)/2)^n$ converges a.e. to χ_F. Hence $\chi_F \in H^\infty(\mu)$ and from the assumption $\mu(F) = 1$. It follows $(a+b)/2 = b$, so $a = b$. $\diamondsuit$

Recall (see [5]) that a probability measure μ is called a *real part representing measure* for $\phi \in \Phi_A$ (Φ_A denotes the *carrier space* of an algebra A) if:

- for all $f \in A, \int \text{Re } f\, d\mu = \text{Re}\,\phi(f)$, and
- for every Borel set $E, \mu(E) = \mu(\sigma E)$.

Remark 16. Note that the measure μ is multiplicative on Re $A \cap A$, since for $f \in \text{Re } a \cap A, \int f\, d\mu = \int \text{Re } f\, d\mu = \text{Re } \phi(f) = \phi(f)$. The last equality follows from the general fact that if an algebra B is of strictly real type, $\mathcal{R}_4$, then for every $\phi \in \Phi_B, \phi(f) \in \mathbb{R}$ for $f \in B$ - see [3] for details. It is obvious that Re $A \cap A$ is of R_4 type.

Lemma 17. *If μ is a real part representing measure for a homomorphism $\phi \in \Phi_A$ then it is antisymmetric.*

Proof. Take any real function $f \in H^\infty(\mu)$. We have to show that f is constant. By Remark 16 μ is multiplicative on the $L^1(\mu)$ – closure of Re $A \cap A$; we will denote this closure $H^1(\mu)^r$. Now take a Borel set F such that $\chi_F \in H^\infty(\mu)$. Then $\chi_F \in H^1(\mu)^r$, so $\mu(F)^2 = \mu(\chi_F)^2 = = \mu(\chi_F^2) = \mu(\chi_f) = \mu(F)$. Hence $\mu(F) = 0$ or $\mu(F) = 1$. From the preceding lemma, μ is antisymmetric. $\diamondsuit$

Let $\mathcal{S}'$ denote the cover of X by supports of real part representing measures. From the above lemma, $\mathcal{S}' \subseteq \mathcal{R}_3$. If $\mathcal{R}_4$ had been a subfamily of $\mathcal{S}'$,we would have known that $\mathcal{S}'$ localizes X. But $\mathcal{S}'$ cannot contain $\mathcal{R}_4$ because $\mathcal{S}'$ consists of σ- invariant sets only. In order to have a localizing family we will add to $\mathcal{S}'$ some other sets.

Definition 18. Let Y be any subset of X. If a set Y_σ fulfills $Y_\sigma \cup \cup \sigma(Y_\sigma) = Y$, we will call Y_σ a σ - *generating subset* for Y. If moreover, Y_σ does not contain any Z_σ with $Z_\sigma \cup \sigma(Z_\sigma) = Y$, we will say that Y_σ is *a minimal σ -generating subset* for Y.

Of course Y is σ-generating for itself.

Let now $\mathcal{S} = \{Y_\sigma : Y \in \mathcal{S}'\}$. We will prove that the family $\mathcal{S}$ localizes A if A is large enough.

Recall that there are various methods of defining a Shilov boundary of a real function algebra A. We will use the following. If A is a real function algebra on (X,σ) then $S \subseteq X$ is called a *boundary if* $S = \sigma(S)$ and if Re f assumes its maximum on S for all $f \in A$. The *Shilov boundary* $S(A)$ of A is defined as the smallest closed boundary of A.

It can be shown ([4]), Cor. 3.8) that the Shilov boundary of A coincides with the Shilov boundary of its complexification, $S(A) = = S(A + iA)$.

A complex function algebra B is said to be *relatively maximal* ([7]) if for any subalgebra B' of $C(\Phi_B)$ containing B and such that $S(B) = S(B')$ it follows $B = B'$. Following this definition we will call a real function algebra A *relatively maximal* if its complexification $B = A + iA$ is relatively maximal.

Remark 19. Let us call a real function algebra A *weakly relatively maximal* if for any subalgebra A' of $C(\Phi_A)$ containing A and such that $S(A) = S(A')$ it follows $A = A'$. It is easy to see that if A is relatively maximal then it is weakly relatively maximal. (For the proof take any $A' \supseteq A, A'$ a subalgebra of $C(\Phi_A), S(A) = S(A')$. Then $B' = A' + iA'$ is a complex function algebra, $B' \supseteq B = A + iA$ and from [4] Cor.3.8 $S(B') = S(A') = S(A) = S(B)$, so it follows $B = B'$ hence $A = A'$.) It is not clear whether the converse holds true.

Corollary 2 in [7] states that if a complex function algebra B is relatively maximal and $X = S(B)$ then the cover of X by supports of representing measures localizes A.

Theorem 20. *If a real function algebra A is relatively maximal and $X = S(A)$ then $\mathcal{S}$ localizes A.*

Proof. Let $B = A+iA$. B is relatively maximal and $S(A) = S(B) = X$ (by assumption and [4] Cor. 3.8). Hence by [7] Cor. 2 the family

$$\mathcal{U} = \{\text{supp}\mu : \mu \text{ is representing for } B\}$$

localizes B. Let μ_σ be a measure on X defined by $\mu_\sigma(E) = \mu(\sigma E)$ for all Borel subset of X and $m = (\mu+\mu_\sigma)/2$. m is a real part representing measure for A ([4], Cor. 3.4) and suppμ is a σ - generating subset for $Y = \text{supp}m$. It follows $\mathcal{U} \subseteq \mathcal{S}$ so $\mathcal{S}$ localizes B. Let $f \in C(X,\sigma), f|_S \in$

$\in A|_S^-$ for $S \in \mathcal{S}$. Then $f + if \in C(X), (f + if)|_S \in (A + iA)|_S^-$, so $f + if \in B$. Hence $f \in A$. $\diamond$

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Mathematica Pannonica
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SIMULTANEOUS EXTENSIONS OF PROXIMITIES, SEMI-UNIFORMITIES, CONTIGUITIES AND MEROTOPIES IV*

Á. **Császár**

Department of Analysis, Eötvös Loránd University, H-1088 Budapest, Múzeum krt. 6 - 8, Hungary.

J. **Deák**

Mathematical Institute of the Hungarian Academy of Sciences, H-1053 Budapest, Reáltanoda u. 13 - 15, Hungary.

Received November 1990

AMS Subject Classification: (1985) 54 E 15, 54 E 05, 54 E 17

Keywords: (Riesz/Lodato) proximity, (Riesz/Lodato) semi-uniformity, (Riesz/Lodato) contiguity, (Riesz/Lodato) merotopy, Cauchy filter, extension.

Abstract: Given compatible merotopies in a semi-uniform or contiguity space (or semi-uniformities in a proximity space), we are looking for a common extension of these structures.

§§ 0 and 1 can be found in Part I [1], §§ 2 to 4 in Part II [2], §§ 5 and 6 in Part III [3]. See § 0 for terminology, notations and conventions.

* Research supported by Hungarian National Foundation for Scientific Research, grant no. 1807.

7. Extending a family of merotopies in a semi-uniform space

A. WITHOUT SEPARATION AXIOMS

7.1. A family of merotopies in a semi-uniform space has a coarsest and a finest extension; we are going to construct both.
Notation. For an entourage U, let

$$\mathsf{c}^0(U)=\{C: C^2\subset U\},\qquad \mathsf{c}^1(U)=\Big\{\{x,y\}: xUy, yUx\Big\}.$$

Recall from 0.4 that $U(\mathsf{c})=\bigcup_{C\in\mathsf{c}} C^2$ for a cover c; this notation will be used for arbitrary collections $\mathsf{c}\subset \exp X$. ◇
Lemma.
a) $\mathsf{c}^0(U)$ *and* $\mathsf{c}^1(U)$ *are covers.* $U(\mathsf{c})$ *is an entourage iff* c *is a cover.*
b) $\mathsf{c}^k(U\cap V)=\mathsf{c}^k(U)\cap\mathsf{c}^k(V)$ $(k=1,2)$.
c) $U(\mathsf{c}^k(U))=U\cap U^{-1}$ $(k=1,2)$.
d) *For a cover* c, $\mathsf{c}^1(U(\mathsf{c}))$ *refines* c, *and* $\mathsf{c}\subset\mathsf{c}^0(U(\mathsf{c}))$.
e) *If* c *is a topology on* X, *and* U *is symmetric and open then* $U(\mathrm{int}_c\mathsf{c}^0(U))=U$.

Proof. e) $U(\mathrm{int}_c\mathsf{c}^0(U))\subset U(\mathsf{c}^0(U))=U$. Conversely, if xUy then $V\times W\subset U$ for some c-open neighbourhoods V of x and W of y. We may assume $V^2\subset U$, $W^2\subset U$, since xUx, yUy; $W\times V\subset U$ by the symmetry. Thus $C=V\cup W\in\mathsf{c}^0(U)$, and, C being c-open, $C\in\mathrm{int}_c\mathsf{c}^0(U)$, $(x,y)\in C^2\subset U(\mathrm{int}_c(\mathsf{c}^0(U))$. ◇
Remark. Saying that c is *finer* than d instead of c refines d (which is, of course, in conflict with established terminology), the content of this trivial lemma can be interpreted as follows: any symmetric entourage U can be induced by coverings; $\mathsf{c}^0(U)$ is the coarsest and $\mathsf{c}^1(U)$ the finest one (more precisely, one of the coarsest, respectively finest ones); if U is open then $\mathrm{int}\,\mathsf{c}^0(U)$ is the coarsest open cover inducing U.

7.2 Recall the following notations:

$$\mathsf{c}_i^0=\{C_i^0: C_i\in\mathsf{c}_i\}\quad (i\in I,\ \mathsf{c}_i\in\mathsf{M}_i),$$
$$C_i^0=C_i\cup X_i^r,\quad X_i^r=X\setminus X_i;$$

$s\mathcal{U}$ denotes the collection of the symmetric elements of $\mathcal{U}$.
Definition. For a family of merotopies in a semi-uniform space,

a) Let M^0 be the merotopy for which c_i^0 $(i \in I,\ \mathsf{c}_i \in \mathsf{M}_i)$ and $\mathsf{c}^0(U)$ $(U \in \mathcal{U})$ form a subbase B^0.

b) Let M^1 consist of those covers c of X for which

$$\mathsf{c}|X_i \in \mathsf{M}_i \quad (i \in I); \tag{1}$$

$$U(\mathsf{c}) \in \mathcal{U}. \tag{2}$$

The more precise notations $\mathsf{M}^k(\mathcal{U}, \mathsf{M}_i) = \mathsf{M}^k(\mathcal{U}, \{\mathsf{M}_i : i \in I\})$ will be used when necessary; $\mathsf{M}^k(\mathcal{U}) = \mathsf{M}^k(\mathcal{U}, \emptyset)$ $(k = 0, 1)$. ◊

The elements of B^0 are covers, so it is indeed a subbase for a merotopy. It does not change B^0 if $\mathcal{U}$ is replaced by $s\mathcal{U}$ in the definition (since $\mathsf{c}^0(U)$ depends only on $U \cap U^{-1}$). Replacing $\mathcal{U}$ and each M_i by subbases, we still obtain a subbase for M^0 (Lemma 7.1 b) and $\mathsf{c}_i^0(\cap)\mathsf{d}_i^0 = (\mathsf{c}_i(\cap)\mathsf{d}_i)^0$). If $I = \emptyset$ then $\mathsf{B}^0 = \{\mathsf{c}^0(U) : U \in s\mathcal{U}\}$ is a base, not just a subbase (Lemma 7.1 b)). M^1 is clearly a merotopy. The next Lemma gives an alternative description of M^1; in particular, if $I = \emptyset$ then $\mathsf{B}^1 = \{\mathsf{c}^1(U) : U \in s\mathcal{U}\}$ is a base for M^1.

Lemma *The covers of the form*

$$\mathsf{c}_1(U) \cup \bigcup_{i \in 1} \mathsf{c}_i \quad (U \in s\mathcal{U}, \mathsf{c}_i \in \mathsf{M}_i \quad (i \in I)) \tag{3}$$

make up a base B^1 *for* M^1.

Proof. If c is as in (3) then $\mathsf{c}|X_i \supset \mathsf{c}_i$, thus (1) holds; $U(\mathsf{c}) \supset U(\mathsf{c}^1(U)) = U$, thus (2) holds, too. This means that $\mathsf{B}^1 \subset \mathsf{M}^1$. Conversely, any $\mathsf{c} \in \mathsf{M}^1$ is refined by (3) taken with $\mathsf{c}_i = \mathsf{c}|\mathsf{M}_i$ and $U = U(\mathsf{c})$. ◊

Theorem. *Any family of merotopies in a semi-uniform space has extensions;* M^0 *is the coarsest and* M^1 *the finest one.*

Proof. 1° M^0 *is coarser than* M^1. It is enough to show that $\mathsf{B}^0 \subset \mathsf{M}^1$, i.e. that (1) and (2) hold for the covers c_i^0 and $\mathsf{c}^0(U)$. It follows from the accordance that c_i^0 satisfies (1) (this fact was already used in the proof of Theorem 3.1). (2) is satisfied, too, since the compatibility implies that $U(\mathsf{c}_i) = U|X_i$ with some $U \in \mathcal{U}$, and from

$$C_i^{02} = C_i^2 \cup (C_i \times X_i^r) \cup (X_i^r \times C_i) \cup X_i^{r2}$$

we obtain $U(\mathsf{c}_i^0) = U|X_i \cup (X^2 \setminus X_i^2)$, so that $U \subset U(C_i^0)$. $\mathsf{c}^0(U)|X_i = \mathsf{c}^0(U|X_i)$ is clear from the definition, thus (1) holds for $\mathsf{c}^0(U)$ (since, assuming $U \in s\mathcal{U}$, $U|X_i = U(\mathsf{c}_i)$ for some $\mathsf{c}_i \in \mathsf{M}_i$, which refines $\mathsf{c}^0(U|X_i)$ by Lemma 7.1 d)); (2) follows from Lemma 7.1 c).

2° M^0 *and* M^1 *are compatible.* According to 1°, it is enough to check that $\mathcal{U}(\mathsf{M}^1) \subset \mathcal{U} \subset \mathcal{U}(\mathsf{M}^0)$. The first inclusion is evident from (2). If $U \in s\mathcal{U}$ then $\mathsf{c}^0(\mathcal{U}) \in \mathsf{M}^0$, so $U(\mathsf{c}^0(U)) \in \mathcal{U}(\mathsf{M}^0)$; hence $U \in \mathsf{M}^0$ by Lemma 7.1 c).
3° M^0 *and* M^1 *are extensions.* By 1° and 2°, we have only to see that $\mathsf{M}^1|X_i \subset \mathsf{M}_i \subset \mathsf{M}_0|X_i$. The first inclusion is clear from (1), the second one from $\mathsf{c}_i^0|X_i = \mathsf{c}_i$.
4° M^0 *is coarsest,* M_1 *is finest.* Let M be an extension. Any $\mathsf{c} \in \mathsf{M}$ satisfies (1) and (2) by the definition of an extension, thus $\mathsf{M} \subset \mathsf{M}^1$. For $\mathsf{c}_i \in \mathsf{M}_i$, there is a $\mathsf{c} \in \mathsf{M}$ with $\mathsf{c}|X_i = \mathsf{c}_i$; c refines c_i^0, thus $\mathsf{c}_i^0 \in \mathsf{M}$. Given a $U \in s\mathcal{U}$, there is a $\mathsf{c} \in \mathsf{M}$ with $U = U(\mathsf{c})$ (see 0.4), and then $\mathsf{c}^0(U) \supset \mathsf{c}$ by Lemma 7.1 d), thus $\mathsf{c}^0(U) \in \mathsf{M}$, too. Hence $B^0 \subset \mathsf{M}$, implying $\mathsf{M}^0 \subset \mathsf{M}$. ◊

B. RIESZ MEROTOPIES IN A SEMI-UNIFORM SPACE

7.3 If a family of merotopies in a semi-uniform space has a Riesz extension then the semi-uniformity is Riesz, and the trace filters are Cauchy (with respect to the merotopies). The merotopies are also Riesz, but this is included in the statement that the trace filters are Cauchy. The above conditions are sufficient, too.
Definition. For a family of merotopies in a semi-uniform space, let

$$\mathsf{M}_R^1 = \{\mathsf{c} \in \mathsf{M}^1 : \ \operatorname{int} \mathsf{c} \text{ is a cover of } X\}. \ \Diamond$$

(Compare with Definition 3.2.)
Theorem. *A family of merotopies in a Riesz semi-uniform space has a Riesz extension iff the trace filters are Cauchy; if so then* M^0 *is the coarsest and* M_R^1 *the finest Riesz extension.*
Proof. The necessity is obvious. Assume conversely that the trace filters are Cauchy. Now M^0 is Riesz, since $\operatorname{int}\mathsf{c}$ is a cover for each $\mathsf{c} \in B^0$. Indeed, $\operatorname{int}\mathsf{c}_i^0$ is a cover by the Cauchy property, while if $U \in \mathcal{U}$ then $\Delta \subset \operatorname{int} U$ implies that for any $x \in X$, there is a $C \in \mathrm{v}(x)$ with $C^2 \subset U$, and it follows from $C \in \mathsf{c}^0(U)$ that $\operatorname{int} \mathsf{c}^0(U)$is a cover, too.

M^0 is the coarsest Riesz extension by Theorem 7.2. If M is a Riesz extension then $\mathsf{M} \subset \mathsf{M}^1$ (Theorem 7.2), therefore $\mathsf{M} \subset \mathsf{M}_R^1$. In particular, $\mathsf{M}^0 \subset \mathsf{M}_R^1$; this and the evident inclusion $\mathsf{M}_R^1 \subset \mathsf{M}^1$ imply that M_R^1 is an extension (again Theorem 7.2). It follows from the definition that, being compatible, M_R^1 is Riesz. ◊

Remark. Given a semi-uniformity $\mathcal{U}$ and a $U \in s\mathcal{U}$, there is in general no finest one (in the sense of Remark 7.1) among the covers c inducing U for which int c is cover: take the Euclidean uniformity on $\mathbb{R}$, and $U = \mathbb{R}^2$; observe that $U = U(\mathsf{c}(\varepsilon))$ $(\varepsilon > 0)$ where

$$\mathsf{c}(\varepsilon) = \{]x, x + \varepsilon[\cup \{y\} : x, y \in \mathbb{R}\}.$$

So we cannot hope for a characterization of M^1_R similar to Lemma 7.2.

C. LODATO MEROTOPIES IN A SEMI-UNIFORM SPACE

7.4 If a family of merotopies in a semi-uniform space has a Lodato extension then the semi-uniformity and the merotopies are Lodato, the trace filters are Cauchy, and 3.6 (1) holds. These conditions are not sufficient, see Examples 7.12.

Definition. For a family of Lodato merotopies in a Lodato semi-uniform space,

a) Let $\mathsf{M}^1_L = \{\mathsf{c} \in \mathsf{M}^1 : \text{int}\,\mathsf{c} \in \mathsf{M}^1\}$.

b) If the trace filters are Cauchy then let M^0_L be the merotopy for which $\{\text{int}\,\mathsf{c} : \mathsf{c} \in \mathsf{M}^0\}$ is a base. $\diamondsuit$

The open covers in M^1 form a base for M^1_L. In b), int c is a cover, because the trace filters are Cauchy and $\mathcal{U}$ is Lodato; these covers form a base for a merotopy, since int c $(\cap)$ int d = int $(\mathsf{c}(\cap)\mathsf{d})$. The following covers make up a subbase B^0_L for M^0_L:

$$\text{int}\,\mathsf{c}^0_i \quad (i \in I,\ \mathsf{c}_i \in \mathsf{M}_i,\ \mathsf{c}_i \text{ is } c_i\text{-open});$$

$$\text{int}\,\mathsf{c}_0(U) \quad (U \in s\mathcal{U}, U \text{ is open}).$$

Observe that

(1) $$\text{int}\,\mathsf{c}^0(U) = \{C : C^2 \subset U, \quad C \text{ is open}\}.$$

Remark. There is a simple reason for the similarity with Definitions 3.4, 3.5 and 5.14: If μ is a collection of compatible merotopies in a topological space such that $\mathsf{M} \subset \mathsf{M'} \subset \mathsf{M''}$ and $\mathsf{M}, \mathsf{M''} \in \mu$ imply $\mathsf{M'} \in \mu$, there is a coarsest $\mathsf{M}^0 \in \mu$ (a finest $\mathsf{M}^1 \in \mu$), and there exists a Lodato merotopy in μ then M^0_L (M^1_L) defined as above is the coarsest (finest) Lodato merotopy in μ. (The proof is straightforward.) Analogous statements hold for contiguities and semi-uniformities.

Lemma. *A family of Lodato merotopies in a Lodato semi-uniform space has a Lodato extension iff the trace filters are Cauchy and* $\mathsf{M}^0_L \subset$

$\subset \mathsf{M}_L^1$; *if so then* M_L^0 *is the coarsest and* M_L^1 *the finest Lodato extension.*
Proof. The above remark applied to the collection of all extensions (Theorem 7.2) gives that if there are Lodato extensions then M_L^0 is the coarsest and M_L^1 the finest one; therefore $\mathsf{M}_L^0 \subset \mathsf{M}_L^1$. Assume conversely that the trace filters are Cauchy and $\mathsf{M}_L^0 \subset \mathsf{M}_L^1$. Then Theorem 7.2 and the trivial inclusions $\mathsf{M}^0 \subset \mathsf{M}_L^0$ and $\mathsf{M}_L^1 \subset \mathsf{M}^1$ yield that M_L^0 and M_L^1 are extensions. Being compatible, they are clearly Lodato. $\diamondsuit$

7.5 Remark. Lemma 7.4 remains valid if $\mathsf{M}_L^0 \subset \mathsf{M}_L^1$ is replaced by $\mathsf{M}_L^0 \subset \mathsf{M}^1$ (or $\mathsf{M}^0 \subset \mathsf{M}_L^1$). The proof is the same.

7.6 Lemma. *A family of merotopies in a semi-uniform space has a Lodato extension iff*

(i) *the semi-uniformity and the merotopies are Lodato;*
(ii) $U(\operatorname{int} \mathsf{c}_i^0) \in \mathcal{U} \quad (i \in I, \mathsf{c}_i \in \mathsf{M}_i)$;
(iii) $(\operatorname{int} \mathsf{c}_i^0)|X_j \in \mathsf{M}_j \quad (i, j \in I, \mathsf{c}_i \in \mathsf{M}_i)$;
(iv) $(\operatorname{int} \mathsf{c}^0(U))|X_i \in \mathsf{M}_i \quad (U \in s\mathcal{U}, i \in I)$.

Remarks. a) (ii) implies that each $\operatorname{int} \mathsf{c}_i^0$ is a cover, i.e. that the trace filters are Cauchy.

b) In comparison with Lemmas 5.17 and 6.8, Condition (iv) is completely new; we shall later see that it is not superfluous.

Proof. 1° *Necessity.* (i) is clear. (iii) follows from Theorem 3.6. If there are Lodato extensions then M_L^0 is one of them by Lemma 7.4, $\operatorname{int} \mathsf{c}_i^0 \in \mathsf{M}_L^0$ by definition, thus, M_L^0 being compatible, (ii) holds; (iv) follows from $\mathsf{M}_L^0|X_i = \mathsf{M}_i$ and $\operatorname{int} \mathsf{c}^0(U) \in \mathsf{M}_L^0$.

2° *Sufficiency.* The assumptions of Definition 7.4 are fulfilled, so, according to Remark 7.5, it is enough to check that $\mathsf{M}_L^0 \subset \mathsf{M}^1$, i.e. that $B_L^0 \subset \mathsf{M}^1$. This means four conditions, from which three are just (ii), (iii) and (iv), and the fourth, namely $U(\operatorname{int} \mathsf{c}^0(U)) \in \mathcal{U}$, holds by Lemma 7.1 e). $\diamondsuit$

Corollary. *A single Lodato merotopy* M_0 *in a Lodato semi-uniform space has a Lodato extension iff* $U(\operatorname{int} \mathsf{c}_0^0) \in \mathcal{U}$ *for each* c_0*-open* $\mathsf{c}_0 \in \mathsf{M}_0$, *and* $(\operatorname{int} \mathsf{c}^0(U))|X_0 \in \mathsf{M}_0$ *for each open* $U \in s\mathcal{U}$. $\diamondsuit$

The first assumption cannot be replaced by the Cauchy property of the trace filters, and the second one cannot be dropped either, see Examples 7.12.

7.7 Corollary. *Any Lodato semi-uniformity* $\mathcal{U}$ *can be induced by Lodato merotopies;* $\mathsf{M}_L^0(\mathcal{U})$ *is the coarsest and* $\mathsf{M}_L^1(\mathcal{U})$ *the finest one.* $\diamondsuit$

B_L^0 (consisting in this special case of the covers given in 7.4 (1)) is a base for $\mathsf{M}_L^0(\mathcal{U})$).

It can happen that $\mathsf{M}_L^0(\mathcal{U}) \neq \mathsf{M}^0(\mathcal{U})$ for a Lodato semi-uniformity $\mathcal{U}$. (In a proximity space, $I \neq \emptyset$ was needed for an analogous example, see Lemma 5.15 and Example 5.17.)

Example. On $X = \mathbb{R}$, take the semi-uniformity $\mathcal{U}$ for which $\{U(k) : k \in \mathbb{N}\}$ is a base, where

$$U(k) = \{(x,y) : \ |x-y| < 1/k\} \cup \bigcup\{Q_{mn} : \ m,n > k\},$$
$$Q_{mn} =]m - \frac{1}{m+n}, \ m + \frac{1}{m+n}[\times]\, n - \frac{1}{m+n}, \ n + \frac{1}{m+n}[.$$

c is the Euclidean topology, thus $U(k)$ is open, and $\mathcal{U}$ is Lodato. We claim that

$$\mathsf{c} = \operatorname{int} \mathsf{c}^0(u(1)) \in \mathsf{M}_L^0(\mathcal{U}) \backslash \mathsf{M}^0(\mathcal{U}).$$

Indeed, if c belonged to $\mathsf{M}^0(\mathcal{U})$ then there were a $k \in \mathbb{N}$ with $\mathsf{d} = \mathsf{c}^0(U(k))$ refining c. This is, however, impossible since $A = \{n \in \mathbb{N} : n > k\} \in \mathsf{d}$, but there is no open set $G \supset A$ such that $G^2 \subset U(1)$. ◇

7.8 $\mathsf{M}_L^0(\mathcal{U})$, $\mathsf{M}_L^1(\mathcal{U})$ and $\mathsf{M}_R^1(\mathcal{U})$ can be different:

Example. Take the Euclidean uniformity $\mathcal{U}$ on $X = \mathbb{R}$, and let $f(x) = x + (1+|x|)^{-1}$. Then

(1) $$\mathsf{d} = \{]x, f(x)[\, \cup\,]y, f(y)[: x, y \in X\} \in \mathsf{M}_L^1(\mathcal{U}) \backslash \mathsf{M}_L^0(\mathcal{U}),$$
$$\{\{x,y\} : x,y \in X\} \cup \{]x, f(x)[: x \in X\} \in \mathsf{M}_R^1(\mathcal{U}) \backslash \mathsf{M}_L^1(\mathcal{U}). \ ◇$$

7.9 Condition (iii) is not superfluous in Lemma 7.6:

Example. Let $\mathcal{U}$ be the Euclidean uniformity on $\times = \mathbb{R} \times [0,1[$, $X_0 = \mathbb{R} \times \{0\}$, $X_1 = X_0^r$, $\mathsf{M}_i = \mathsf{M}_L^i(\mathcal{U})|X_i$. 7.5 (ii) and (iv) are satisfied, since M_0 and M_1 separately have extensions. But (iii) fails for $i = 1$, $j = 0$,

$$\mathsf{c}_1 = \{D \times]\, 0, \ 1[: D \in \mathsf{d}\} \in \mathsf{M}_1$$

with d from 7.8 (1). ◇

7.10 Corollary. *A family of merotopies in a Lodato semi-uniform space has a Lodato extension iff* $\{\mathsf{M}_i, \mathsf{M}_j\}$ *has a Lodato extension for any* $i, j \in I$. ◇

7.11 Corollary. *A family of merotopies in a Lodato semi-uniform space has a Lodato extension iff it has a Lodato extension in* (X, c), *and each* M_i *has a Lodato extension in* $(X, \mathcal{U})$.

Proof. Theorem 3.6 and Lemma 7.6. ♢

7.12 Theorem. *A family of Lodato merotopies given on open-closed subsets in a Lodato semi-uniform space has Lodato extensions.*
Proof. By Corollaries 3.8 and 7.11 it is enough to check that each M_i separately has a Lodato extension, i.e. that

(1) $$U(\operatorname{int} \mathsf{c}_i^0) \in \mathcal{U} \quad (\mathsf{c}_i \in \mathsf{M}_i \quad \text{is} \quad c_i\text{-open}),$$

(2) $$(\operatorname{int} \mathsf{c}^0(U))|X^i \in \mathsf{M}_i \quad (U \in s\mathcal{U} \quad \text{is open}).$$

X_i being closed, we have int $\mathsf{c}_i^0 = \mathsf{c}_i^0$; $U(\mathsf{c}_i^0) \in \mathcal{U}$, because $\mathsf{c}_i^0 \in \mathsf{M}^0$, which is compatible. Thus (1) holds indeed. On the other hand, the openness of X_i implies that

$$(\operatorname{int} \mathsf{c}_0(U))|X_i = \operatorname{int}_i (\mathsf{c}^0(U)|X_i)$$

(see 7.4 (1)). Now $\mathsf{c}^0(U)|X_i \in \mathsf{M}_i$, since $\mathsf{c}^0(U)$ belongs to the extension M^0. Thus, M_i being Lodato, (2) is satisfied, too. ♢

It is not enough to assume that the sets are open and the trace filters Cauchy, or that the sets are closed. The next examples (with $|I| = 1$) have the additional property that there exists a Lodato extension in $(X, \delta(\mathcal{U}))$.
Examples. a) With X , X_0 and M_0 from Example 5.20, M_0 is compatible with $\mathcal{U}|X_0$, where $\mathcal{U}$ is the Euclidean uniformity on X. $\mathcal{U}$ and M_0 are Lodato, and X_0 is open. The trace filters are Cauchy; in fact, M_0 has a Lodato extension in $(X, \delta(\mathcal{U}))$ (see 5.20 and Corollary 5.17). The second condition of Corollary 7.6 holds (because X_0 is open), but the first one fails for $\mathsf{c}_0(1)$: no set of the form $(]-\varepsilon, \varepsilon[\times\{0\})^2 \cap x$ is contained by $U(\operatorname{int} \mathsf{c}_0(1)^0)$.

b) Let X and $\mathcal{U}$ be as in Example 7.7, $X_0 = \mathbb{N}$, $\mathsf{M}_0 = \mathsf{M}^0(\mathcal{U}_0)$ $\mathcal{U}_0 = \mathcal{U}|X_0$. Now $\mathcal{U}$ and M_0 are Lodato (the latter because c_0 is discrete), and X_0 is closed. M_0 has a Lodato extension in $(X, \delta(\mathcal{U}))$ (Theorem 5.22), but it does not have one in $(X, \mathcal{U})$: $(\operatorname{int} \mathsf{c}^0(U(1)))|X_0 \notin \mathsf{M}_0$, since this cover consists of finite sets, while M_0 is contigual. ♢

8. Extending a family of semi-uniformities in a proximity space

A. WITHOUT SEPARATION AXIOMS

8.1 Results are, and proofs could be, analogous to those for merotopies in a proximity space (§ 5). The foll]owing simple observation will save us from doing all over again:
Lemma. *For a family of semi-uniformities in a proximity space,* $\{\mathsf{M}^0(U_i) : i \in I\}$ *is a family of merotopies in the same space. The trace filters are* U_i*-Cauchy iff they are* $\mathsf{M}^0(U_i)$*-Cauchy.*
Proof. The accordance follows from $C^0(U|X_i) = \mathsf{c}^0(U)|X_i$. ◇

8.2 Definition. For a family of merotopies in a proximity space, let:

$$U^0 = \mathcal{U}(\mathsf{M}^0(\delta, \mathsf{M}^0(U_i))). \ \diamond$$

The following entourages constitute a subbase $\mathcal{B}$ for $\mathcal{U}^0$:

$$U_i^0 = U_i \cup (X^2 \backslash X_i^2) = U((\mathsf{c}^0(U_i)^0) \qquad (i \in I, U_i \in \mathcal{U}_i);$$

$$U_{A,B} = A^{r2} \cup B^{r2} = U(\mathsf{c}_{A,B}) \qquad (A\, \bar{\delta}\, B).$$

Theorem. *A family of semi-uniformities in a proximity space can always be extended;* $\mathcal{U}^0$ *is the coarsest extension.*
Proof. It follows from Theorem 5.4 and Lemma 8.1 that $\mathcal{U}^0$ is an extension. Let $\mathcal{U}$ be another extension; then $\mathsf{M}^0(\mathcal{U})$ is an extension of the merotopies $\mathsf{M}^0(\mathcal{U}_i)$, thus $\mathsf{M}^0 \subset \mathsf{M}^0(\mathcal{U})$ (Theorem 5.4), implying $\mathcal{U}^0 = \mathcal{U}(\mathsf{M}^0) \subset \mathcal{U}(\mathsf{M}^0(\mathcal{U})) = \mathcal{U}$. ◇

It follows from Example 5.3 that there is in general no finest compatible (Riesz/Lodato) semi-uniformity in a (Riesz/Lodato) proximity space.

B. RIESZ SEMI-UNIFORMITIES IN A PROXIMITY SPACE

8.3 Theorem. *A family of semi-uniformities in a Riesz proximity space has a Riesz extension iff the trace filters are Cauchy; if so then* $\mathcal{U}^0$ *is the coarsest Riesz extension.*
Proof. If the conditions are fulfilled then $\mathcal{U}^0$ is Riesz by Lemma 8.1 and Theorem 5.9. ◇

C. LODATO SEMI-UNIFORMITIES IN A PROXIMITY SPACE

8.4 Although the results are analogous to those for Lodato merotopies, we cannot keep on applying the results of § 5, since $\mathsf{M}^0(\mathcal{U}_i)$ is in general not Lodato (Example 7.7), while it can occur that $\{\mathsf{M}^0_L(\mathcal{U}_i) : i \in I\}$ is not a family of merotopies (it is not accordant):

Example. With X and $\mathcal{U}$ from Example 7.7, let $\delta = \delta(\mathcal{U})$, $X_0 = \mathbb{N}$, $X_1 = X$, $\mathcal{U}_i = \mathcal{U}|X_i$. Now $\{\mathcal{U}_0, \mathcal{U}_1\}$ is a family of semi-uniformities having a Lodato extension (namely $\mathcal{U}$), but $\mathsf{M}^0_L(\mathcal{U}_0)$ and $\mathsf{M}^0_L(\mathcal{U}_1)$ are not accordant: if they were then $\mathsf{M}^0_L(\mathcal{U})$ would be a Lodato extension of $\mathsf{M}^0_L(\mathcal{U}_0)$, contradicting Example 7.12 b). ◇

Remark. An open filter (in particular, a trace filter) is $\mathcal{U}_i$-Cauchy iff it is $\mathsf{M}^0_L(\mathcal{U}_i)$-Cauchy. This observation makes it possible to apply the results of § 5 C in the special case $|I| \leq 1$.

8.5 Definition. The entourage U is a *δ-entourage* if $A \,\delta\, B$ implies that there are $x \in A$, $y \in B$ with xUy. ◇

U is a δ-entourage iff $A \,\bar{\delta}\, U[A]^r \quad (A \subset X)$.

Lemma. *A cover* $\mathfrak{c}$ *is a δ-cover iff* $U(\mathfrak{c})$ *is a δ-entourage.* ◇

8.6 Lemma. *For a semi-uniformity* U *on* X, $\delta(\mathcal{U})$ *is coarser than* δ *iff every* $U \in \mathcal{U}$ *is a δ-entourage iff* $\mathcal{U}$ *has a base consisting of δ-entourages.* ◇

8.7 Lemma. *If* U *and* V *are δ-entourages and* $V = U(\mathfrak{f})$ *with a finite cover* $\mathfrak{f}$ *then* $U \cap V$ *is a δ-entourage.*

Proof. Take a cover $\mathfrak{c}$ such that $U \cap U^{-1} = U(\mathfrak{c})$, and use Lemmas 5.2 and 8.5. ◇

8.8 Definition. For a family of Lodato semi-uniformities in a Lodato proximity space with Cauchy trace filters, let $\{\operatorname{Int} U : U \in \mathcal{B}\}$ be a subbase for $\mathcal{U}^0_L$ (with $\mathcal{B}$ from 8.2). ◇

The Cauchy property implies that Int U is indeed an entourage. Copying the argument from 5.14 to 5.17 and 5.22, we obtain:

Lemma. *A family of semi-uniformities in a proximity space has a Lodato extension iff*

(i) *the proximity and the semi-uniformities are Lodato;*

(ii) $\bigcap_{i \in F} \operatorname{Int} U^0_i$ *is a δ-entourage whenever* $\emptyset \neq F \subset I$ *is finite, and* $U_i \in \mathcal{U}_i \quad (i \in F)$;

(iii) $(\operatorname{Int} U^0_i)|X_j \in \mathcal{U}_j \quad (i, j \in I, U_i \in \mathcal{U}_i)$.

If these conditions are satisfied then $\mathcal{U}_L^0$ is the coarsest Lodato extension. $\diamondsuit$

(When showing that $\bigcap_{i\in F} \mathrm{Int} U_i^0 \cap \bigcap_{k=1}^{n} U_{A_k,B_k}$ is a δ-entourage, apply Lemma 8.7 n times.)

Corollary. *A single Lodato semi-uniformity $\mathcal{U}_0$ in a Lodato proximity space has a Lodato extension iff* Int U_0^0 *is a δ-entourage for each ($c_0 \times c_0$-open) $U_0 \in \mathcal{U}_0$.* $\diamondsuit$

Theorem. *A family of Lodato semi-uniformities given on closed subsets in a Lodato proximity space has Lodato extensions; $\mathcal{U}^0 = \mathcal{U}_L^0$ is the coarsest one.* $\diamondsuit$

8.9 The condition in Corollary 8.8 cannot be replaced by the weaker assumption that the trace filters are Cauchy:

Examples. a) Let

$$X_0 = \{(1/k, 1/n) : k, n \in \mathbb{N}, k \leq n\},\ X = X_0 \cup \{(1/k, 0) : k \in \mathbb{N}\}.$$

With the Euclidean proximity δ on X, X_0 is open. For $x = (x', x''), y = = (y', y'')$, $x, y \in X$ and $\varepsilon > 0$, define

$$(1) \qquad xU_0(\varepsilon)y \text{ iff } |x' - y'| < \varepsilon,\ |x'' - y''| < \varepsilon,\ (x' \neq y' \Rightarrow x'' \neq y''),$$

and let $\{U_0(\varepsilon) : \varepsilon > 0\}$ be a base for $\mathcal{U}_0$. Each $U_0(\varepsilon)$ is an open δ_0-entourage, and $\mathcal{U}_0$ is clearly finer than the Euclidean semi-uniformity on X_0, thus $\mathcal{U}_0$ is a compatible Lodato semi-uniformity. The trace filters are Cauchy, but $\mathrm{Int}\, U_0(1)^0$ is not a δ-entourage (let A and B be disjoint infinite subsets of X_0^r).

b) Let everything be as above, but replace the last condition in (1) by

$$(x' = x'',\ y' \neq y'' \Rightarrow x'' < y''),\ (x' \neq x'',\ y' = y'' \Rightarrow y'' < x'').$$

Now the sets $A = X_0^r$ and $B = \{(1/n, 1/n) : n \in \mathrm{N}\}$ show that $\mathrm{Int}\, U_0(1)^0$ is not a δ-entourage. $\diamondsuit$

Similarly to 5.18 the condition of Corollary 8.8 can be split into two parts. The above examples show that neither of these parts is sufficient in itself.

8.10 Condition (iii) cannot be dropped from Lemma 8.8, see Example 2.10; (ii) cannot be replaced by the weaker assumption that each $\mathrm{Int}\, U_i^0$ is a δ-entourage:

Example. Taking X, X_0, X_1 and δ from Example 5.20, let $\{U_i(\varepsilon) : \varepsilon > 0\}$ be a base for $\mathcal{U}_i$ on X_i, where, with $x = (x', x'')$ and $y = (y', y'')$,

$$\begin{aligned} xU_1(\varepsilon)y \text{ iff } & |x' - y'| < \varepsilon, \ |x'' - y''| < \varepsilon, \\ & (x'', y'' < \varepsilon, x' < 0 < y' \Rightarrow -x' < y'), \\ & (x'', y'' < \varepsilon, y' < 0 < x' \Rightarrow -y' < x'), \\ xU_0(\varepsilon)y \text{ iff } & (-x', -x'')U_1(\varepsilon)(-y', -y'') \end{aligned}$$

The reasoning from 5.20 can be easily adapted. ◊

9. Extending a family of merotopies in a contiguity space

A. WITHOUT SEPARATION AXIOMS

9.1 In the problems investigated so far, a family of structures always had an extension if no separation property was required; this is not the case for merotopies in a contiguity space. It will be easier to describe the counterexample after some definitions and lemmas.

Definition. In a contiguity space (X, Γ),

a) A cover c of X is a Γ-*cover* if any finite cover refined by c belongs to Γ.

b) (See e.g. [4].) A collection $\mathsf{n} \subset \exp X$ is Γ-*near* if it is finite and $\mathsf{n}^r = \{N^r : N \in \mathsf{n}\} \notin \Gamma$. ◊

A finite cover is a Γ-cover iff it belongs to Γ. It follows easily from the axioms that Co2 could be replaced by

Co2″. if n is Γ-near and each $N \in \mathsf{n}$ is the union of a finite collection $\mathsf{a}(N)$ then there are $A(N) \in \mathsf{a}(N)$ such that $\{A(N) : N \in \mathsf{n}\}$ is Γ-near.

(Compare with P5 in 0.2, or rather with its more complicated form that can be obtained by induction. Observe that $A\,\delta(\Gamma)\,B$ iff $\{A, B\}$ is Γ-near.)

Lemma. *A cover* c *is a* Γ*-cover iff* $\mathsf{c} \cap \sec \mathsf{n} \neq \emptyset$ *for each* Γ*-near collection* n.

Proof. c is not a Γ-cover iff it refines some finite $\mathsf{f} \notin \Gamma$, i.e. iff there is a Γ-near collection n such that each $C \in \mathsf{c}$ is the subset of some

$N^r \in \mathsf{n}^r$. ◇

Compare this lemma with the definition of a δ-cover (5.1). By the observation made before the lemma, any Γ-cover is a $\delta(\Gamma)$-cover. Conversely, any δ-cover is a $\Gamma^1(\delta)$-cover (indeed, if c is a δ-cover then any finite cover refined by c is a δ-cover, too, so it belongs to $\Gamma^1(\delta)$ by definition).

9.2 Lemma. *For a merotopy* M *on* X, $\Gamma(\mathsf{M})$ *is coarser than* Γ *iff each element of* M *is a* Γ*-cover iff* M *has a base consisting of* Γ*-covers.* ◇

9.3 Lemma. *If* c *is a* Γ*-cover and* $\mathsf{f} \in \Gamma$ *then* $\mathsf{C}\,(\cap)\,\mathsf{f}$ *is a* Γ*-cover.*
Proof. Given a Γ-near collection n, we need $C \in \mathsf{c}$ and $D \in \mathsf{f}$ such that $C \cap D \in \sec \mathsf{n}$ (Lemma 9.1). By Co″2, it can be assumed that each element of n is contained by some element of the partition generated by f. As f is a Γ-cover, there is a $D \in \mathsf{f} \cap \sec \mathsf{n}$, implying $\bigcup \mathsf{n} \subset D$. Taking now a $C \in \mathsf{c} \cap \sec \mathsf{n}$, we have $C \cap D \in \sec \mathsf{n}$. ◇

For Γ-covers c and d, $\mathsf{c}\,(\cap)\,\mathsf{d}$ is not necessarily a Γ-cover: in Example 5.2, take $\Gamma = \Gamma^1(\delta)$.

9.4 Definition. For a family of merotopies in a contiguity space, let M^0 be the merotopy for which Γ and the covers c_i^0 $(i \in I, \mathsf{c}_i \in \mathsf{M}_i)$ form a subbase B. ◇

Γ could be replaced here by a subbase.

Lemma. *A family of merotopies in a contiguity space has an extension iff*

(1) $\left(\bigcap_{i \in F}\right) \mathsf{c}_i^0$ *is a* Γ*-cover whenever* $\emptyset \neq F \subset I$ *is finite and* $\mathsf{c}_i \in \mathsf{M}_i$ $(i \in F)$; *if so then* M^0 *is the coarsest extension.*

Remark. Compare (1) with (ii) of Lemma 5.17.

Proof. 1° *Necessity.* Let M be an extension. Then $\mathsf{c}_i \in \mathsf{M}_i = \mathsf{M}|X_i$, thus $\mathsf{c}_i^0 \in \mathsf{M}$, and $(\bigcap_{i \in F})\mathsf{c}_i^0 \in \mathsf{M}$, hence it is a Γ-cover by Lemma 9.2.

2° *Sufficiency.* We show that M^0 is an extension. Each element of M^0 is refined by a cover of the form $\mathsf{c} = ((\bigcap_{i \in F} \mathsf{c}_i^0)\,(\cap)\,\mathsf{f}$, where $\mathsf{c}_i \in \mathsf{M}_i$ and $\mathsf{f} \in \Gamma$. It follows from (1) and Lemma 9.3 that c is a Γ-cover; hence $\Gamma(\mathsf{M}^0) \subset \Gamma$ by Lemma 9.2. On the other hand, $\Gamma \subset \mathsf{B} \subset \mathsf{M}^0$ implies $\Gamma \subset \Gamma(\mathsf{M}^0)$. As $\mathsf{M}^0|X_i \supset \mathsf{M}_i$ is evident, we have only to check that $\mathsf{M}^0|X_i \subset \mathsf{M}_i$, i.e. that $\mathsf{B}|X_i \subset \mathsf{M}_i$. It was already used in other proofs that, in consequence of the accordance, $\mathsf{c}_j^0|X_i \in \mathsf{M}_i$; if $\mathsf{f} \in \Gamma$ then $\mathsf{f}|X_i \in \Gamma_i \subset \mathsf{M}_i$.

3° M^0 *is the coarsest extension.* It is clear that any extension has to contain B. ◊

Theorem. *A family of merotopies given on disjoint subsets in a contiguity space can always be extended.* M^0 *is the coarsest extension.*

Proof. To prove that (1) holds, it is enough to show (by Lemma 9.1) that if n is Γ-near then there are $C_i \in \mathsf{c}_i$ such that $\bigcap_{i\in F} C_i^0 \in \sec \mathsf{n}$. Take an index $k \notin I$, and define $X_k = (\bigcup_{i\in F} X_i)^r$, $J = F\{\cup k\}$. By Co″2, we may assume that each $N \in \mathsf{n}$ is the subset of some $X_{j(N)}$ with $j(N) \in J$. For any $i \in F$ fixed, take a $C_i \in \mathsf{c}_i$ that meets each $N \in \mathsf{n}$ lying in X_i; this is possible because a subcollection of a Γ-near collection is Γ-near, a Γ-near collection in X_i is Γ_i-near, and c_i is a Γ_i-cover. Now $\bigcap_{i\in F} C_i^0 = X_k \cup \bigcup_{i\in F} C_i$ meets each $N \in \mathsf{n}$. ◊

There is, in general, no finest compatible merotopy in a contiguity space: replace δ by $\Gamma^1(\delta)$ in Example 5.3 (if there existed a finest merotopy compatible with $\Gamma^1(\delta)$ then it would be the finest one among the merotopies compatible with δ).

9.5 Disjointness is essential in Theorem 9.4. In fact, for $n = 2, 3, \ldots$, there is a family of n merotopies in a contiguity space that has no extension, although any subfamily of cardinality $n-1$ has one:

Example. Let $2 \le n \in \mathbb{N}$, $Y_s = \mathbb{N} \times \{s\}$ $(1 \le s \le 2n)$, $I = \{1, \ldots, n\}$, $K = \{n+1, \ldots, 2n\}$, $X = \bigcup_{s=1}^{2n} Y_s$, $X_i = Y_i \cup \bigcup_{k\in K} Y_k$. Take the proximity δ on X for which $A\delta B$ iff either $A \cap B \neq \emptyset$ or both A and B are infinite. For $i \in I$, let $\mathsf{M}^0(\delta_i) \cup \{\mathsf{d}_i\}$ be a subbase for M_i on X_i, where

$$\mathsf{d}_i = \Big\{\{(m_s, s) : s \in \{i\} \cup K\} : m_s \in \mathbb{N}, m_{n+i} < m_{n+i+1}\Big\} \cup$$

$$\cup \Big\{ \bigcup_{k\in K} Y_k \Big\} \cup \{Y_i\},$$

and, in the definition of d_n, m_{2n+1} is identified with m_{n+1}. d_i is a δ_i-cover, thus $\delta(\mathsf{M}_i) = \delta_i$ by Lemmas 5.2 and 5.1 (because $\mathsf{M}^0(\delta_i)$ is compatible with δ_i, and it has a base consisting of finite covers). If $i, j \in I$, $i \neq j$ then $X_{ij} = \bigcup_{s\in K} Y_s \in d_i$, thus

$$\mathsf{M}_i|X_{ij} = \mathsf{M}^0(\delta_i)|X_{ij} = (\mathsf{M}^0(\delta)|X_i)|X_{ij} = \mathsf{M}^0(\delta)|X_{ij} = \mathsf{M}^0(\delta|X_{ij})$$

(Lemma 5.3 c)). Hence $\{\mathsf{M}_i : i \in I\}$ is a family of merotopies in (X, δ). Define $\Gamma_i = \Gamma(\mathsf{M}_i)$. Now $\{\Gamma_i : i \in I\}$ is clearly a family of contiguities in (X, δ), so we can take the coarsest extension $\Gamma = \Gamma_0$ (Definition and Theorem 6.2). $\{\mathsf{M}_i : i \in I\}$ is a family of merotopies in (X, Γ). We claim that $\mathsf{y} = \{Y_s : s \in I \cup K\}$ is Γ-near but $\mathsf{c} \cap \sec \mathsf{y} = \emptyset$ for $\mathsf{c} = (\bigcap_{i \in I})\mathsf{d}_i^0$; according to Lemmas 9.1 and 9.4, this implies that the family of merotopies cannot be extended.

To prove that y is Γ-near, it is enough to check that $\mathsf{f} \cap \sec \mathsf{y} \neq \emptyset$ for each $\mathsf{f} \in \Gamma$ (because this condition does not hold for $\mathsf{f} = \mathsf{y}^r$). f is refined by a cover $\mathsf{g}(\cap)(\bigcap_{i \in I})\mathsf{f}_i^0$ with $\mathsf{g} \in \Gamma^0(\delta)$ and $\mathsf{f}_i \in \Gamma_i$ (see the definition of Γ^0; it is enough to take only one f_i from each Γ_i, because the operations $(\cap)$ and ^{0} commute). $\mathsf{f}_i \in \mathsf{M}_i$, thus there is a finite $\mathsf{g}_i \in \mathsf{M}^0(\delta_i)$, i.e. a $\mathsf{g}_i \in \Gamma^0(\delta_i)$, such that $\mathsf{g}_i(\cap)\mathsf{d}_i$ refines f_i. If $A\bar{\delta}B$ then either A or B is finite, thus $\mathsf{c}_{A,B}$ contains a cofinite set; hence there is a cofinite $H \in \mathsf{g}$ (see the definition of $\Gamma^0(\delta)$). Similarly, there are sets $H_i \in \mathsf{g}_i$ cofinite in X_i. Pick a $\nu \in \mathbb{N}$ such that

$$H^r \cup \bigcup_{i \in I}(X_i \backslash H_i) \subset \{1, \ldots, \nu\} \times (I \cup K).$$

Consider the sets

$$D_1(\mu) = \{(\nu + 1, 1)\} \cup \{(\mu + k, k) : k \in K\} \in \mathsf{d}_1 \quad (\mu > \nu).$$

$D_1(\mu) \subset H_1 \in \mathsf{g}_1$, thus $D_1(\mu) \in \mathsf{g}_1(\cap)\mathsf{d}_1$. As this cover refines the finite f_1, there are $\mu > \nu, \eta > \mu + n$ and $E_1 \in \mathsf{f}_1$ such that $D_1(\mu), D_1(\eta) \subset E_1$. For $1 \neq i \in I$, define

$$D_i = \{(\nu + 1, i), (\eta + n + 1)\} \cup \{(\mu + k, k) : n + 1 \neq k \in K\}.$$

$D_i \in \mathsf{d}_i$, and also $D_i \subset H_i \in \mathsf{g}_i$, thus $D_i \in \mathsf{g}_i(\cap)\mathsf{d}_i$; hence $D_i \subset E_i$ with some $F_i \in \mathsf{f}_i$. Now

$$(\nu + 1, 1), \ldots, (\nu + 1, n), (\eta + n + 1, n + 1), (\mu + n + 2, n + 2), \ldots,$$
$$\ldots (\mu + 2n, 2n) \in H \cap \textstyle\bigcap_{i \in I} E_i^0 \in \mathsf{g}(\cap)(\bigcap_{i \in I})\mathsf{f}_i^0.$$

So there is indeed an element of f meeting each Y_s.

On the other hand, $\mathsf{c} \cap \sec \mathsf{y} = \emptyset$ is evident: any $C \in \mathsf{c}$ is of the form $\bigcap_{i \in I} D_i^0$ with suitable $D_i \in \mathsf{d}_i$; if $D_i = \bigcup_{k \in K} Y_k$ for some i then $C \cap Y_i = \emptyset$; if $D_i = Y_i$ for some i then $C \cap Y_k = \emptyset (k \in K)$; otherwise, $C \cap Y_k \neq \emptyset$ $(k \in K)$ would lead to n inequalities that cannot hold at the same time.

So we have proved that the merotopies cannot be extended. Any $n-1$ have, however, an extension; for reasons of symmetry, it is enough to show that this holds for $\mathsf{M}_1, \ldots, \mathsf{M}_{n-1}$, i.e. that, with I_0 denoting $\{1, \ldots, n-1\}, \mathsf{b} = (\bigcap_{i \in I_0})\mathsf{c}_i^0$ is a Γ-cover if $\mathsf{c}_i \in \mathsf{M}_i$ (Lemma

9.4). c_i is refined by $\mathsf{f}_i(\cap)\mathsf{d}_i$ with some finite $\mathsf{f}_i \in \mathsf{M}_i$; therefore $(\bigcap_{i\in I_0})\mathsf{f}_i^0(\cap)(\bigcap_{i\in I_0})\mathsf{d}_i^0$ refines b. The covers f_i^0 and d_i^0 are Γ-covers by Theorem and Lemma 9.4 (applied to M_i). f_i^0 being finite, it belongs to Γ, so we have only to prove that $(\bigcap_{i\in I_0})\mathsf{d}_i^0$ is a Γ-cover as well (Lemma 9.3).

Let n be Γ-near; sets $D_i \in \mathsf{d}_i$ have to be chosen such that $\bigcap_{i\in I_0} D_i^0 \in \sec \mathsf{n}$ (Lemma 9.1). According to Co2$''$ we may assume that for $N \in \mathsf{n}, N \subset Y_{s(N)}$ with some $s(N) \in I \cup K$. Consider the set $S = \{s(N) : N \in \mathsf{n}\}$ of indices. If $S \cap I_0 = \emptyset$ or $S \cap K = \emptyset$ then $D_i = \bigcup_{k\in K} Y_k$ $(i \in I_0)$, respectively $D_i = Y_i$ $(i \in \in I_0)$ will do. So we may assume that $S \cap I_0 \neq \emptyset \neq S \cap K$. Define

$$Z_s = \bigcap\{N \in \mathsf{n} : N \subset Y_s\} \quad (s \in S).$$

We claim that $Z_s \neq \emptyset$.

Indeed, let $j = s$ if $s \in I$, and $j \in S \cap I$ arbitrary if $s \in K$; d_j^0 being a Γ-cover, there is an $E_j \in \mathsf{d}_j$ with $E_j^0 \in \sec \mathsf{n}$. Clearly $Y_j \neq \neq E_j \neq \bigcup_{k\in K} Y_k$ (as E_j has to meet both sets). Hence E_j (so also E_j^0) meets Y_s in a single point, which lies necessarily in Z_s. We can deduce from this that Z_s is in fact infinite for $s \in S \cap K$:

Assume it is finite, and apply Co2$''$with $\mathsf{a}(N) = \{Z_s, N\backslash Z_s\}$ for $N \subset Y_s$ and $\mathsf{a}(N) = \{N\}$ otherwise. $A(N) = Z_s$ is impossible, since $\mathsf{c}^* = \{Z_s, Z_s^r\} \in \Gamma^0(\delta) \subset \Gamma$, so it is a Γ -cover; but $Z_s^r \cap A(N) = = Z_s^r \cap Z_s = \emptyset$, and $Z_s \cap A(\mathsf{M}) = Z_s \cap \mathsf{M} = \emptyset$ for $\mathsf{M} \subset Y_i$, $\mathsf{M} \in \mathsf{n}$, $i \in S \cap I$ (there is such an M because $S \cap I_0 \neq \emptyset$); hence $\mathsf{c}^* \cap \sec\{A(N) : N \in \mathsf{n}\} = \emptyset$, contradicting Lemma 9.1. Therefore $A(N) = N\backslash Z_s$ for $N \subset Y_s$, and the result of the foregoing paragraph, applied to $\{A(N) : N \in \mathsf{n}\}$ instead of n, yields $\bigcap\{N\backslash Z_s : N \in \mathsf{n}, N \subset Y_s\} \neq \emptyset$, a contradiction.

Pick now points $x_s = (\mu_s, s)$ for $s \in S \cup K$ such that $x_s \in Z_s$ if $s \in S$, and $\mu_s < \mu_{s+1}$ if $2n \neq s \in K$. The requires sets D_i $(i \in I_0)$ can be defined as follows: $D_i = \bigcup_{k\in K} Y_k$ if $i \notin S$; $D_i = \{x_s : s \in \{i\} \cup K\}$ if $i \in S$. ◇

B. RIESZ MEROTOPIES IN A CONTIGUITY SPACE

9.6 Lemma. *A family of merotopies in a contiguity space has a Riesz extension iff the trace filters are Cauchy, the contiguity is Riesz, and 9.4 (1) holds; if so then* M^0 *is the coarsest Riesz extension.*

Proof. The necessity of the conditions is clear. Conversely, if they are fulfilled then M^0 is an extension by Lemma 9.4, and it is Riesz, since $\operatorname{int} \mathsf{c}$ is a cover whenever $\mathsf{c} \in \mathsf{B}$ (for $\mathsf{c} \in \Gamma$ because Γ is Riesz, for c_i^0 because the trace filters are Cauchy). ◇

Theorem. *A family of merotopies given on disjoint subset in a Riesz contiguity space has a Riesz extension iff the trace filters are Cauchy.* ◇

C. LODATO MEROTOPIES IN A CONTIGUITY SPACE

9.7 If a family of merotopies in a contiguity space has a Lodato extension then the contiguity and the merotopies are Lodato, (ii) and (iii) from Lemma 5.17 hold, as well as 9.4 (1). We shall see that these conditions are sufficient if "Γ-cover" is substituted for "δ-cover" in (ii) (and then 9.4 (1) is superfluous), but not otherwise.

Definition. For a family of Lodato merotopies in a Lodato contiguity space with Cauchy trace filters, let $\{\operatorname{int} \mathsf{c} : \mathsf{c} \in \mathsf{B}\}$ be a subbase for M_L^0 (with B from Definition 9.4). ◇

Cf. Definition 5.14. $\{\operatorname{int} \mathsf{c} : \mathsf{c} \in \mathsf{M}^0\}$ is a base for M_L^0 ; the following covers form a subbase B_L for M_L^0 : the open elements of Γ , and $\operatorname{int} \mathsf{c}_i^0$ $(i \in I, \mathsf{c}_i \in \mathsf{M}_i, \mathsf{c}_i$ is c_i-open). M_L^0 is a Lodato merotopy compatible with c (just like in Lemma 5.14).

Lemma. *A family of merotopies in a contiguity space has a Lodato extension iff*

(i) *the contiguity and the merotopies are Lodato;*

(ii) $(\bigcap_{i \in F}) \operatorname{int} \mathsf{c}_i^0$ *is a Γ-cover whenever $\emptyset \neq F \subset I$ is finite and* $\mathsf{c}_i \in \mathsf{M}_i$ $(i \in F)$;

(iii) $(\operatorname{int} \mathsf{c}_i^0)|X_j \in \mathsf{M}_j$ $(i, j \in I, \mathsf{c}_i \in \mathsf{M}_i)$.

If these conditions are satisfied then M_L^0 is the coarsest Lodato extension.

Remark. See Remarks 5.17.

Proof. 1° *Necessity.* (i) is clear. If M is a Lodato extension then $\mathsf{c}_i^0 \in \mathsf{M}$ and $\operatorname{int} \mathsf{c}_i^0 \in \mathsf{M}$, implying $\mathsf{c} = (\bigcap_{i \in F}) \operatorname{int} \mathsf{c}_i^0 \in \mathsf{M}$, thus c is a Γ-cover by Lemma 9.2. (iii) follows from Theorem 3.6.

2° *Sufficiency.* We are going to show that M_L^0 is a Lodato extension (the conditions in its definition are satisfied, since the Cauchy property follows from (ii)). M^0 is an extension by Lemma 9.4 (as 9.4 (1) follows from (ii)). $\mathsf{M}^0 \subset \mathsf{M}_L^0$, so $\Gamma(\mathsf{M}_L^0) \supset \Gamma$ and $\mathsf{M}_L^0|X_i \supset \mathsf{M}_i$. It follows from (ii) and Lemma 9.3 that the elements of B_L are Γ-covers; hence

$\Gamma(\mathsf{M}_L^0) \subset \Gamma$ (Lemma 9.2). $\mathsf{B}_L|X_i \subset \mathsf{M}_i$ (for int c_j^0 by (iii), for the others by the compatibility), thus $\mathsf{M}_L^0|X_i \subset \mathsf{M}_i$, too. M_L^0 is clearly Lodato; it is the coarsest one, see Remark 7.4. ◇

The following three weaker conditions together cannot stand in lieu of (ii): 9.4 (1), 5.7 (ii), and each int c_i^0 is a Γ-cover (Example a) below). Condition (iii) cannot be dropped either (Example b)).

Examples. a) (A modification of Example 5.20.) Let $T = \{-1/n, 1/n : n \in \mathbb{N}\}, X = T\times]-1,1[, X_0 = T\times]-1,0[, X_1 = T\times]0,1[$, and take the Euclidean contiguity Γ on X, i.e. the one induced by the Euclidean merotopy (whose definition was given at the end of Example 3.8). Denoting the Euclidean closure on $\mathbb{R}^2$ by c^*, n is Γ-near iff $\bigcap_{N\in\mathsf{n}} \mathsf{c}^*(N) \neq \emptyset$ (because X is bounded in $\mathbb{R}^2$). Let $\{\mathsf{c}_i(\varepsilon) : 0, \varepsilon \leq 1\}$ be a base for M_i on X_i, where

$$\mathsf{c}_1(\varepsilon) = \{(]p, p+\varepsilon[\times]q, q+\varepsilon[) \cap X_1 : (p \in \mathbb{R}, q > 0) \text{ or } (0 \notin]p, p+\varepsilon[, q = 0)\} \cup$$

$$\cup\{C_1(k,n) : k, n \in \mathbb{N},\ k, n > 1/\varepsilon\} \cup \{D_1(\varepsilon', \varepsilon'') : 0 < \varepsilon'' < \varepsilon' < \varepsilon\},$$

$$C_1(k,n) = \{-1/k, 1/n\}\times]0,\varepsilon[,$$

$$D_1(\varepsilon', \varepsilon'') = ((]-\varepsilon'', 0[\cup]\varepsilon'', \varepsilon[)\times]0,\varepsilon[) \cap X_1,$$

$$\mathsf{c}_0(\varepsilon) = \{-C_1 : C_1 \in \mathsf{c}_1(\varepsilon)\},\ \ -C_1 = \{(-p, -q) : (p,q) \in C_1\}.$$

M_1 is compatible, because if f is a finite cover refined by $\mathsf{c}_1(\varepsilon)$ then there is an $E \in \mathsf{f}$ that contains infinitely many of the sets

$$(]-\varepsilon/2, \varepsilon/2[\times]1/m, 1/m+\varepsilon[) \cap X_1 \qquad (m \in \mathbb{N}),$$

therefore $Q(\varepsilon) = (]-\varepsilon/2, \varepsilon/2[\times]0,\varepsilon[) \cap X_1 \subset E$, i.e. the merotopy N_1 with the base $\{\mathsf{c}_1(\varepsilon) \cup \{Q(\varepsilon)\} : 0 < \varepsilon < 1\}$ induces the same contiguity as M_1 ; one can, however, easily see that N_1 is the Euclidean merotopy on X_1. $\mathsf{c}_1(\varepsilon)$ is c_1-open, thus M_1 is Lodato. Analogously, M_0 is compatible and Lodato, too. The merotopies are evidently accordant; (iii) holds bacause int $\mathsf{c}_i(\varepsilon)^0|X_{1-i} = \{X_{1-i}\}$.

To check that $\mathsf{c} = \text{int}\, \mathsf{c}_i(\varepsilon)^0$ is a Γ-cover, take a near collection n; we need a $C \in \mathsf{c} \cap \sec \mathsf{n}$. Pick a $z \in \bigcap_{N\in\mathsf{n}} \mathsf{c}^*(N)$. If $z = (0,0)$ then $C = \text{int}\, D_i(\varepsilon', \varepsilon'')^0$ will do with suitable ε' and ε'' ; the other cases are trivial. $\mathsf{c}_0(\varepsilon)^0\ (\cap)\ \mathsf{c}_1(\varepsilon)^0$ is a Γ-cover by Lemma and Theorem 9.4. The sets $C_1(k,n)$ and $-C_1(n,k)$ guarantee that int $\mathsf{c}_0(\varepsilon)^0\ (\cap)$ int $\mathsf{c}_1(\varepsilon)^0$ is a δ-cover. Thus all the three weaker versions of (ii) are fulfilled.

Nevertheless, (ii) fails for $\mathsf{c}_0(1)$ and $\mathsf{c}_1(1)$: take $\mathsf{n} = \{N_1, N_2, N_3\}$,

$$N_1 = \{(-1/n, 0) : n \in \mathbb{N}\}, N_2 = \{(1/2n, 0) : n \in \mathbb{N}\},$$

$$N_3 = \{(1/(2n+1), 0) : n \in \mathbb{N}\}.$$

b) (A modification of Example 5.19.) Let X, X_0, X_1 and M_0 be as in Example 3.8, but replace $\mathsf{c}_1(\varepsilon)$ in the definition of M_1 by

$$\mathsf{d}_1(\varepsilon) = \mathsf{c}_1(\varepsilon) \cup \{(H \times]0, \varepsilon[) \cap X_1 : H \subset]0, \varepsilon[\text{ is finite } \}.$$

(In 5.19, we did the same with $|H| = 2$.) $\{\mathsf{M}_0, \mathsf{M}_1\}$ is a family of Lodato merotopies in the Euclidean contiguity space on X; the modification was needed to make M_1 compatible. (ii) holds, but (iii) fails, just like in 5.19. (Use Lemma 9.3 instead of Lemma 5.2.) $\diamondsuit$

9.8 Corollary. *A single Lodato merotopy* M_0 *in a Lodato contiguity space has a Lodato extension iff* $\operatorname{int} \mathsf{c}_0^0$ *is a* Γ*-cover whenever* $\mathsf{c}_0 \in \mathsf{M}_0$. $\diamondsuit$

It is not enough to assume that the trace filters are Cauchy, not even when X_0 is open (take Example 5.18 b) with the Euclidean contiguity on X). In fact, the condition that $\operatorname{int} \mathsf{c}_0^0$ is a $\delta(\Gamma)$-cover (i.e. that there is a Lodato extension in $(X, \delta(\Gamma))$) is not sufficient either:

Example. With X, X_1, M_1 from Example 5.19 a), and the Euclidean contiguity Γ on X, $\operatorname{int} \mathsf{d}_1(\varepsilon)^0$ is a $\delta(\Gamma)$-cover ($\delta(\Gamma)$ is the same as δ in 5.19 a)), but it is not a Γ-cover (let n consist of three disjoint infinite subsets of X_1^r). $\diamondsuit$

9.9 Theorem. *A family of Lodato merotopies given on disjoint closed subsets in a Lodato contiguity space always has Lodato extensions;* $\mathsf{M}^0 = \mathsf{M}_L^0$ *is the coarsest one.*

Proof. M^0 is an extension by Theorem 9.4. For any c_i-open c_i, $\operatorname{int} \mathsf{c}_i^0 = \mathsf{c}_i^0$, thus $\mathsf{M}^0 = \mathsf{M}_L^0$; M_L^0 is always Lodato. $\diamondsuit$

Example 9.5 shows that the statement of this theorem is false for intersecting sets, even for open-closed ones. M^0 and M_L^0 can be different in general; e.g. with $\{\mathsf{M}_1\}$ from Example 9.7 b), $\operatorname{int} \mathsf{c}_1(1)^0 \notin \mathsf{M}^0$.

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Mathematica Pannonica
3/1 (1992), 77 – 81

ON UNITAL EXTENSIONS OF NEAR-RINGS AND THEIR RADICALS

Stefan **Veldsman**

Department of Mathematics, University of Port Elizabeth, PO Box 1600, Port Elizabeth 6000, South Africa.

Received November 1990

AMS Subject Classification: 16 Y 30, 16 N 80

Keywords: near-ring, unital extension, radical.

Abstract: Not every near-ring can be embedded as an ideal in a near-ring with an identity. A necessary and sufficient condition on a near-ring N for such an extension $\overline{N}$ to exist is known. The construction of $\overline{N}$ is not canonical in the sense that the quotient $\overline{N}/N$ is not fixed for a given N. We modify this extension to one (resembling the Dorroh extension of rings) for which the quotient is always fixed. For radicals with hereditary semisimple classes, the radical of N and the radical of this extension coincide if and only if the ring of integers has zero radical.

1. Introduction

Not every near-ring has a unital extension. Betsch [1] gave an example of such a near-ring on a non-commutative group and asks whether such near-rings on commutative groups exist. We provides such examples in section 1 below. Subsequently Betsch gives a necessary and sufficient condition on a near-ring N to have a unital extension $\overline{N}$. He also gives an explicit description of this near-ring $\overline{N}$. In section 1 we provide an alternative construction of the near-ring $\overline{N}$. This construction, which generalizes the well-known Dorroh extension of a ring, has the advantage that it makes it easy to compare the radicals of N and its unital extension (section 2).

1. Unital extensions of near-rings

All near-rings considered are 0-symmetric and right distributive near-rings.

Example 1.1. *There exists near-rings with commutative underlying groups which are never left ideals nor right ideals in a near-ring with an identity:*

Let G be any group which contains an element $e \neq 0$ with order not 2. Let N be the near-ring on G with multiplication defined by:

$$nm = \begin{cases} n & \text{if } m \neq 0 \\ 0 & \text{if } m = 0. \end{cases}$$

Let $\overline{N}$ be a near-ring with an identity 1 such that $N \subseteq \overline{N}$. If N is a right ideal in $\overline{N}$, then $e(e+1) \in N$. Thus $e(e+1) = (e(e+1))(-e) = e(e-e) = 0$. If N is a left ideal in $\overline{N}$, then $e(e+1) - e = e(e+1) - e1 \in N$. Thus $e(e+1) - e = (e(e+1) - e)(-e) = e(e-e) - e = -e$ and whence $e(e+1) = 0$. Hence, if N is either a left or a right ideal of $\overline{N}$, then $e(e+1) = 0$. Consequently, since $e + e \neq 0$, we have $0 = 0e = (e(e+1))e = e(e+e) = e$. But this contradicts the choice of $e \neq 0$. ◇

In [1], Betsch has given a necessary and sufficient condition on a near-ring to have a unital extension. This condition on a near-ring N is:

(BC) There exists a faithful N-group Γ (hence N is considered as a subnear-ring of $M_0(\Gamma)$) such that:

- (i) The mapping $x \longrightarrow -1+x+1$ of $M_0(\Gamma)$ into itself induces an automorphism of N (1 is the identity map on Γ).
- (ii) For all $n, m \in N$ and $a \in \mathbb{Z}$ ($\mathbb{Z}$ the integers), $n(m+a1) \in N$ (the cyclic subgroup of $M_0(\Gamma)$ generated by 1 is considered as an $\mathbb{Z}$ -module).

The near-ring $\overline{N}$ is a subnear-ring of $M_0(\Gamma)$ and is given by $\overline{N} = \{n + a1 | n \in N, a \in \mathbb{Z}\}$. This near-ring $\overline{N}$ is not canonical in the sense that for a near-ring N satisfying the condition (BC), $\overline{N}/N$ need not be fixed. It can be verified that $\overline{N}/N$ is always either one of the rings $\mathbb{Z}$

(integers) or $\mathbb{Z}_a$ (integers mod a) for some $a \geq 1$. When comparing the radicals of N and $\overline{N}$, it is useful to know the radical of $\overline{N}/N$. Since this quotient is not fixed, it is not always straightforward to compare the respective radicals. In order to fix the quotient, we propose a slightly modified construction, denoted by $D(N)$, such that for any near-ring N satisfying the condition (BC), $D(N)/N \cong \mathbb{Z}$. Furthermore, if N is a ring, the faithful N-group Γ can be chosen such that $D(N)$ is the usual unital extension of N (i.e. the Dorroh extension of N, cf [3]). Although this may not be the most ecconomical embedding, this construction enables us to give an easy criterion for comparing the radicals of N and $D(N)$ (Theorem 2.1 below).

Theorem 1.2. *Let N be near-ring which satisfies the condition* (BC). *Then there exists a unital extension $D(N)$ of N such that $D(N)/N \cong$ $\cong \mathbb{Z}(\mathbb{Z}$ is the ring of integers.)*

Proof. Let Γ be the faithful N-group provided by our assumption BC on N (hence $N \hookrightarrow M_0(\Gamma)$). On the cartesian product $N \times \mathbb{Z}$ define addition and multiplication by:

$$(n, a) + (m, b) = (n + a1 + m - a1, a + b)$$

$$(n, a)(m, b) = ((n + a1)(m + b1) - (ab)1, ab)$$

At the outset, we must verify that these operations are well defined. Since $n \longrightarrow -1+n+1$ is an automorphism of N (1 is the identity map on Γ), it follows that $a1 + m - a1 \in N$ for all $a \in \mathbb{Z}, m \in N$. Furthermore, $(n + a1)(m + b1) - (ab)1 = n(m + b1) + a1(m + b1) - (ab)1$. The first term is in N from the second part of the condition (BC); hence we only concern ourselves with the last two terms.

Suppose $a > 0$ (a similar argument takes care of the case $a < 0$). Then

$$a1(m + b1) - (ab)1 = (m + b1) + \ldots + (m + b1) - (ab)1 = m + (b1+$$
$$+m - b1) + (2b1 + m - 2b1) + \ldots + ((ab)1 + m - (ab)1) + (ab)1 - (ab)1$$

which is in N.

It can be verified that $+$ defines a group structure on $N \times \mathbb{Z}$ with additive identity (0,0) and the additive inverse of (n, a) given by $(-a1 - n + a1, -a)$. Furthermore, the multiplication is associative and distributive over the addition, hence we have a near-ring which we denote by $D(N)$. Clearly $N \cong \{(n, 0) | n \in N\} \triangleleft D(N), D(N)/N \cong \mathbb{Z}$ and (0,1) is the multiplicative identity of $D(N)$. $\diamondsuit$

If R is a ring, then R satisfies condition (BC) with $\Gamma = D(R)^+$, where $D(R)$ here denotes the usual Dorroh extension of the ring R. In this case, the addition in the above construction simplifies to $(n,a) + +(m,b) = (n+a1+m-a1, a+b) = (n+m, a+b)$ and the multiplication becomes $(n,a)(m,b) = ((n+a1)(n+b1)-ab1, ab)+(nm+bn+am, ab)$. Hence the above construction coincides with Dorroh extension of the ring R for this choice of Γ.

A sufficient "internal" condition on a near-ring N which implies the condition (BC) is given by:

Proposition 1.3. *Let N be a near-ring which contains a left ideal L with $(L:N)_N = 0$ such that:*

1. *For any $N \in N, a \in \mathbb{Z}$, there exists an $p \in N$ such that $-ak+nk+ak-pk \in L$ for all $k \in N$.*
2. *For any $n,m \in N, a \in \mathbb{Z}$, there exists an $p \in N$ such that $n(mk+ak)-pk \in L$ for all $k \in N$.*

Then N satisfies condition (BC).

Proof. Since L is a left ideal of N with $(L:N)_N = 0, \Gamma := N/L$ is a faithfull N-group via $n(x+L) = nx+L$. Embed N in $M_0(\Gamma)$ by $\varphi : N \longrightarrow M_0(\Gamma)$ defined by $\varphi(n) = \varphi_n : \Gamma \longrightarrow \Gamma, \varphi_n(x+L) = nx+L$. Let $f : M_o(\Gamma) \longrightarrow M_0(\Gamma)$ be the function defined by $f(x) = -1+x+1$. By condition 1 above, f induces an automorphism of $N \cong \varphi(N)$. Moreover, condition 2 above yields the requirement (ii) of (BC). $\diamondsuit$

The converse of the above proposition is not true: Consider any non-zero ring R with $R^2 = 0$.

2. The radical of the unital extension D(R).

Radical classes will be in the sense of Kurosh and Amitsur, cf [4] or Wiegandt [5]. The *semisimple class* of a radical $\mathcal{R}$ is the class $S\mathcal{R} = \{N|\mathcal{R}(N) = 0\}$. $S\mathcal{R}$ is *hereditary* if $I \triangleleft N \in S\mathcal{R}$ implies $I \in S\mathcal{R}$. As is well known, $S\mathcal{R}$ is hereditary if and only if $\mathcal{R}(I) \subseteq \mathcal{R}(N)$ for all near-rings N and $I \triangleleft N$. The variety of 0-symmetric near-rings contains many examples of radicals with hereditary semisimple classes, for example, J_2, J_3 and $\mathcal{G}$ (the Brown-McCoy radical class). Many more examples can be found in [4]. Some useful properties of a radical class $\mathcal{R}$ required here are:

(1) $\mathcal{R}(N/I) = 0$ implies $\mathcal{R}(N) \subseteq I$ for $I \triangleleft N$;

(2) $\mathcal{R}(\mathcal{R}(N)) = \mathcal{R}(N)$ for all N;
(3) $\mathcal{R}(N/\mathcal{R}(N)) = 0$ for all N.

Our final result generalizes the corresponding result from the variety of rings (cf De la Rosa and Heyman [2]), albeit with some restrictions. This is necessitated by the fact that, contrary to the case for rings, not every semisimple class of near-rings is necessarily hereditary and not every near-ring has a unital extension.

Theorem 2.1. *Let $\mathcal{R}$ be a radical class with a hereditary semisimple class. Then $\mathcal{R}(N) = \mathcal{R}(D(N))$ for all near-rings N which satisfy the condition* (BC) *if and only if $\mathcal{R}(\mathbb{Z}) = 0$.*

Proof. If $\mathcal{R}(\mathbb{Z}) = 0$ and $D(N)$ exists for the near-ring N, then $\mathcal{R}(D(N)/N) = \mathcal{R}(\mathbb{Z}) = 0$; hence $\mathcal{R}(D(N)) \subseteq \mathcal{R}(N)$. But $\mathcal{SR}$ hereditary implies $\mathcal{R}(N) \subseteq \mathcal{R}(D(N))$ which yields $\mathcal{R}(D(N)) = \mathcal{R}(N)$. Conversely, suppose $\mathcal{R}(D(N)) = \mathcal{R}(N)$ for all near-rings N which satisfy the condition (BC). In particular, since $\mathbb{Z}$ is a ring, so is $A := \mathcal{R}(\mathbb{Z})$ and $\mathcal{R}(D(A)) = \mathcal{R}(A) = \mathcal{R}(\mathcal{R}(\mathbb{Z})) = \mathcal{R}(\mathbb{Z}) = A$. Since $\mathbb{Z} \cong D(A)/A = D(A)/\mathcal{R}(D(A))$, we have $\mathcal{R}(\mathbb{Z}) = \mathcal{R}(D(A)/\mathcal{R}(D(A))) = 0$. ◇

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Mathematica Pannonica
3/1 (1992), 83 – 89

A CONTINUOUS AND A DISCRETE VARIANT OF WIRTINGER'S INEQUALITY

Horst **Alzer**

5220 Waldbröl, Morsbacher Str. 10, Germany.

Dedicated to Professor Dr. Dr. h. c. mult. Edmund Hlawka on occasion of his 75th birthday

Received March 1991

AMS Subject Classification: 26 D 15

Keywords: Wirtinger's inequality, inequalities for integrals and sums.

Abstract. We prove: If f is a real-valued continuously differentiable function with period 2π and $\int_0^{2\pi} f(x)dx = 0$, then

$$\frac{6}{\pi} \max_{0\le x\le 2\pi} f(x)^2 \le \int_0^{2\pi} f'(x)^2\, dx,$$

and, if $z_1, \ldots, z_n (n \ge 2)$ are complex numbers with $\sum_{k=1}^{n} z_k = 0$, then

$$\frac{12n}{n^2-1} \max_{1\le k\le n} |z_k|^2 \le \sum_{k=1}^{n} |z_{k+1} - z_k|^2,$$

where $z_{n+1} = z_1$. The constants $6/\pi$ and $12n/(n^2-1)$ are best possible.

1. Introduction

In 1916 a remarkable result of W. Wirtinger, which compares the integral of a square of a function with that of the square of its first derivative, was published in W.Blaschke's book "Kreis und Kugel" [2, p. 105]:

Theorem A. *Let f be a real-valued function with period 2π and $\int_0^{2\pi} f(x)dx = 0$. If $f' \in L^2$, then*

$$\int_0^{2\pi} f(x)^2 dx \leq \int_0^{2\pi} f'(x)^2 dx \tag{1.1}$$

with equality holding if and only if

$$f(x) = A\cos(x) + B\sin(x) \qquad (A, B \in \mathbb{R}).$$

The following discrete analogue of Wirtinger's inequality was proved for the first time in 1950 by I.J. Schoenberg [11].

Theorem B. *If $z_1, \ldots, z_n (n \geq 2)$ are complex numbers with $\sum_{k=1}^n z_k = 0$, then*

$$4\sin^2 \frac{\pi}{n} \sum_{k=1}^n |z_k|^2 \leq \sum_{k=1}^n |z_{k+1} - z_k|^2, \tag{1.2}$$

where $z_{n+1} = z_1$. Equality holds in (1.2) if and only if $z_k = A\cos\frac{2\pi k}{n} + B\sin\frac{2\pi k}{n}$, $\quad (k = 1, \ldots, n;\ A, B \in \mathbb{C})$.

Theorem A and Theorem B have evoked the attention of many mathematicians and in the past years different proofs, intriguing extensions and refinements as well as many related results were discovered [1 – 13]; see in particular [1], [8, pp. 141 – 154] and the references therein.

The aim of this paper is to present variants of inequalities (1.1) and (1.2). More precisely we shall answer the questions: What is the best possible constant α such that

$$\alpha \max_{0 \leq x \leq 2\pi} f(x)^2 \leq \int_0^{2\pi} f'(x)^2 dx$$

holds for all real-valued functions $f \in C^1$ fulfilling the conditions of Theorem A; and what is the best possible constant β_n such that

$$\beta_n \max_{1 \leq k \leq n} |z_k|^2 \leq \sum_{k=1}^n |z_{k+1} - z_k|^2$$

is valid for all complex numbers $z_1, \ldots, z_n$ satisfying the assumptions of Theorem B? Furthermore in both inequalities we determine all cases of equality.

2. The continuous case

In this section we establish a counterpart of Wirtinger's inequality (1.1).

Theorem 1. *If f is a real-valued continuously differentiable function with period 2π and $\int_0^{2\pi} f(x)dx = 0$, then*

$$\frac{6}{\pi} \max_{0 \le x \le 2\pi} f(x)^2 \le \int_0^{2\pi} f'(x)^2 dx. \tag{2.1}$$

Equality holds in (2.1) if and only if

$$f(x) = c\,[\,3(\frac{x-\pi}{\pi})^2 - 1\,] \quad (0 \le x \le 2\pi)$$

where c is a real constant.

Proof. We may assume

$$\max_{0 \le x \le 2\pi} f(x)^2 = f(x_0)^2 > 0, \quad 0 \le x_0 < 2\pi.$$

Then we have the following integral identity:

$$\int_{x_0}^{x_0+2\pi} \Big[\frac{f'(x)}{f(x_0)} - \frac{3}{\pi^2}(x - x_0 - \pi)\Big]^2 dx = \tag{2.2}$$

$$= \int_{x_0}^{x_0+2\pi} \Big[\frac{f'(x)}{f(x_0)}\Big]^2 dx - \frac{6}{\pi^2 f(x_0)} \int_{x_0}^{x_0+2\pi} f'(x)\ (x - x_0 - \pi) dx +$$

$$+ \frac{9}{\pi^4} \int_{x_0}^{x_0+2\pi} (x - x_0 - \pi)^2 dx = \frac{1}{f(x_0)^2} \int_{x_0}^{x_0+2\pi} f'(x)^2 dx - \frac{6}{\pi},$$

where the third integral of (2.2) has been calculated by integration by parts and by using the assumptions $f(x_0) = f(x_0 + 2\pi)$ and $\int_{x_0}^{x_0+2\pi} f(x)dx = 0$.

Hence we obtain

$$\int_0^{2\pi} f'(x)^2 dx = \int_{x_0}^{x_0+2\pi} f'(x)^2 dx \ge \frac{6}{\pi} \max_{0 \le x \le 2\pi} f(x)^2.$$

We discuss the cases of equality. Let $f(x) = c\,[3(\frac{x-\pi}{\pi})^2 - 1]$ $(0 \le$ $\le x \le 2\pi; c \in \mathbb{R})$. Simple calculations reveal that f^2 attains its maximum at 0 which implies

$$\int_0^{2\pi} f'(x)^2 dx = \frac{24c^2}{\pi} = \frac{6}{\pi} \max_{0\le x\le 2\pi} f(x)^2.$$

If equality holds in (2.1) then we obtain from the identity above:

$$f'(x) = \frac{3f(x_0)}{\pi^2}(x - x_0 - \pi) \quad (x_0 \le x \le x_0 + 2\pi)$$

which leads to

$$f(x) = \frac{3f(x_0)}{2\pi^2}(x - x_0 - \pi)^2 + c' \quad (c' \in \mathbb{R}).$$

Setting $x = x_0$ we get $c' = -\frac{1}{2}f(x_0)$; thus we have

$$f(x) = \frac{1}{2}f(x_0)[\frac{3}{\pi^2}(x - x_0 - \pi)^2 - 1] \qquad (x_0 \le x \le x_0 + 2\pi)$$

or

$$f(x) = \begin{cases} \frac{1}{2}f(x_0)[3(\frac{x-x_0+\pi}{\pi})^2 - 1], & 0 \le x \le x_0 \\ \frac{1}{2}f(x_0)[3(\frac{x-x_0-\pi}{\pi})^2 - 1], & x_0 \le x \le 2\pi. \end{cases}$$

Since f is differentiable at $x_0 \in [0, 2\pi)$ we conclude $x_0 = 0$; this yields

$$f(x) = \frac{1}{2}f(0)\,[3(\frac{x-\pi}{\pi})^2 - 1] \qquad (0 \le x \le 2\pi). \diamondsuit$$

3. The discrete case

Now we provide a variant of Schoenberg's inequality (1.2), respectively a discrete analogue of (2.1).

Theorem 2. *If* $z_1, \dots, z_n$ $(n \ge 2)$ *are complex numbers with* $\sum_{k=1}^n z_k = 0$, *then*

$$\frac{12n}{n^2-1} \max_{1\le k\le n} |z_k|^2 \le \sum_{k=1}^{n} |z_{k+1} - z_k|^2, \tag{3.1}$$

where $z_{n+1} = z_1$. *Equality holds in* (3.1) *if and only if*

$$z_k = \begin{cases} c\left[1 + \frac{6(k-r)(k+n-r)}{n^2-1}\right], & 1 \le k \le r-1, \\ c\left[1 + \frac{6(k-r)(k-n-r)}{n^2-1}\right], & r \le k \le n, \end{cases}$$

where $r \in \{1, \ldots, n\}$ and c is a complex constant.

Proof. Let $\max_{1 \le k \le n} |z_k| = |z_r| > 0$. Using the assumptions $z_{n+1} = z_1$ and $\sum_{k=1}^{n} z_k = 0$ we obtain after several elementary (but tedious) calculations the following identity:

$$\sum_{k=1}^{r-1} \left| \frac{z_{k+1} - z_k}{n z_r} - \frac{12(k+n-r) - 6(n-1)}{n(n^2-1)} \right|^2 + \tag{3.2}$$

$$+ \sum_{k=r}^{n} \left| \frac{z_{k+1} - z_k}{n z_r} - \frac{12(k-r) - 6(n-1)}{n(n^2-1)} \right|^2 =$$

$$= \sum_{k=1}^{n} \left| \frac{z_{k+1} - z_k}{n z_r} \right|^2 + \frac{36}{[n(n^2-1)]^2} \left\{ \sum_{k=1}^{r-1} (2k+n-2r+1)^2 + \right.$$

$$\left. + \sum_{k=r}^{n} (2k-n-2r+1)^2 \right\} - \frac{12}{n^2(n^2-1)} \mathrm{Re} \left\{ \frac{1}{z_r} \sum_{k=1}^{r-1} (z_{k+1} - \right.$$

$$\left. - z_k)(2k+n-2r+1) + \frac{1}{z_r} \sum_{k=r}^{n} (z_{k+1} - z_k)(2k-n-2r+1) \right\} =$$

$$= \frac{1}{n^2 |z_r|^2} \sum_{k=1}^{n} |z_{k+1} - z_k|^2 - \frac{12}{n(n^2-1)}$$

which implies

$$\sum_{k=1}^{n} |z_{k+1} - z_k|^2 \ge \frac{12n}{n^2-1} \max_{1 \le k \le n} |z_k|^2.$$

It remains to discuss the cases of equality. Let $r \in \{1, \ldots, n\}, c \in \mathbb{C}$ and let

$$z_k = \begin{cases} c\left[1 + \frac{6(k-r)(k+n-r)}{n^2-1}\right], & 1 \le k \le r-1, \\ c\left[1 + \frac{6(k-r)(k-n-r)}{n^2-1}\right], & r \le k \le n. \end{cases}$$

Then we have

$$\max_{1 \le k \le n} |z_k| = |z_r| = |c|$$

which leads to

$$\sum_{k=1}^{n} |z_{k+1} - z_k|^2 = \frac{12n}{n^2-1} |c|^2 = \frac{12n}{n^2-1} \max_{1 \le k \le n} |z_k|^2.$$

Now we assume that equality holds in (3.1). Then we conclude from (3.2):

$$\frac{z_{k+1} - z_k}{n z_r} = \begin{cases} \frac{12(k+n-r)-6(n-1)}{n(n^2-1)}, & 1 \le k \le r-1, \\ \frac{12(k-r)-6(n-1)}{n(n^2-1)}, & r \le k \le n. \end{cases} \tag{3.3}$$

Let $1 \le k \le r$; because of $z_{n+1} = z_1$ we obtain from (3.3):

$$z_k - z_r = \sum_{j=r}^{n}(z_{j+1} - z_j) + \sum_{j=1}^{k-1}(z_{j+1} - z_j) = \frac{6(k-r)(k+n-r)}{n^2-1} z_r;$$

and if $r \le k \le n$, then (3.3) yields

$$z_k - z_r = \sum_{j=r}^{k-1}(z_{j+1} - z_j) = \frac{6(k-r)(k-n-r)}{n^2-1} z_r.$$

This completes the proof of Theorem 2. ◇

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Mathematica Pannonica
3/1 (1992), 91 – 115

POWERS AND GENERALIZED CARDINAL NUMBERS FOR HCH-OBJECTS - BASIC NOTIONS

Maciej **Wygralak**

Institute of Mathematics, Adam Mickiewicz University, 60-769 Poznań, Matejki 48/49, Poland.

Received August 1990

AMS Subject Classification: 03 E 70, 94 A 17, 03 B 50, 94 D 05

Keywords: Vaguely defined objects, Lukasiewicz logic, generalized cardinals.

Abstract: In this paper we present an introduction to a theory of powers and (generalized) cardinal numbers which is based on the infinite-valued Lukasiewicz logic and refers to so-called HCH-objects, i.e. to objects which in general cannot be mathematically modelled using the notion of a set. We focus here our attention on the notion of equipotency for HCH-objects and the construction of generalized cardinals and their basic properties. Problems related to order and operations on the generalized cardinals will be discussed in [24,25].

1. Introduction and notations

The purpose of this paper is to present mathematical base of a theory of powers and generalized cardinal numbers for hardly characterizable objects, shortly HCH-objects. By HCH-objects we mean here parts of some infinite universal set $\mathcal{U}$ which maybe are vaguely defined and do not need to be sets themselves, i.e. which in general cannot be mathematically modelled, without essential distortions, using the classical notion of a (sub)set (cf. semisets [18]). However, we assume that each HCH-object can be described, at least in a subjective way, by means of a function $\mathcal{U} \to \mathcal{L}$ or using a pair of such the functions ($\mathcal{L}$ denotes a suitable lattice). These functions will be called generalized characteristic functions or membership functions. This way sets, fuzzy sets ([27]), intuitionistic fuzzy sets ([16]) and generally $\mathcal{L}$-fuzzy sets ([4]; cf. Heyting algebra valued sets in [9]), twofold fuzzy sets ([3]), rough sets ([13]), and partial sets ([10]) become special cases of HCH-objects. HCH-objects which are not sets will be called proper HCH-objects.

So, although the given definition of an HCH-object is rather informal, it is sufficiently good for our purposes because in a way it makes possible to bring together those more or less different notions what is very convenient for the presentation of the theory (see e.g. Section 8).

If $A : \mathcal{U} \to \mathcal{L}$, then obj($A$) denotes the HCH-object 'embedded' in $\mathcal{U}$ and described (characterized) by means of A. Since obj(A) is not necessarily a set we shall write $x \epsilon \text{obj}(A)$ instead of $x \in \text{obj}(A)$; obviously, obj(A) is a set if $A(x) \in \{0,1\}$ for each x from $\mathcal{U}$. Then $[x \epsilon \text{obj}(A)] := A(x)$, where $[s]$ denotes the truth value of a sentence s (obviously, $[s] \in \mathcal{L}$) and the symbol := stands always for 'equals by definition'. Each value $A(x)$ will be called membership grade of x in obj(A). Moreover, we accept the following definitions:

$$[\neg s] := [s] \to 0,$$
$$[r \& s] := [r] \wedge [s],$$
$$[r \mid s] := [r] \vee [s],$$
$$[r \Rightarrow s] := [r] \to [s],$$
$$[r \Leftrightarrow s] := [r \Rightarrow s \;\&\; s \Rightarrow r],$$
$$[\forall\!\!\!/ x \in \mathcal{U} : s(x)] := \bigwedge_{a \in \mathcal{U}} [s(x/a)],$$
$$[\exists\!\!\mid x \in \mathcal{U} : s(x)] := \bigvee_{a \in \mathcal{U}} [s(x/a)],$$

where

(a) $\neg$, &, |, $\Rightarrow$, $\Leftrightarrow$ are logical symbols of negation, conjunction, disjunction, implication, and equivalence, respectively;

(b) $\forall\!\!\!/$ and $\exists\!\!\mid$ denote general and existential many-valued quantifiers and $s(x/a)$ is the usual substitution notation (classical quantifiers will be denoted by $\forall$ and $\exists$);

(c) $\wedge$, $\bigwedge$ ($\vee$, $\bigvee$, resp.) denote the operation of the greatest lower

bound (least upper bound, resp.) for two arguments or their arbitrary number;
(d) $\to$ denotes many-valued implication operator; we additionally assume that it fulfills two properties: $b \to c = 1$ iff $b \leq c$,$1 \to b = b$ for each b, $c \in \mathcal{L}$.

Generalized inclusion obj $(A) \underset{\sim}{\subset}$ obj(B) and equality obj$(A) \approx$ $\approx$obj(B) of two HCH-objects are respectively defined by the conditions

$$\forall x \in \mathcal{U} : x \in \text{obj}(A) \Rightarrow x \in \text{obj}(B),$$

and

$$\text{obj}(A) \underset{\sim}{\subset} \text{obj}(B) \ \& \ \text{obj}(B) \underset{\sim}{\subset} \text{obj}(A).$$

Of course, the usual two-valued inclusion and equality one defines by

$$\text{obj}(A) \subset \text{obj}(B) \text{ iff } A \subset B \text{ with } A \subset B \text{ iff } \forall x \in \mathcal{U} : A(x) \leq B(x),$$

$$\text{obj}(A) = \text{obj}(B) \text{ iff } A = B \text{ with } A = B \text{ iff } \forall x \in \mathcal{U} : A(x) = B(x).$$

We at once see that

$$\text{obj}(A) \subset \text{obj}(B) \text{ iff } [\text{obj}(A) \underset{\sim}{\subset} \text{obj}(B)] = 1,$$
$$\text{obj}(A) = \text{obj}(B) \text{ iff } [\text{obj}(A) \approx \text{obj}(B)] = 1.$$

The conditions defining union and intersection of two HCH-objects are also quite natural, namely

$$\text{obj}(A) \cup \text{obj}(B) = \text{obj}(C) \text{ iff } C = A \cup B, \text{ where } (A \cup B)(x) :=$$
$$:= A(x) \vee B(x),$$
$$\text{obj}(A) \cap \text{obj}(B) = \text{obj}(D) \text{ iff } D = A \cap B, \text{ where } (A \cap B)(x) :=$$
$$:= A(x) \wedge B(x).$$

So, the sentence $x \in \text{obj}(A) \cup \text{obj}(B)$ ($x \in \text{obj}(A) \cap \text{obj}(B)$, resp.) has the same truth value as the sentence $x \in \text{obj}(A) \mid x \in \text{obj}(B)$ ($x \in \text{obj}(A)$ & & $x \in \text{obj}(B)$, resp. .

Nowadays (proper) HCH-objects play an important role in many branches of mathematics, computer and information sciences, social sciences, engineering, etc. It is quite clear that in many situations there is a necessity of having (as precise and adequate as possible) handy quantitative information about an HCH-object. So, it would be very useful to have for HCH-objects some counterpart of cardinal numbers. Such a reasonable counterpart will be constructed here and will be called generalized cardinal numbers (shortly gcn's).

In this paper we like to present a detailed discussion devoted to such basic questions as equipotency of HCH-objects, gcn's - their construction and elementary properties, the notion of finiteness for HCH-objects. Problems of comparing and ordering for gcn's will be presented in [24], operations on them are studied in [25]. We construct the theory for quite arbitrary HCH-objects and use here the infinite-valued Łukasiewicz logic; some results for HCH-objects with finite supports are already placed in [21,22]. So, we put $\mathcal{L} := \mathcal{J}$, where $\mathcal{J} := [0,1]$, whereas $\rightarrow$ is the Łukasiewicz implication operator, i.e. $b \rightarrow c := 1 \wedge 1 - b + c$; of course, $\wedge$, $\bigwedge$ and $\vee$, $\bigvee$ denote then usual operations of minimum, infimum, maximum, and supremum of numbers from the closed unit interval. It is however possible to construct an analogous intuitionistic theory of powers and gcn's for HCN-objects using triangular norms and φ-operators or applying intuitionistic logic with $\mathcal{L} :=$ complete Heyting algebra (see [26]).

As regards the notation and terminology, we decide to use troughout this paper the following additional rules:

(a) Sets are denoted by script capitals (e.g. $\mathcal{D}, \mathcal{J}, \mathcal{U}$) and some multi-letter symbols defined in the sequel of the paper; as usual $\emptyset$ denotes the empty set.

(b) Capitals in italic denote the membership functions. The functions E and U are defined as follows: $\forall x \in \mathcal{U} : E(x) = 0,\ U(x) = 1$.

(c) The letters i,j,.....,p,q denote both the finite and transfinite numbers.

(d) Small Greek letters with or without subscripts (e.g. α, $\beta_{f,g}$) will denote the generalized cardinal numbers related to HCH-objects.

(e) If $A : \mathcal{U} \rightarrow \mathcal{J}$, then $\text{supp}(\text{obj}(A)) := \text{supp}(A) := \{x \in \mathcal{U} : A(x) \neq \neq 0\}$; so, the so-called support of obj(A) and support of A are identically defined. Moreover $A_t := \{x \in \mathcal{U} : A(x) \geq t\}$for $t \in \mathcal{J}_o := (0,1]$; A_t will be called t-level set of A and obj(A).

(f) $PS(\mathcal{D}) := \{0,1\}^{\mathcal{D}}$, $GP(\mathcal{D}) := \mathcal{J}^{\mathcal{D}}$, $P_i(\mathcal{D}) := \{\mathcal{B} \subset \mathcal{D} : \text{card}\ \mathcal{B} = i\}$.

(g) $1_{\mathcal{D}}$ denotes the characteristic function of $\mathcal{D} \subset \mathcal{U}$, i.e. $1_{\mathcal{D}}(x) = 1$ if $x \in \mathcal{D}$ else $1_{\mathcal{D}}(x) = 0$. So, $E = 1_{\emptyset}$.

(h) CN denotes the set of all the cardinals i such that card $\mathcal{U} \geq i$, $\text{betw}(i,j) := \{k \in CN : i \leq k \leq j\}$ for $i,j \in CN$.

(i) i^+ denotes the succesor of i. Thus $i^+ = i + 1$ for finite i.

(j) For the simplicity of the presentation, in the examples placed in Section 8 we will accept the Continuum Hypothesis and use some special notation for elements $P \in GP(CN)$. Namely

$$P = (v_0, v_1, \ldots.., v_r, (v) \parallel w_1, w_2)$$

means that $P(i) = v_i$ for $i \in \text{betw}(0, r), P(i) = v$ for each finite $i > r$, $P(\aleph_0) = w_1$, $P(\diamondsuit) = w_2$, $P(i) = 0$ for $i > \diamondsuit$. For instance, if $P = (1, 1, 0.5, (0.3) \parallel 0.3, 0.1)$, then we have $P(0) = P(1) = 1$, $P(2) = 0.5$, $P(i) = 0.3$ for $i \in$ betw $(3, \aleph_0)$, $P(\diamondsuit) = 0.1$, and $P(i) = 0$ for $i > \diamondsuit$.

2. Towards generalized cardinal numbers

In the earlier many-valued theories of cardinality presented in [1], [5,6], [11] one assumes at the beginning that the notion of cardinality is unknown even for sets; gcn's are then constructed via many-valued bijections,i.e. via direct adaptation of the classical construction of cardinals. Unfortunately, such an approach is not successful and appears not very useful in practice because the obtained theories become essentially dependent on the chosen definitions of such the bijections and, on the other hand, respective calculations of powers are extremely difficult even in the case of small finite supports (see also [2], [7], [19,20] for a review of some other early approaches). In the theory proposed here we use quite different approximative approach in which we try to make a good use of the already existing ordinary cardinals and apply some axiomatic method that generates various types of gcn's. So, we have then the possibility to choose such a type which is most suitable in a concrete application inside or outside mathematics. Moreover, we assume that our information about any membership function can be imprecise or incomplete.

Let us consider the family composed of all the subsets of $\mathcal{U}$. We define classical cardinals in the ordinary way. So, for any $\mathcal{A} \subset \mathcal{U}$ we have card $\mathcal{A} = i$ iff $\exists \mathcal{B} \in P_i(U) : \mathcal{A} = \mathcal{B}$, i.e. the power of $\mathcal{A}$ equals i iff $\mathcal{A}$ belongs to respective family of equipotent sets. It is quite clear that for each fixed $\mathcal{A}$ the sentence $\exists \mathcal{B} \in P_i(\mathcal{U}) : \mathcal{A} = \mathcal{B}$ is true (in other words: has positive truth value) for exactly one cardinal number i. However, if we deal with HCH-objects, then in general the many-valued counterpart $\exists \mathcal{B} \in P_i(\mathcal{U})$: obj$(A) \approx$ obj$(1_{\mathcal{B}})$ attains positive truth values for different i's. So, one can say that obj(A) belongs "to a degree" to many families of equipotent sets. Thus the power of obj(A) cannot be represented by one cardinal number but should be expressed by means of an HCH-object 'embedded' in CN and having the membership grades identical with respective truth values of the above given many-valued sentence. Then it is quite natural to consider as equipotent such HCH-objects obj(A), obj(B) in $\mathcal{U}$ which are related to identical HCH-objects in CN. Since our information about A can be imprecise or incomplete we additionally assume that $f(A) \subset A \subset g(A)$, where f and g are some approximating functions (see Section 3). Therefore we finally use the condition

$$\exists \mathbb{B} \in P_i(\mathcal{U}): \quad \mathrm{obj}(f(A)) \underset{\sim}{\subset} \mathrm{obj}(1_{\mathbb{B}}) \quad \&$$
$$\& \ \exists \mathbb{C} \in P_i(\mathcal{U}): \quad \mathrm{obj}(1_{\mathbb{C}}) \underset{\sim}{\subset} \mathrm{obj}(g(A))$$

which in some cases can be rewritten in a simpler from (see Remark 6.5). In the main, in this paper we focus our attention on such properties of gcn's which are independent on the choice of (f, g) and on the power of supp(obj(A)). Simple proofs are given in outline. Although $\Rightarrow$, & and | are basically understood as many-valued connectives, in the sentences or conditions containing exclusively the classical (two-valued) quantifiers, relations or predicates they will be throughout interpreted as respective classical connectives.

3. Approximation of the membership functions

Let A denote a membership function characterizing some HCH-object in $\mathcal{U}$. As we mentioned in previous section, we suppose that in general A can be given imprecisely or incompletely. So, we approximate A by means of two other functions $f(A)$ and $g(A)$, i.e. we approximate obj(A) by obj$(f(A))$ and obj$(g(A))$, where $f, g : GP(\mathcal{U}) \to GP(\mathcal{U})$. However, we assume that either at least one of the functions f, g is a function to $PS(\mathcal{U}) \subset GP(\mathcal{U})$ (i.e. at least one of the HCH-objects obj$(f(A))$ and obj$(g(A))$ is in a way simpler than obj(A)) or $f = g =$id with id denoting the identity function (i.e. our information about A is assumed to be perfect). Moreover we accept the following additional axioms about f and g:

(A1) $\forall A \in GP(\mathcal{U}) : f(A) \subset A \subset g(A)$,
(A2) $\forall A, B \in GP(\mathcal{U}) \forall x, y \in \mathcal{U} : A(x) \leq B(y) \Rightarrow f(A)(x) \leq f(B)(y) \&$
$\quad \& \ g(A)(x) \leq g(B)(y)$,

(A3) $\forall A \in PS(\mathcal{U}) : f(A), g(A) \in PS(\mathcal{U})$.

The family of all the pairs (f, g) of approximating functions fulfilling these postulates, but excluding the trivial (E, U), will be denoted by $\mathcal{F}$. As regards some interpretation of the axioms, (A1) means that $f(A)$ and $g(A)$ are always the lower and upper approximations of A,(A2) says that both $f(A)(x)$ and $g(A)(x)$ depend only on $A(x)$. Finally, (A3) is also quite natural and states that if obj(A) is a set, then both obj($f(A)$) and obj($g(A)$) are sets too. As consequences of (A1)$-$(A3) we get some simple but useful properties which are listed in the following theorem and corollaries.

Theorem 3.1. *For each* $(f, g) \in \mathcal{F}$ *and each* $A, B \in GP(\mathcal{U})$ *we have*

(A2)$'$ $A(x) = B(y)$ *implies* $f(A)(x) = f(B)(y)$ *and* $g(A)(x) = g(B)(y)$;

(A4) $f(A \overset{\cup}{\cap} B) = f(A) \overset{\cup}{\cap} f(B)$, $g(A \overset{\cup}{\cap} B) = g(A) \overset{\cup}{\cap} g(B)$;

(A5) $A \subset B$ *implies* $f(A) \subset f(B)$ *and* $g(A) \subset g(B)$;

(A6) $A(x) = 0$ *implies* $f(A)(x) = 0$ *and* $g(A)(x) \in \{0, 1\}$,
$A(x) = 1$ *implies* $f(A)(x) \in \{0, 1\}$ *and* $g(A)(x) = 1$;

(A6)$'$ $f \equiv E$ *or* $(f(A)(x) = 1$ *iff* $A(x) = 1)$, $g \equiv U$ *or* $(g(A)(x) = 0$ *iff* $A(x) = 0)$;

(A7) *If* $A \in PS(\mathcal{U})$, *then* $f(A) = A$ *or* $f(A) = E$ *and* $g(A) = A$ *or* $g(A) = U$;

(A7)$'$ $f(E) = E$, $g(U) = U$, $f(U), g(E) \in \{E, U\}$.

Proof. We get (A2)$'$ using twice (A2). (A4) is a direct consequence of (A2), (A2)$'$ and the definition of $\cup$ and $\cap$. (A5) is implied again by (A2). (A6) follows from (A1), (A2)$'$, (A3) and implies (A6)$'$. Finally, (A7) follows from (A6), (A6)$'$ and implies (A7)$'$. This completes the proof. ◇

Corollary 3.2 *For each* $(f, g) \in \mathcal{F}$ *and* $A \in GP(\mathcal{U})$ *we have*

(a) *if* $f : GP(\mathcal{U}) \to PS(\mathcal{U})$, *then* $f \equiv E$ *or* $f(A) = 1_{A_1}$;

(b) *if* $g : GP(\mathcal{U}) \to PS(\mathcal{U})$, *then* $g \equiv U$ *or* $g(A) = 1_{\mathrm{supp}(A)}$.

Proof. Both (a) and (b) are immediate consequences of (A6)$'$. ◇

The corollary given above is very useful when one proves other theorems because it shows how look the possible pairs $(f, g) \in \mathcal{F}$.

Corollary 3.3. *For each* $(f, g) \in \mathcal{F}$ *and* $A \in GP(\mathcal{U})$ *we have*

(a) $f(A) \supset 1_{A_1}$ *or* $f \equiv E$.

(b) $g(A) \subset 1_{\mathrm{supp}(A)}$ *or* $g \equiv U$.

Proof. Again, it follows directly from (A6)$'$. ◇

4. Equipotent HCH-objects

Now we are ready to introduce the notion of equipotency for HCH-objects. Let $f(A)_t$, $g(A)_t$ denote the t-level sets of $f(A)$ and $g(A)$, respectively.

Definition 4.1. We write $A \sim_{f,g} B$ and say that two HCH-objects obj(A) and obj(B) in $\mathcal{U}$ are *equipotent* (in other words: are of the *same power*) with respect to a pair $(f, g) \in \mathcal{F}$ iff the conditions

$$\bigwedge\{t : \text{card } f(A)_t \leq i\} = \bigwedge\{t : \text{card } f(B)_t \leq i\},$$

$$\bigvee\{t : \text{card } g(A)_t \geq i\} = \bigvee\{t : \text{card } g(B)_t \geq i\}$$

are fulfilled by each cardinal number i.

If $(f, g) \in \mathcal{F}$ is fixed, one can write simply $A \sim B$. It is quite obvious but very important that $\sim_{f,g}$ is an equivalence relation for each $(f, g) \in \in \mathcal{F}$. Using very puristic notation we should rather write obj$(A) \sim_{f,g}$ obj(B) but the form $A \sim_{f,g} B$ does not lead to misunderstanding and is more convenient in use. Also, it is justified by the fact that operations or relations for HCH-objects are often defined by means of respective operations or relations over membership functions.

As concerns the condition defining the equipotency of HCH-objects, we at once see that it is a weakened form of the following (both versions are equivalent for HCH-objects with finite supports):

$$\forall t \in (0, 1) : \text{ card } f(A)_t = \text{ card } f(B)_t \& \text{ card } g(A)_t = \text{ card } g(B)_t.$$

But any definition describing the equipotency via equalities of powers of some t-level sets is dangerous since it makes the equipotency to much dependent on a finite number of membership values even if we deal with HCH-objects with infinite supports. So, we refuse it. By using infima, suprema and inequalities, the proposed definition reflects instead two facts: first that $f(A)$ and $g(A)$ are lower and upper approximations of A, and second that using an approximative approach we should accept as equipotent not only such HCH-objects whose respective t-level sets are equipotent but also such ones which for each $t \in \mathcal{J}_o$ have 'the same amount' of elements with membership values equal to t or lying as near to t as one likes.

5. Relativity of the equipotency for HCH-objects

Let $f_i(A) := \bigvee\{t : \text{card } f(A)_t \geq i\}$, $g_i(A) := \bigvee\{t : \text{card } g(A)_t \geq \geq i\}$ and $a_i := \bigvee\{t : \text{ card } A_t \geq i\}$ for $A \in GP(\mathcal{U})$ and $(f,g) \in \mathcal{F}$ (in the same way one defines for instance numbers r_i for some $R \in GP(\mathcal{U})$). One can easily check that $f_i(A)$ is nonincreasing with respect to i and $f_i(A) = 0$ for $i >$ card supp(A); of course, analogous properties are satisfied by $g_i(A)$ and a_i. Also, one can easily prove that for each $A, B \in GP(\mathcal{U})$, $(f,g) \in \mathcal{F}$ and any cardinal number i we have

$$A \subset B \Rightarrow f_i(A) \leq f_i(B) \,\&\, g_i(A) \leq g_i(B)$$

and

$$f_i(A) \leq a_i \leq g_i(A).$$

So, $A \subset B$ implies $a_i \leq b_i$ for each i. Moreover the following properties will be useful: $f_i(A) = 1$ for $i \leq$ card $f(A)_1$, $f_1(A) = \bigvee\{f(A)(x) : x \in \mathcal{U}\}$, and $g_o(A) = 1$.

5.1. Useful characterizations of the equipotency

We notice that the equipotency condition can be rewritten as

$A \sim_{f,g} B$ iff $g_i(A) = g_i(B)$ & $f_{i^+}(A) = f_{i^+}(B)$ for each $i \in CN$.
So,

$$A \sim_{f,g} B \quad \text{iff} \quad g(A) \sim_{E,id} g(B) \;\;\&\;\; f(A) \sim_{id,U} f(B)\ .$$

Hence

$$A \sim_{id,id} B \quad \text{iff} \quad A \sim_{E,\text{id}} B \;\&\; A \sim_{id,U} B;$$

we even have

$$A \sim_{id,id} B \;\text{ iff }\; A \sim_{E,id} B \;\text{ iff }\; A \sim_{id,U} B$$

i.e. if (f,g) equals (id,id), (E,id) or (id,U), then $A \sim_{f,g} B$ iff $a_i = b_i$ for each $i \in CN$. Moreover, the following implications hold:

(a) if $f(\cdot) = 1_{(\cdot)_1}$ and $g =$ id, then $A \sim_{f,g} B$ implies card $A_1 =$ card B_1;

(b) if $f =$ id and $g(\cdot) = 1_{\text{supp}(\cdot)}$, then $A \sim_{f,g} B$ implies card supp$(A) =$ card supp(B) ;

(c) if $f \equiv E$ and $g(\cdot) = 1_{\text{supp}(\cdot)}$, then $A \sim_{f,g} B$ iff card supp$(A) =$ = card supp(B);

(d) if $f(\cdot) = 1_{(\cdot)_1}$ and $g \equiv U$, then $A \sim_{f,g} B$ iff card $A_1 =$ card B_1 ;

(e) if $f(\cdot) = 1_{(\cdot)_1}$ and $g(\cdot) = 1_{\text{supp}(\cdot)}$, then $A \sim_{f,g} B$ iff card $A_1 =$ $=$ card B_1 & card supp$(A) =$ card supp(B) .

So, the spectrum of possible conditions characterizing the equipotency of HCH-objects is rather wide; of course, these and other characterizations can be enhanced for HCH-objects with finite supports. The most interesting postulate is however that $a_i = b_i$ for each $i \in CN$ since we like to have the equipotency independent on an order of elements in HCH-objects . Really, let us notice that if A and B have finite supports, then this condition means that functions A and B attain the same values (with regard to their repetitions) but maybe in different points. This follows from the observation that if supp(D) is finite, then d_i is the i-th value in the sequence of positive membership grades $D(x)$ (including their repetitions) ordered in a nonincreasing way with $d_o := 1$ and $d_i := 0$ for $i >$ card supp(D). Finally, let us notice that if $(f, g) = (E, U)$ were an element of $\mathcal{F}$, then all the HCH-objects in $\mathcal{U}$ are equipotent with respect to such (f, g).

It is possible that obj(A) and obj(B) are equipotent with respect to some $(f, g) \in \mathcal{F}$ but simultaneously they are not equipotent with respect to some other $(f^*, g^*) \neq (f, g)$ (one can easily give respective examples for instance for $(f, g) = (1_{(\cdot)_1}, 1_{\text{supp}(\cdot)})$ and $(f^*, g^*) = (\text{id}, \text{id})$). This fact is however not so surprising because we deal here with HCH-objects whose nature is vague. Using two different pairs of approximating functions we apply in essence two different criteria to evaluate the powers of those objects . This is analogous to the situation well-known in our common life when two persons compare two things which are vague in a way and they get different results.

5.2. Some criteria of choice for the approximating functions

Since the family $\mathcal{F}$ is rather rich and, on the other hand, the equipotency or nonequipotency of two HCH-objects depends in general case on the choice of $(f, g) \in \mathcal{F}$, it is essential to ask how to choose (f, g) in 'proper' way; there is no problem with $A, B \in PS(\mathcal{U})$ because then either $A \sim_{f,g} B$ for each (f, g) or the HCH-objects are nonequipotent with respect to each (f, g) from $\mathcal{F}$. Obviously, total instructions are not possible. However, we like to present some approaches starting from different motivations.

APPROACH 1. We choose (f,g) taking into account how looks the condition characterizing $\sim_{f,g}$.
APPROACH 2. If $(f,g) \neq (\mathrm{id},\mathrm{id})$, then $f(A)$ and $g(A)$ can be interpreted as components of a twofold fuzzy set (see Section 7). So, our choice of (f,g) from among pairs differing from (id,id) depends on that what elements in $\mathcal{U}$ are considered to be sure and possible elements of an HCH-object .
APPROACH 3. We choose $f = g = \mathrm{id}$ if A is known exactly. Otherwise we suppose that $F \subset A \subset G$ and that only F and G are given. The choice of (f,g) depends then on the form of F and G. For instance, if we know only all the points x such that $A(x) > 0$ and $A(x) = 1$, we choose $f(\cdot) = 1_{(\cdot)_1}$ and $g(\cdot) = 1_{\mathrm{supp}(\cdot)}$. If the lower approximation of A is given only, one can take $f = \mathrm{id}$, $g \equiv U$ (or $g(\cdot) = 1_{\mathrm{supp}\ (\cdot)}$ provided that we know all the points x such that $A(x) > 0$).
APPROACH 4. Some properties and the form of generalized cardinal numbers are dependent on the used pair (f,g). So, one can choose (f,g) so as to get such gcn's that have the most convenient form and properties from the viewpoint of a concrete application.

6. The operator GCN and its basic properties

Now we are going to define an operator which will be used in Section 7 to generate the generalized cardinal numbers. That is why it is denoted by **GCN**. More precisely, let

GCN: $GP(\mathcal{U}) \times GP(\mathcal{U}) \to GP(CN)$ and let $\mathbf{GCN}(F,G)(i)$ be equal to

$$[\exists \mathcal{Y} \in P_i(\mathcal{U}) : \mathrm{obj}(F) \subset \mathrm{obj}(1_{\mathcal{Y}})] \wedge [\exists \mathcal{Z} \in P_i(\mathcal{U}) : \mathrm{obj}(1_{\mathcal{Z}}) \underset{\sim}{\subset} \mathrm{obj}(G)]$$

provided that $F \subset G$. So, we have
Theorem 6.1. *For each $F, G \in GP(\mathcal{U})$ such that $F \subset G$ and each $i \in CN$*

$$\mathbf{GCN}(F,G)(i) = \bigvee_{\mathcal{Y} \in P_i(\mathcal{U})} \bigwedge_{x \in \mathcal{Y}} G(x) \ \wedge \bigvee_{\mathcal{Y} \in P_i(\mathcal{U})} \bigwedge_{x \notin \mathcal{Y}} 1 - F(x).$$

Proof. This equality is obvious because for each $i \in CN$ and $\mathcal{Y} \in P_i(\mathcal{U})$ we get

$$[\mathrm{obj}(F) \underset{\sim}{\subset} \mathrm{obj}(1_{\mathcal{Y}})] = \bigwedge_{x \in \mathcal{U}} F(x) \longrightarrow 1_{\mathcal{Y}}(x) = \bigwedge_{x \notin \mathcal{Y}} 1 - F(x)$$

and

$$[\mathrm{obj}(1_{\mathcal{Y}}) \underset{\sim}{\subset} \mathrm{obj}(G)] = \bigwedge_{x \in \mathcal{U}} 1_{\mathcal{Y}}(x) \longrightarrow G(x) = \bigwedge_{x \in \mathcal{Y}} G(x).$$

Remark 6.2. One can easily notice that using (instead of the Lukasiewicz implication operator) a φ-operator induced by a triangular norm or putting $\mathcal{L} :=$ complete Heyting algebra we obtain the formula

$$\mathbf{GCN}(F, G)(i) = \bigvee_{\mathcal{Y} \in P_i(\mathcal{U})} \bigwedge_{x \in \mathcal{Y}} G(x) \wedge \bigvee_{\mathcal{Y} \in P_i(\mathcal{U})} \bigwedge_{x \notin \mathcal{Y}} F(x) \longrightarrow 0.$$

We see that $i > \mathrm{card}(\mathcal{U})$ implies $P_i(\mathcal{U}) = \emptyset$ and then $\mathbf{GCN}(F, G)(i) = 0$. This is why we always restrict ourselves to cardinals belonging to CN. Moreover, it is quite clear that the following simplification is possible:

$$\mathbf{GCN}(F, G)(i) = \bigvee_{\mathcal{Y} \in P_i(\mathrm{supp}\ (G))} \bigwedge_{x \in \mathcal{Y}} G(x) \wedge \bigvee_{\{\mathcal{Y} \in P_i(\mathcal{U}) : F_1 \subset \mathcal{Y}\}} \bigwedge_{x \notin \mathcal{Y}} 1 - F(x).$$

Hence $\mathbf{GCN}(F, G)(i) = 0$ for each $i \notin \mathrm{betw}(\mathrm{card} F_1, \mathrm{card\ supp}\ (G))$. Finally if $i \geq \mathrm{card\ supp}\ (\mathcal{F})$, there exists $\mathcal{Y} \in P_i(\mathcal{U})$ such that $F_1 \subset \subset \mathrm{supp}\ (F) \subset \mathcal{Y}$. But then we get $\bigwedge\{1 - F(x) : x \notin \mathcal{Y}\} = 1$. So, for each $i \geq \mathrm{card\ supp}\ (F)$ we obtain

$$\mathbf{GCN}(F, G)(i) = \bigvee_{\mathcal{Y} \in P_i(\mathrm{supp}\ (G))} \bigwedge_{x \in \mathcal{Y}} G(x).$$

As a next corollary from Theorem 6.1. we have

Theorem 6.3. *$D \subset F \subset G \subset H$ implies* $\mathbf{GCN}(F, G) \subset \mathbf{GCN}(D, H)$.

Proof. $D \subset F \subset G \subset H$ implies $P_i(\mathrm{supp}\ (G)) \subset P_i(\mathrm{supp}\ (H))$ and $\{\mathcal{Y} \in P_i(\mathcal{U}) : F_i \subset \mathcal{Y}\} \subset \{\mathcal{Y} \in P_i(\mathcal{U}) : D_1 \subset \mathcal{Y}\}$. Using Th. 6.1 and the previous corollaries following therefrom, we at once obtain the final thesis. $\diamond$

Corollary 6.4. $\mathbf{GCN}(A, A) \subset \mathbf{GCN}(f(A), g(A))$ *for each* $A \in GP(\mathcal{U})$ *and* $(f, g) \in \mathcal{F}$.

Proof. It follows directly from (A1) and Th. 6.3. $\diamond$

So, if $A \in GP(\mathcal{U})$ is fixed and we consider $\mathbf{GCN}(f(A), g(A))$ with different pairs $(f, g) \in \mathcal{F}$, then the least possible energy measure (see e.g. [8], [12]) occurs when $f = g = \mathrm{id}$. In other words, the least deviation of $\mathbf{GCN}(f(A), g(A))$ from a function of the form $1_{\{i\}}$ for some $i \in CN$, i.e. the least deviation from a membership function related to a classical cardinal number, is attained for $f = g = \mathrm{id}$.

From now on, we shall always use $F = f(X)$ and $G = g(X)$ as arguments of the operator **GCN**, where f and g are some approximating functions defined in Section 3 and $X \in GP(\mathcal{U})$.

Remark 6.5. One can show that for each $A \in GP(\mathcal{U})$ if $(f, g) \neq \neq (\mathrm{id}, \mathrm{id})$, then

$$\mathbf{GCN}(f(A), g(A))(i) = [\exists|\mathcal{Y} \in P_i(\mathcal{U}) : \mathrm{obj}(f(A)) \underset{\sim}{\subset} \mathrm{obj}(1_{\mathcal{Y}}) \underset{\sim}{\subset} \underset{\sim}{\subset} \mathrm{obj}(g(A))] = \bigvee_{\mathcal{Y} \in P_i(\mathcal{U})} (\bigwedge_{x \in \mathcal{Y}} g(A)(x) \wedge \bigwedge_{x \notin \mathcal{Y}} 1 - f(A)(x)) \cdot$$

The same holds if $(f, g) \in \mathcal{F}$ is quite arbitrary and supp (A) is finite.

Now we like to express $\mathbf{GCN}(f(A), g(A))(i)$ in a form more simple and convenient than that following from Th. 6.1.

Theorem 6.6. *For each* $(f, g) \in \mathcal{F}$, $A \in GP(\mathcal{U})$ *and* $i \in CN$ *we have*

$$\mathbf{GCN}(f(A), g(A))(i) = g_i(A) \wedge 1 - f_{i^+}(A).$$

Proof. Let $L_{g(A),i} := \bigvee_{\mathcal{Y} \in P_i(\mathrm{supp}\ (g(A)))} \bigwedge_{x \in \mathcal{Y}} g(A)(x)$. We shall prove that $L_{g(A),i} = g_i(A)$. Let us fix $i \in CN$ and suppose that $L_{g(A),i} < g_i(A)$. Then there exists t^* such that $\mathrm{card} g(A)_{t^*} \geq i$ and $L_{g(A),i} < t^*$. But one can choose $\mathcal{Y}^* \in P_i(\mathrm{supp}\ (g(A)))$ such that $\mathcal{Y}^* \subset g(A)_{t^*}$. Hence $\bigwedge\{g(A)(x) : x \in \mathcal{Y}^*\} \geq t^*$ what leads to a contradiction.

Now, suppose that $L_{g(A),i} > g_i(A)$. Then, again, there exists $\mathcal{Y}^*$ such that card $\mathcal{Y}^* = i$ and $g_i(A) < \bigwedge\{g(A)(x) : x \in \mathcal{Y}^*\}$. Let $t^* := g_i(A)$. If card $g(A)_t \geq i$ for each t, then $t^* = 1$ and the previous inequality cannot be true. So, we can assume that there exists t such that card $g(A)_t < i$. Moreover, card $g(A)_{t_*} < i$ for each $t_* > t^*$. But $g(A)(x) > t^*$ for each $x \in \mathcal{Y}^*$. Hence $g(A)(x) \geq t_* > t^*$ for each $x \in \mathcal{Y}^*$ and some $t_* > t^*$, i.e. $\mathcal{Y}^* \subset g(A)_{t_*}$ what implies that card $\mathcal{Y}^* \leq$ card $g(A)_{t_*} < i$ and gives this way a contradiction. So, $L_{g(A),i} = g_i(A)$. The equality $\bigvee_{\{\mathcal{Y} \in P_i(\mathcal{U}) : f(A)_1 \subset \mathcal{Y}\}} \bigwedge_{x \notin \mathcal{Y}} 1 - f(A)(x) = = 1 - f_{i^+}(A)$ can be proved in an analogous way. This completes the proof. ◇

Remark 6.7. In the proof of Th. 6.6. we obtained two important equalities which imply that

$$[\exists|\mathcal{Y} \in P_i(\mathcal{U}) : \mathrm{obj}(1_{\mathcal{Y}}) \underset{\sim}{\subset} \mathrm{obj}(g(A))] = \bigvee\{t : \mathrm{card}\ g(A)_t \geq i\}$$

and

$$[\exists|\mathcal{Y} \in P_i(\mathcal{U}) : \mathrm{obj}(f(A)) \underset{\sim}{\subset} \mathrm{obj}(1_{\mathcal{Y}})] = 1 - \bigwedge\{t : \mathrm{card}\ f(A)_t \leq i\}.$$

Thus, again, the equipotency condition could be rewritten in another equivalent form. Using it one can formulate a generalized (i.e. many-valued) version of the equipotency definition for HCH-objects and introduce this way a notion of HCH-objects equipotent "to a degree $a \in \mathcal{J}$". However, we shall use here only the sharp two-valued Def. 4.1 which is quite sufficient if one likes to construct an applicable and useful theory. On the other hand, this definition accepts also some vagueness and subjectivity of the equipotency by the presence of the approximating functions which after all can be chosen from $\mathcal{F}$ quite arbitrary (cf. Section 5.2).

Applying Cor. 3.2 and Th. 6.6. one can express the membership values to obj(**GCN**$(f(A), g(A))$ in more explicit way. It suffices to consider the following variants of pairs $(f, g) \in \mathcal{F} : f = g = \mathrm{id}$, g is arbitrary and $f \equiv E$ or $f(\cdot) = 1_{(\cdot)_1}$, f is arbitrary and $g \equiv U$ or $g(\cdot) = 1_{\mathrm{supp}\ (\cdot)}$. We easily notice that

$$\mathbf{GCN}(f(A), g(A))(i) = \begin{cases} 1 - f_{i^+}(A) & \textit{if } i < z_{A,f,g}\ , \\ g_i(A) & \textit{otherwise}, \end{cases}$$

where $z_{A,f,g} := \bigwedge\{i \in CN : g_i(A) + f_{i^+}(A) \leq 1\}$.

Theorem 6.8. *For each $A \in GP(\mathcal{U})$ the following properties are fulfilled:*

(a) $$\mathbf{GCN}(E, g(A))(i) = g_i(A) \quad \textit{for each } i \in CN.$$

(b) $$\mathbf{GCN}(1_{A_1}, g(A))(i) = \begin{cases} 0 & \textit{if } i < \mathrm{card}\ A_1, \\ 1 & \textit{if } i = \mathrm{card}\ A_1, \\ g_i(A) & \textit{otherwise}. \end{cases}$$

(c) $$\mathbf{GCN}(f(A), U)(i) = 1 - f_{i^+}(A)\ \textit{for each } i \in CN.$$

(d) $$\mathbf{GCN}(f(A), 1_{\mathrm{supp}\ (A)})(i) = \begin{cases} 1 - f_{i^+}(A) & \textit{if } i < \mathrm{card\ supp}\ (A), \\ 1 & \textit{if } i = \mathrm{card\ supp}\ (A), \\ 0 & \textit{otherwise}. \end{cases}$$

(e) $\mathbf{GCN}(A, A)(i) = a_i \wedge 1 - a_{i^+}$ *for each $i \in CN$ with a_i defined in Section 5.*

Proof. It is an immediate consequence of Th. 6.6 and definitions of $f_i(A), g_i(A)$ and a_i. ♢

Now we are going to present very specific property of the operator **GCN** which holds exclusively for $f = g = \mathrm{id}$ (cf. Cor. 6.4). One can check that for each $A \in GP(\mathcal{U})$ there exists such i that $\mathbf{GCN}(A, A)(i) \geq$

≥ 0.5. It follows from Th. 6.8 that this holds for any other pair (f, g) from $\mathcal{F}$, too. However

Theorem 6.9. *For each $A \in GP(\mathcal{U})$ there exists at most one cardinal number i such that* $\mathbf{GCN}(A, A)(i) > 0.5$.

Proof. It suffices to observe that $\mathbf{GCN}(A, A)(i) > 0.5$ only if $t = 0.5$ is an internal point of $\{t : \text{card } A_t = i\}$. Such the cardinal number is unique if exists. $\diamondsuit$

Using Th. 6.8 we notice that the property described in Th. 6.9 does not hold for pairs $(f, g) \neq (\text{id}, \text{id})$. Finally, we like to formulate some decomposition theorem which will be useful in proving other facts.

Theorem 6.10. *For each $(f, g) \in \mathcal{F}$ and $A \in GP(\mathcal{U})$ we have*

$$\mathbf{GCN}(f(A), g(A)) = \mathbf{GCN}(E, g(A)) \cap \mathbf{GCN}(f(A), U).$$

Proof. This is quite clear since from Th. 6.8 and Th. 6.6 it follows that for each $i \in CN$ we get $\mathbf{GCN}(f(A), g(A))(i) = g_i(A) \wedge 1 - f_{i^+}(A)$, $\mathbf{GCN}(E, g(A))(i) = g_i(A)$, and $\mathbf{GCN}(f(A), U)(i) = 1 - f_{i^+}(A)$. $\diamondsuit$

So, we have for instance

Corollary 6.11. $\mathbf{GCN}(A, A) = \mathbf{GCN}(E, A) \cap \mathbf{GCN}(A, U)$, $\mathbf{GCN}(1_{A_1}, A) = \mathbf{GCN}(E, A) \cap \mathbf{GCN}(1_{A_1}, U)$, $\mathbf{GCN}(1_{A_1}, 1_{\text{supp }(A)}) = \mathbf{GCN}(E, 1_{\text{supp }(A)}) \cap \mathbf{GCN}(1_{A_1}, U)$, and $\mathbf{GCN}(A, 1_{\text{supp }(A)}) = \mathbf{GCN}(E, 1_{\text{supp }(A)}) \cap \mathbf{GCN}(A, U)$.

7. The generalized cardinal numbers

First of all, we like to formulate a property which is a key-stone of the presented theory, namely

Theorem 7.1. *For each $(f, g) \in \mathcal{F}$ and $A, B \in GP(\mathcal{U})$ the following equivalence holds*

$$\mathbf{GCN}(f(A), g(A)) = \mathbf{GCN}(f(B), g(B)) \textit{ iff } A \sim_{f,g} B.$$

Proof. Let us fix some arbitrary (f, g) from $\mathcal{F}$ and A, B from $GP(\mathcal{U})$. It is quite obvious that $A \sim_{f,g} B$ implies $\mathbf{GCN}(f(A), g(A)) = \mathbf{GCN}(f(B), g(B))$. So, assume $\mathbf{GCN}(f(A), g(A)) = \mathbf{GCN}(f(B), g(B))$. If $f \equiv E$, then from Th. 6.8 we obtain $g_i(A) = g_i(B)$ for each $i \in CN$. Obviously, $f_{i^+}(A) = f_{i^+}(B) = 0$ for all i from CN. Thus $A \sim_{f,g} B$. If $f(\cdot) = 1_{(\cdot)_1}$, then from Th. 6.8 we get again $g_i(A) = g_i(B)$ for $i >$

card A_1 = card B_1. From the definitions of $g_i(D)$ and $f_i(D)$ it follows however that $g_i(A) = g_i(B) = 1$ for $i \leq$ card A_1. On the other hand, $f_{i^+}(D) = 1$ if $i <$ card D_1 else $f_{i^+}(D) = 0$. So, $f_{i^+}(A) = f_{i^+}(B)$ for each $i \in CN$. Hence $A \sim_{f,g} B$. Our thesis for $g \equiv U$ and $g(\cdot) = 1_{\text{supp }(\cdot)}$ can be proved in quite similar way. Finally, if $f = g = \text{id}$, it suffices to show that $a_i = b_i$ for each $i \in CN$ what is however again a simple exercise and therefore omitted. $\diamondsuit$

Thus the values of the operator **GCN** fulfill the axiomatic definition of cardinal numbers proposed by A.Tarski ([17], see also [14]). These values will be just called generalized cardinal numbers (gcn's) and denoted by small Greek letters equipped with indexing pair f, g emphasizing which approximating functions have been used. If $\mathbf{GCN}(f(A), g(A)) = \alpha_{f,g} \in GP(CN)$, then we shall write $\text{Gcard}_{f,g}(A) = \alpha_{f,g}$ and say that the power of obj(A) equals $\alpha_{f,g}$ with respect to $(f,g) \in \mathcal{F}$. Obviously, then $\text{Gcard}_{f,g}(A)(i) = \alpha_{f,g}(i) = g_i(A) \wedge 1 - f_{i^+}(A)$.

Let us observe that the Tarski's definition gives us in essence two equivalent possibilities: the first one has been already described, the second and in fact more proper variant is instead to consider the HCH-object obj($\mathbf{GCN}(f(A), g(A))$) in CN as a gcn, i.e. as a tool describing the power of obj(A). Then we should rather write $\text{Gcard}_{f,g}(A) = \text{obj}(\alpha_{f,g})$; moreover, this would be in a way a generalization of the idea of S.Gottwald from [7] who proposed to express the power of a fuzzy set by means of a set composed of some cardinal numbers. However, obj(A) = obj(B) iff $A = B$. This in fact gives us free hand to choose any of those two variants. We have chosen the first one which is more convenient from the practical viewpoint. On the other hand, operations and relations on HCH-objects resolve themselves anyway to operations and relations on respective generalized characteristic functions .

It follows from Th. 7.1 that $\text{Gcard}_{f,g}(A) = \text{Gcard}_{f,g}(B)$ iff $A \sim_{f,g} B$. Moreover the following equivalence is quite obvious

$$\alpha_{f,g} = \beta_{f,g} \text{ iff } \exists A, B \in GP(\mathcal{U}) : \text{Gcard}_{f,g}(A) = \alpha_{f,g} \ \& \\ \& \ \text{Gcard}_{f,g}(B) = \beta_{f,g} \ \& \ A \sim_{f,g} B.$$

So, the equality of gcn's can be defined quite naturally by

$$\alpha_{f,g} = \beta_{f,g} \text{ iff } \alpha_{f,g}(i) = \beta_{f,g}(i) \text{ for each } i \in CN.$$

If the pair $(f,g) \in \mathcal{F}$ is fixed, we write simply Gcard $(A) = \alpha$. From Th. 6.10 we get at once

$$\mathrm{Gcard}_{f,g}(A) = \mathrm{Gcard}_{E,g}(A) \cap \mathrm{Gcard}_{f,U}(A)$$

and

$$\mathrm{Gcard}_{f,g}(A) = \mathrm{Gcard}_{E,id}(g(A)) \cap \mathrm{Gcard}_{id,U}(f(A)).$$

Hence for instance

$$\mathrm{Gcard}_{\mathrm{id,id}}(A) = \mathrm{Gcard}_{\mathrm{E,id}}(A) \cap \mathrm{Gcard}_{\mathrm{id,U}}(A),$$
$$\mathrm{Gcard}_{1_{(\cdot)_1},\mathrm{id}}(A) = \mathrm{Gcard}_{\mathrm{E,id}}(A) \cap \mathrm{Gcard}_{\mathrm{id},U}(1_{A_1}),$$
$$\mathrm{Gcard}_{1_{(\cdot)_1},1_{\mathrm{supp}\ (\cdot)}}(A) = \mathrm{Gcard}_{E,\mathrm{id}}(1_{\mathrm{supp}\ (\mathrm{A})}) \cap \mathrm{Gcard}_{\mathrm{id},U}(1_{A_1}),$$
$$\mathrm{Gcard}_{\mathrm{id},1_{\mathrm{supp}\ (\cdot)}}(A) = \mathrm{Gcard}_{E,\mathrm{id}}(1_{\mathrm{supp}\ (A)}) \cap \mathrm{Gcard}_{\mathrm{id},U}(A),$$
$$\mathrm{Gcard}_{E,1_{\mathrm{supp}\ (\cdot)}}(A) = \mathrm{Gcard}_{E,id}(1_{\mathrm{supp}\ (A)}) \cap \mathrm{Gcard}_{\mathrm{id},U}(E).$$

Obviously, using Th. 6.8 one can automatically express $\mathrm{Gcard}_{f,g}(A)(i)$ for the five basic groups of pairs $(f,g) \in \mathcal{F}$. Let

$$\mathbf{GCN}_{f,g} := \{\alpha \in GP(CN) : \mathrm{Gcard}_{f,g}(D) = \alpha \text{ for some } D \in GP(\mathcal{U})\}.$$

Theorem 7.2. (a) *For each* $(f,g) \in \mathcal{F}$ *all the elements* $\alpha \in \mathbf{GCN}_{f,g}$ are convex, i.e. $\alpha(j) \geq \alpha(i) \wedge \alpha(k)$ *for* $i \leq j \leq k$.
(b) *If* $f \equiv E(f(\cdot) = 1_{(\cdot)_1}$, *resp.*), *then each* $\alpha \in \mathbf{GCN}_{f,g}$ *is antitonic (is antitonic on its support, resp.).For* $g \equiv U(g(\cdot) = 1_{supp\ (\cdot)}$, *resp.*) *each* $\alpha \in \mathbf{GCN}_{f,g}$ *is isotonic (is isotonic on its support, resp.)* .
(c) *If* $(f,g) \neq (\mathrm{id},\mathrm{id})$, *then each element* $\alpha \in \mathbf{GCN}_{f,g}$ *is normal, i.e. there exists* $i \in CN$ *such that* $\alpha(i) = 1$.
Proof. All the results are simple corollaries of Th. 6.8. ◇

It is quite obvious that $\alpha \in \mathbf{GCN}_{f,g}$ is in general case nonmonotonic for $f = g = \mathrm{id}$ (see however the formula preceding Th. 6.8). Also, it is not normal in general but, on the other hand, fulfills an interesting property described by Th. 6.9. Cleary, if α is normal for $f = g = \mathrm{id}$, then its support has exactly one element.

If $(f,g) \neq (\mathrm{id},\mathrm{id})$, then obviously $f(D) \subset 1_{g(D)_1}$ for each $D \in GP(\mathcal{U})$. So, in that case $\mathrm{Gcard}_{f,g}(D)$ is simultaneously equal to the power of the twofold fuzzy set $\Omega = (f(D), g(D))$ (see [3]). Thus gcn's constructed by means of pairs $(f,g) \neq (\mathrm{id},\mathrm{id})$ refer not only to fuzzy sets but also to twofold fuzzy sets (see Section 8.1).

8. Examples and comments

8.1. Let $n :=$ card supp (A), $m :=$ card A_1 and $\text{Gcard}_{f,g}(A) = \alpha$ for some fixed $(f, g) \in \mathcal{F}$. As previously, $a_i := \bigvee\{t : \text{card } A_t \geq i\}$. Using Cor. 3.2 and Th. 6.8 one can present how looks the gcn representing the power of obj(A) with respect to eight main pairs of approximating functions . Then we obtain the following formulae:

(Pair#1: $f = g = \text{id}$)

$$\alpha(i) = a_i \wedge 1 - a_{i+} = \begin{cases} 1 - a_{i+} & \text{if } i < z_{A,f,g}, \\ a_i & \text{otherwise} \end{cases}.$$

(Pair#2: $f \equiv E,\ g = \text{id}$)

$$\alpha(i) = a_i, \text{ where } a_0 = 1 \text{ and } a_i = 0 \text{ for } i > n.$$

(Pair#3: $f(\cdot) = 1_{(\cdot)_1},\ g = \text{id}$)

$$\alpha(i) = \begin{cases} 1 & \text{if } i = m, \\ a_i & \text{if } m < i \leq n, \\ 0 & \text{otherwise.} \end{cases}$$

(Pair#4: $f = \text{id},\ g(\cdot) = 1_{\text{supp }(\cdot)}$)

$$\alpha(i) = \begin{cases} 1 - a_{i+} & \text{if } m \leq i < n, \\ 1 & \text{if } i = n, \\ 0 & \text{otherwise.} \end{cases}$$

(Pair#5: $f = \text{id},\ g \equiv U$)

$$\alpha(i) = \begin{cases} 0 & \text{if } i < m, \\ 1 - a_{i+} & \text{if } m \leq i < n, \\ 1 & \text{otherwise.} \end{cases}$$

(Pair#6: $f \equiv E,\ g(\cdot) = 1_{\text{supp }(\cdot)}$)

$$\alpha(i) = \begin{cases} 1 & \text{if } i \leq n, \\ 0 & \text{otherwise.} \end{cases}$$

(Pair#7: $f(\cdot) = 1_{(\cdot)_1},\ g(\cdot) = 1_{\text{supp }(\cdot)}$)

$$\alpha(i) = \begin{cases} 1 & \text{if } i \in \text{ betw } (m, n), \\ 0 & \text{otherwise.} \end{cases}$$

(Pair#8: $f(\cdot) = 1_{(\cdot)_1},\ g \equiv U$)

$$\alpha(i) = \begin{cases} 0 & \text{if } i < m, \\ 1 & \text{otherwise.} \end{cases}$$

From Th. 6.3 we get at once some inclusions, for instance

$$\alpha_{\#1} \subset \alpha_{\#3} \subset \alpha_{\#2} \subset \alpha_{\#6},$$

$$\alpha_{\#1} \subset \alpha_{\#4} \subset \alpha_{\#5} \subset \alpha_{\#8};$$

$$\alpha_{\#3} \subset \alpha_{\#7} \subset \alpha_{\#8}, \alpha_{\#4} \subset \alpha_{\#7} \subset \alpha_{\#6}.$$

One can easily formulate different simple conditions for having $\alpha_{\#i} = \alpha_{\#j}$. Let us notice that if the pair (E, U) were an element of $\mathcal{F}$, then $\alpha(i) = 1$ for each $i \in CN$.

Using Pair#2 we get gcn's defined for fuzzy sets by L.A.Zadeh ([28]; cf.[15] and see also [20] for a review of early approaches). Pair#3 generates instead gcn's of the type introduced by D.Dubois and H.Prade ([2]) also for fuzzy sets. Finally, $(f, g) = (\mathrm{id}, \mathrm{id})$ gives gcn's defined for fuzzy sets by the author in [20]. Pair#7 generates gcn's identical to the partial cardinal numbers of D.Klaua ([10]) and seems to be suitable (like Pair#6 and #8) for rough sets (see [13],[23]). So, the presented theory brings together a lot of early approaches to gcn's although they have been started from different motivations and have been proposed for different kinds of HCH-objects such as fuzzy sets, twofold fuzzy sets, partial sets and rough sets.

8.2. Let $B \in PS(\mathcal{U})$, $q :=$ card supp (B) and $\mathrm{Gcard}_{f,g}(B) = \beta_{f,g}$ for $(f, g) \in \mathcal{F}$. Then

$$\beta_{f,g} = \begin{cases} 1_{\{q\}} & \text{if } f \not\equiv E \text{ and } g \not\equiv U, \\ 1_{\{i \in CN : i \leq q\}} & \text{if } f \equiv E, \\ 1_{\{i \in CN : i \geq q\}} & \text{if } g \equiv U. \end{cases}$$

Hence $\mathrm{Gcard}_{f,g}(B) = 1_{\mathrm{betw}(\mathrm{card}\ f(B)_1, \mathrm{card}\ g(B)_1)}$ for each $(f, g) \in \mathcal{F}$. These results suggest some interpretation of the values $\mathrm{Gcard}_{f,g}(A)(i)$ for $A \in GP(\mathcal{U})$, namely: if respectively $f \equiv E, g \equiv U, f \not\equiv E$ and $g \not\equiv U$, then one can consider $\mathrm{Gcard}_{f,g}(A)(i)$ to be the degree to which obj(A) has at least, at most, exactly i elements, respectively. As a second corollary we obtain the following formulae ($k :=$ card $\mathcal{U}$):

$$\mathrm{Gcard}_{f,g}(E) = \begin{cases} 1_{\{0\}} \\ 1_{\{0\}} \\ 1_{CN} \end{cases} \text{ and } \mathrm{Gcard}_{f,g}(U) = \begin{cases} 1_{\{k\}} & \text{if } f \not\equiv E \text{ and } g \not\equiv U, \\ 1_{CN} & \text{if } f \equiv E, \\ 1_{\{k\}} & \text{if } g \equiv U. \end{cases}$$

8.3. Let $\mathcal{U} = \mathbb{R}$, i.e. $CN = \{i : i \leq \Diamond\}$. Moreover, let

$$A(x) = \begin{cases} 1 - x & \text{if } 0 \leq x \leq 1 \\ 0 & \text{otherwise,} \end{cases} \qquad B(x) = 0.8A(x),$$

$$C(x) = \begin{cases} 1 - 1/x & \text{if } x = 2, 3, 4, \ldots, \\ 0 & \text{otherwise,} \end{cases}$$

$$D(x) = \begin{cases} 1 & \text{if } x = 0, 1 \\ 0.9 & \text{if } x = 2 \\ 0.7 & \text{if } x = 3 \\ 0.3 & \text{if } 4 \leq x \leq 5 \\ 0 & \text{otherwise,} \end{cases} \qquad S(x) = \begin{cases} 1 & \text{if } x = 2, 3, 4 \\ 0.2 & \text{if } x = 5 \\ 0.3 & \text{if } x = 6 \\ 0.9 & \text{if } x = 7 \\ 0.6 & \text{if } x = 8 \\ 0 & \text{otherwise.} \end{cases}$$

Further, let $\mathrm{Gcard}_{f,g}(A) = \alpha_{f,g}$, $\mathrm{Gcard}_{f,g}(B) = \beta_{f,g}$, $\mathrm{Gcard}_{f,g}(C) = \gamma_{f,g}$, $\mathrm{Gcard}_{f,g}(D) = \delta_{f,g}$, and $\mathrm{Gcard}_{f,g}(S) = \sigma_{f,g}$. So, we have $a_i = 1$ for each $i \in CN$, $b_i = 1$ for $i = 0$ and $b_i = 0.8$ if $i \in$ betw $(1, \Diamond)$, $c_i = 1$ if $i \leq \aleph_0$ and $c_i = 0$ for $i = \Diamond$, and

$$d_i = \begin{cases} 1 & \text{if } i = 0, 1, 2 \\ 0.9 & \text{if } i = 3 \\ 0.7 & \text{if } i = 4 \\ 0.3 & \text{if } i \in \text{ betw } (5, \Diamond), \end{cases} \qquad s_i = \begin{cases} 1 & \text{if } i = 0, 1, 2, 3 \\ 0.9 & \text{if } i = 4 \\ 0.6 & \text{if } i = 5 \\ 0.3 & \text{if } i = 6 \\ 0.2 & \text{if } i = 7 \\ 0 & \text{if } i \in \text{ betw } (8, \Diamond). \end{cases}$$

Then using the notational rule (j) from Section 1 we get

$\alpha_{\#1} = ((0) \parallel 0, 1)$, $\beta_{\#1} = ((0.2) \parallel 0.2, 0.8)$, $\gamma_{\#1} = ((0) \parallel 1, 0)$,
$\delta_{\#1} = (0, 0, 0.1, 0.3, 0.7, (0.3) \parallel 0.3, 0.3)$,
$\sigma_{\#1} = (0, 0, 0, 0.1, 0.4, 0.6, 0.3, 0.2, (0) \parallel 0, 0)$,

$\alpha_{\#2} = ((1) \parallel 1, 1)$, $\beta_{\#2} = (1, (0.8) \parallel 0.8, 0.8)$, $\gamma_{\#2} = ((1) \parallel 1, 0)$,
$\delta_{\#2} = (1, 1, 1, 0.9, 0.7, (0.3) \parallel 0.3, 0.3)$,
$\sigma_{\#2} = (1, 1, 1, 1, 0.9, 0.6, 0.3, 0.2, (0) \parallel 0, 0)$,

$\alpha_{\#3} = (0, (1) \parallel 1, 1)$, $\beta_{\#3} = (1, (0.8) \parallel 0.8, 0.8)$, $\gamma_{\#3} = ((1) \parallel 1, 0)$,
$\delta_{\#3} = (0, 0, 1, 0.9, 0.7, (0.3) \parallel 0.3, 0.3)$,
$\sigma_{\#3} = (0, 0, 0, 1, 0.9, 0.6, 0.3, 0.2, (0) \parallel 0, 0)$,

$\alpha_{\#4} = ((0) \parallel 0, 1)$, $\beta_{\#4} = ((0.2) \parallel 0.2, 1)$, $\gamma_{\#4} = ((0) \parallel 1, 0)$,
$\delta_{\#4} = (0, 0, 0.1, 0.3, (0.7) \parallel 0.7, 1)$,
$\sigma_{\#4} = (0, 0, 0, 0.1, 0.4, 0.7, 0.8, 1, (0) \parallel 0, 0)$,

$\alpha_{\#5} = ((0) \parallel 0,1)$, $\beta_{\#5} = ((0.2) \parallel 0.2,1)$, $\gamma_{\#5} = ((0) \parallel 1,1)$,
$\delta_{\#5} = (0,0,0.1,0.3,(0.7) \parallel 0.7,1)$,
$\sigma_{\#5} = (0,0,0,0.1,0.4,0.7,0.8,(1) \parallel 1,1)$,

$\alpha_{\#6} = ((1) \parallel 1,1)$, $\beta_{\#6} = ((1) \parallel 1,1)$, $\gamma_{\#6} = ((1) \parallel 1,0)$,
$\delta_{\#6} = ((1) \parallel 1,1), \sigma_{\#6} = (1,1,1,1,1,1,1,1,(0) \parallel 0,0)$,

$\alpha_{\#7} = (0,(1) \parallel 1,1)$, $\beta_{\#7} = ((1) \parallel 1,1)$, $\gamma_{\#7} = ((1) \parallel 1,0)$,
$\delta_{\#7} = (0,0,(1) \parallel 1,1)$, $\sigma_{\#7} = (0,0,0,1,1,1,1,1,(0) \parallel 0,0)$,

$\alpha_{\#8} = (0,(1) \parallel 1,1)$, $\beta_{\#8} = ((1) \parallel 1,1)$, $\gamma_{\#8} = ((1) \parallel 1,1)$,
$\delta_{\#8} = (0,0,(1) \parallel 1,1)$, $\sigma_{\#8} = (0,0,0,(1) \parallel 1,1)$.)

Worth noticing is that for instance $\alpha_{\#1} \in PS(CN)$ although $A \notin$ $\notin PS(\mathcal{U})$. This is because A has continuum of values which lie as near to 1 as one likes.

9. Further properties of the generalized cardinal numbers

One of the most fundamental requirements concerning gcn's is the coincidence with cardinal numbers if we deal with HCH-objects being sets. This is fulfilled.

Theorem 9.1. *For each* $(f,g) \in \mathcal{F}$ *there exists a bijection* $\Psi_{f,g}$: $CN \to PS(CN)$ *such that* $\mathrm{Gcard}_{f,g}(1_{\mathcal{D}}) = \Psi_{f,g}(q)$ $(q := \mathrm{card}\ \mathcal{D})$, *i.e. respective diagram is commutative.*

Proof. It suffices to use Ex. 8.2 and to define

$$\Psi_{f,g}(i) = \begin{cases} 1_{\{i\}} & \text{if } f \not\equiv E \text{ and } g \not\equiv U, \\ 1_{\{j \in CN:\ j \leq i\}} & \text{if } f \equiv E, \\ 1_{\{j \in CN:\ j \geq i\}} & \text{if } g \equiv U. \end{cases} \quad \diamondsuit$$

Corollary 9.2. An immediate consequence of Ex. 8.2 is also that for each pair $(f,g) \in \mathcal{F}$ if $B \in PS(\mathcal{U})$ and $\mathrm{Gcard}_{f,g}(B) = \beta_{f,g}$, then $\beta_{f,g} \in PS(CN)$. It is quite clear that the property $\mathrm{Gcard}_{f,g}(A) = \alpha \in PS(CN)$ holds for each $A \in GP(\mathcal{U})$ iff both f and g are functions to $PS(\mathcal{U})$.

10. Finite HCH-objects

Sets are divided into two disjoint classes: finite sets and infinite ones. Some properties of powers and cardinal numbers refer either to finite or to infinite sets. The others refer instead to all the sets and those are in fact real properties of powers and cardinals, for instance the monotonicity $\mathcal{A} \subset \mathcal{B} \Rightarrow \text{card } \mathcal{A} \leq \text{card } \mathcal{B}$. Let us try to extend the notions of finiteness and infiniteness to HCH-objects.

As we pointed out, the power of an HCH-object depends in general on the choice of $(f,g) \in \mathcal{F}$. However, it seems to be reasonable to accept the following postulates:

(a) The finiteness/infiniteness of an HCH-object does not depend on $(f,g) \in \mathcal{F}$.

(b1) An HCH-object of power less than or equal to power of a finite HCH-object has to be finite, too.

(b2) An HCH-object of power greater than or equal to power of an infinite HCH-object has to be infinite, too.

On the other hand, if we like to define a relation $\preceq$ ordering gcn's and powers of HCH-objects , then $\preceq$ should be monotonic, i.e. $\forall (f,g) \in \mathcal{F} : \text{obj}(A) \subset \text{obj}(B) \Rightarrow \text{Gcard}_{f,g}(A) \preceq \text{Gcard}_{f,g}(B)$; such a relation is defined and investigated in [24]. So, HCH-objects with finite supports must be considered to be finite ones. Really, if obj(A) has a finite support, then obj$(1_{\text{supp }(A)})$ is a finite set and obj$(A) \subset$ obj$(1_{\text{supp }(A)})$. Thus the monotonicity condition and (b1) imply that obj(A) is finite. Therefore the problem how to define finite HCH-objects resolves itself to the following question: which HCH-objects besides those with finite supports (if any) should be considered to be finite. To answer it let us recall the Dedekind's definition of an infinite set: $\mathcal{A}$ is infinite iff $\mathcal{A}$ is equipotent to its proper subset. This definition seems to be suitable for extending it to HCH-objects because it operates only with the notions of equipotency and proper containment and does not go into the nature of the notion of a set. So, let us test the following tentative definition: obj(A) is infinite iff it is equipotent with respect to any $(f,g) \in \mathcal{F}$ to an HCH-object obj(A^*) properly contained in obj(A). Now the problem is how to define the proper containment (denoted here by $\subset\subset$) of two HCH-objects. Let us consider two variants of definitions:

(v1) $\text{obj}(A^*) \subset\subset \text{obj}(A)$ iff $A^* \subset A \,\&\, \exists x \in \mathcal{U} : A^*(x) < A(x)$.

But then if we put e.g. $f \equiv E$ and $g(\cdot) = 1_{\text{supp }(\cdot)}$, even an HCH-object supported by one element would be infinite. So, we reject this variant.

(v2) $\text{obj}(A^*) \subset\subset \text{obj}(A)$ iff $f(A^*) \subset f(A) \& g(A^*) \subset g(A) \& \exists x \in \mathcal{U} : f(A^*)(x) < f(A)(x) \mid g(A^*)(x) < g(A)(x)$.

Let us consider then an example with $\mathcal{U} := \{2, 3, 4, \ldots\}, A(i) := 1/i$, $f(\cdot) = 1_{(\cdot)_1}$, $g = \text{id}$. So, $\text{Gcard}_{f,g}(A)(i) = 1/i$ for $i > 0$. $\text{obj}(A)$ is not equipotent to any $\text{obj}(A^*) \subset\subset \text{obj}(A)$. Thus $\text{obj}(A)$ is finite although its support is infinite. But putting $f \equiv E, g(\cdot) = 1_{\text{supp }(\cdot)}$ we however get that $\text{obj}(A)$ is infinite what contradicts (a).

So, we cannot use simultaneously (a) and the extended Dedekind's definition. Moreover, using the last example one can point out that if we like to consider some HCH-object with infinite support to be finite and even if we apply another definition of the infiniteness, then that HCH-object will be always infinite with respect to $f \equiv E$ and $g(\cdot) = 1_{\text{supp }(\cdot)}$. So, either we reject (a) or we consider an HCH-object to be finite iff its support is finite. We choose the second possibility; clearly, HCH-objects which are not finite will be called infinite. This definition is convenient and, moreover, it appears that just the transition from finite to infinite supports causes the same change of properties of gcn's and powers for HCH-objects as the change of properties of cardinal numbers and powers of sets caused by a transition from sets nonequipotent to their proper subsets to sets equipotent to such subsets. The gcn's related to finite (infinite, resp.) HCH-objects will be called finite (transfinite, resp.) gcn's.

11. Final remarks

In this paper our attention has been focused on the construction and basic properties of powers and gcn's for HCH-objects. It appears that a lot of these properties can be enhanced or even quite new properties can be formulated if we restrict ourselves to finite HCH-objects (see [25] for details) which seem to be important from the viewpoint of applications. However the aim of this presentation was to emphasize some general properties, i.e. properties which are independent on the powers of supports and on the choice of $(f, g) \in \mathcal{F}$.

Another two basic subjects have to be discussed: order and operations. Detailed solutions of these problems are given in [24,25]. For instance, it appears that gcn's form a commutative semiring.

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Mathematica Pannonica
3/1 (1992), 117 – 119

COMPLEX INTERPOLATION AND l_n^1 PROPERTY

Emanuele **Casini**

Dipartimento di Matematica, Università, 40127 Bologna, Piazza di Porta S. Donato 5, Italia.

Received October 1990

AMS Subject Classification: 46 B 20

Keywords: Uniformly non-l_n^1, B-convexity, complex interpolation method.

Abstract: We prove that, for a Banach space, the uniform non-l_n^1 property is preserved under Calderon's complex interpolation method.

Introduction

In this paper we prove that for a Banach space, the *uniform non*$-l_n^1$ property is preserved under the complex interpolation method introduced by Calderon [2].

Let X a complex Banach space and $B(X)$ its unit ball, that is $B(X) = \{x \in X : \|x\| \leq 1\}$. We say that X is *uniformly non*$-l_n^1$ if there exists some $\delta > 0$ such that, for any vectors $x_1, \dots, x_n$ in $B(X)$ there exists a choice of signs $\varepsilon_1, ., \varepsilon_n (\varepsilon_i = 1 \text{ or } -1)$ such that $\|\sum_{i=1}^n \varepsilon_i x_i\| < n(1-\delta)$ (see [4]).

We say that X is *B-convex* if it is *uniformly non*$-l_n^1$ for some integer n ([1],[4]). For $n = 2$, *uniformly non*$-l_n^1$ spaces are also called *uniform non-square.*

By (X_0, X_1), we denote an interpolation pair of complex Banach spaces and by $X_s, (0 < s < 1)$, the intermediate spaces obtained by Calderon's complex interpolation method. We will indicate, as usually, by $\|\cdot\|_0, \|\cdot\|_1, \|\cdot\|_s$, the norms in X_0, X_1, X_s respectively. We will use, also, some notation of Calderon's paper, in particular we will indicate by $F(X)$ the set of all functions $f : S \to X_0 + X_1$(with $S = \{z \in \in \mathrm{C}, 0 \leq \mathrm{Re}\, z \leq 1\}$), continuous in S, analytic in int$S$, with $f(j+it) \in \in X_j$ for $j = 0,1$ and with $f(j+it) \to 0$ as $|t| \to \infty$.

Main result

Theorem. *If X_0 or X_1 is uniformly non–l_n^1 then X_s is uniformly non–l_n^1 for every $s \in (0,1)$.*

Proof. Suppose that X_0 is uniformly non–l_n^1 and, by absurdity, that X_s is not uniformly non–l_n^1. This means that for every $\sigma > 0$ there exist $x_1, \ldots, x_n \in B(X_s)$ such that $\|\frac{1}{n}\sum_i \varepsilon_i x_i| \geq 1 - \sigma$ for every choice of $\varepsilon_i = \pm 1$. For a fixed $\eta > 0$ there exist functions $f_i \in F(X)$ satisfying

a) $$f_i(s) = \frac{x_i}{1+\eta} = x_i';$$

b) $\|f_i\| = \max_{j=0,1}(\sup_{t\in\mathbb{R}}\|f_i(j+it)\|_j) \leq 1 (i = 1, 2, \ldots, n).$

For every choice of $\varepsilon_i = \pm 1$ we define

$$E_{\varepsilon_1\ldots\varepsilon_n} = \{t \in \mathbb{R} : \|\frac{1}{n}\sum_{i=1}^{n}\varepsilon_i f_i(it)\|_0 < 1 - \delta\}$$

(where δ is taken from the definition of uniformly non–l_n^1 of X_0).

We will use the following inequality (see [2] p.117):

$$\lg(\|\frac{1}{n}\sum_i \varepsilon_i x_i'\|_s) \leq \int_{-\infty}^{+\infty} \lg\|\frac{1}{n}\sum_{i=1}^{n}\varepsilon_i f_i(it)\|_0 \mu_0(s,t)dt+$$
$$+\int_{-\infty}^{+\infty} \lg\|\frac{1}{n}\sum_{i=1}^{n}\varepsilon_i f_i(1+it)\|_1 \mu_1(s,t)dt$$

where $\mu_j(s,t)(j = 0,1)$ is the Poisson kernel for the strip. In our case we obtain:

$$\lg\frac{1-\sigma}{1+\eta} \leq \int_{E_{\varepsilon_1\ldots\varepsilon_n}} \lg\|\frac{1}{n}\sum_{i=1}^{n}\varepsilon_i f_i(it)\|_0 \mu_0(s,t)dt+$$

$$+\int\limits_{E^c_{\varepsilon_1\ldots\varepsilon_n}} \lg\|\frac{1}{n}\sum_{i=1}^{n}\varepsilon_i f_i(it)\|_0\mu_0(s,t)dt+$$

$$+\int_{-\infty}^{+\infty}\lg\|\frac{1}{n}\sum_{i=1}^{n}\varepsilon_i f_i(1+it)\|_1\mu_1(s,t)dt.$$

But since for every $t\in\mathbb{R}$: $\|\frac{1}{n}\sum_{i=1}^{n}\varepsilon_i f_i(j+it)\|_j\leq 1$ $(j=0,1)$, we obtain $\lg\frac{1-\sigma}{1+\eta}\leq(1-s)|E_{\varepsilon_1\ldots\varepsilon_n}|\lg(1-\delta)$ (where we set $|A|=$ $=\frac{1}{1-s}\int_A\mu_0(s,t)dt$) that is $\frac{1-\sigma}{1+\eta}\geq(1-\delta)\exp\{(1-s)|E_{\varepsilon_1\ldots\varepsilon_n}|\}$ and, being $\eta>0$ arbitrary, we obtain $\sigma\leq 1-(1-\delta)\exp\{(1-s)|E_{\varepsilon_1\ldots\varepsilon_n}|\}$.

If we choose $\sigma=1-(1-\delta)\exp(\frac{1-s}{2^{n+1}})$ we must have $|E_{\varepsilon_1\ldots\varepsilon_n}|\leq$ $\leq 1/2^{n+1}$. This implies that $|\bigcup E_{\varepsilon_1\ldots\varepsilon_n}|\leq 1/2$ that is $(\bigcup E_{\varepsilon_1\ldots\varepsilon_n})^c\neq$ $\neq\emptyset$ (where the union is taken over all choices of signs). But this is a contradiction since X_0 is uniformly non$-l^1_n$, so our theorem is proved. ◇

Corollary 1. *If X_0 or X_1 is B-convex, then X_s is B-convex for every $s\in(0,1)$.*

We also obtain the following result already proved in [3]:

Corollary 2. *If X_0 or X_1 is uniformly non-square then X_s is uniformly non-square.*

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