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SIMULTANEOUS EXTENSIONS OF PROXIMITIES, SEMI-UNIFORMITIES, CONTIGUITIES AND MEROTOPIES III*

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Abstract: Given compatible merotopies (or contiguities) on some subspaces of a proximity space, we are looking for a common extension of these structures.

§§ 0 and 1 can be found in Part I [1], §§ 2 to 4 in Part II [2]. See § 0 for terminology, notations and conventions. We shall also need the following notations introduced later: $A^r = X \setminus A$ (for $A \subset X$); $\mathsf{M}^0(\Gamma)$ is the merotopy for which the contiguity Γ constitutes a base (cf. 4.1).

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5. Extending a family of merotopies in a proximity space

A. WITHOUT SEPARATION AXIOMS

5.1 A family of merotopies in a proximity space always has an extension; we are going to construct the coarsest one. In general, there is no finest extension, not even for $I = \emptyset$; this could be deduced from the well-known fact that there may fail to exist a finest compatible uniformity in an Efremovich proximity space (see e.g. [5] Ch. I, Ex. 12.), but we shall give a simpler example in 5.3.

Definition. A cover c in a proximity space (X, δ) is a *δ-cover* if $A\delta B$ implies the existence of a $C \in \mathsf{c}$ with $A \cap C \neq \emptyset \neq B \cap C$. ◊

In other words, c is a δ-cover iff for any $A \subset X$, $A\bar{\delta}\mathrm{St}(A, \mathsf{c})^r$. Evidently, any cover refined by some δ-cover is a δ-cover.

Lemma. *For a merotopy* M *on* X*,* $\delta(\mathsf{M})$ *is coarser than* δ *iff every* $\mathsf{c} \in \mathsf{M}$ *is a δ-cover iff* M *has a base consisting of δ-covers.*

Proof. 0.4 (1). ◊

5.2 Notation. For $\mathsf{a} \subset \exp X$, Let pa denote the partition of X generated by a; this means that $S \in \mathsf{pa}$ iff $S = \bigcap_{A \in \mathsf{a}} f(A)$, where, for each $A \in \mathsf{a}$, either $f(A) = A$ or $f(A) = A^r$. ◊

Lemma. *If* c *and* f *are δ-covers, and* f *is finite then* $\mathsf{c}(\cap)\mathsf{f}$ *is a δ-cover as well.*

Proof. By Axiom P5, we may assume when checking the condition in Definition 5.1 that there are A', $B' \in \mathsf{pf}$ with $A \subset A'$, $B \subset B'$. As f is a δ-cover, there is a $D \in \mathsf{f}$ such that $A \cup B \subset A' \cup B' \subset D$. c is also a δ-cover, so we can pick a $C \in \mathsf{c}$ with $A \cap C \neq \emptyset \neq B \cap C$. Now $C \cap D \in \mathsf{c}(\cap)\mathsf{f}$, and $A \cap (C \cap D) \neq \emptyset \neq B \cap (C \cap D)$. ◊

It is not superfluous to assume that f is finite:

Example. Let $X = \mathbb{N}$, $P = \{2n : n \in \mathbb{N}\}$, $Q = P^r$. For disjoint $A, B \subset X$, let $A\delta B$ iff both A and B are infinite. Now

$$\mathsf{c} = \big\{\{p, q\} : p \in P,\ q \in Q,\ p < q\big\} \cup \{P, Q\}$$

and d defined analogously, with $p > q$ substituted for $p < q$, are δ-covers, but $\mathsf{c}(\cap)\mathsf{d}$ is not a δ-cover. ◊

5.3 Definition. For a family of merotopies in a proximity space, let

M^0 be the merotopy for which the following covers form a subspace B:

$$\mathsf{c}_i^0 = \{C_i^0 = C_i \cup X_i^r : C_i \in \mathsf{c}_i\} \quad (i \in I, \mathsf{c}_i \in \mathsf{M}_i);$$
$$\mathsf{c}_{A,B} = \{A^r, B^r\} \quad (A\bar{\delta}B). \ \diamondsuit$$

Recall the covers c_i^0 were already introduced in §3. We shall write $\mathsf{M}^0(\delta, \mathsf{M}_i) = \mathsf{M}^0(\delta, \{\mathsf{M}_i : i \in I\})$ when necessary, e.g. when it has to be distinguished from $\mathsf{M}^0(c, \mathsf{M}_i)$. $\mathsf{M}^0(\delta) = \mathsf{M}^0(\delta, \emptyset)$.

Lemma. *Let (X, δ) be a proximity space.*

a) *For $A\bar{\delta}B$, $\mathsf{c}_{A,B}$ is a δ-cover.*

b) *$\mathsf{M}^0(\delta)$ is the coarsest merotopy compatible with δ.*

c) *For $X_0 \subset X$, $\mathsf{M}^0(\delta)|X_0 = \mathsf{M}^0(\delta|X_0)$.*

d) *A filter on X is $\mathsf{M}^0(\delta)$-Cauchy iff it is δ-compressed.*

Proof. a) If $\emptyset \neq E \subset A$ then $\mathrm{St}(E, \mathsf{c}_{A,B})^r = B$ and $E\bar{\delta}B$; the case $\emptyset \neq$ $\neq E \subset B$ is analogous; finally, if $E \not\subset A$, $E \not\subset B$ then $\mathrm{St}(E, \mathsf{c}_{A,B}) = X$. Thus $\mathsf{c}_{A,B}$ satisfies the condition mentioned after Definition 5.1.

b) By Lemmas 5.2 and 5.1, $\delta(\mathsf{M}^0(\delta))$ is coarser than δ. Conversely, if $A\bar{\delta}B$ then $\mathrm{St}(A, \mathsf{c}_{A,B}) \cap B = \emptyset$, thus $A\bar{\delta}(\mathsf{M}^0(\delta))B$. Hence $\mathsf{M}^0(\delta)$ is compatible.

If M is compatible and $A\bar{\delta}B$ then there is a $\mathsf{c} \in \mathsf{M}$ such that $\mathrm{St}(A, \mathsf{c}) \cap B = \emptyset$. Now c refines $\mathsf{c}_{A,B}$, so $\mathsf{c}_{A,B} \in \mathsf{M}$ and $\mathsf{M}^0(\delta) \subset \mathsf{M}$.

c) Clearly

$$\mathsf{c}_{A,B}|X_0 = \mathsf{c}_{A \cap X_0, B \cap X_0},$$

with the right hand side understood in the fundamental set X_0, and $A\bar{\delta}B$ implies $A \cap X_0 \bar{\delta}_0 B \cap X_0$, while if $A\bar{\delta}_0 B$ then $A\bar{\delta}B$ (where $\delta_0 =$ $= \delta|X_0$).

d) Recall that a filter is Cauchy iff it intersects each elements of a given subbase. $\diamondsuit$

There is, in general, no finest compatible merotopy:

Example. Take (X, δ), c and d from Example 5.2. By Lemmas 5.1, 5.2 and 5.3 b), $M^0(\delta) \cup \{\mathsf{c}\}$ and $M^0(\delta) \cup \{\mathsf{d}\}$ are subbases for compatible merotopies. A finest compatible merotopy would have to contain $\mathsf{c}(\cap)\mathsf{d}$, which is not a δ-cover. $\diamondsuit$

The induced closure is discrete in this example, thus any merotopy compatible with δ is Lodato. Consequently, there does not exist a finest compatible Lodato (or Riesz) merotopy.

5.4 Theorem. *A family of merotopies in a proximity space can always* *be extended;* M^0 *is the coarsest extension.*

Proof. 1° *$\delta(\mathsf{M}^0)$ is finer than δ.* This follows from $\mathsf{M}^0(\delta) \subset \mathsf{M}^0$ and Lemma 5.3 b).

2° *$\delta(\mathsf{M}^0)$ is coarser than δ.* It is enough to show that if $\emptyset \neq F \subset I$ is finite and $\mathsf{c}_i \in \mathsf{M}_i$ $(i \in F)$ then $\mathsf{c} = (\bigcap_{i \in F})\mathsf{c}_i^0$ is a δ-cover, since $\mathsf{c}_{A,B}$ is a δ-cover by Lemma 5.3 a), so Lemma 5.2 yields that the elements of M^0 are δ-covers, and then Lemma 5.1 can be applied.

Let $A\delta B$; a $C \in \mathsf{c}$ with $A \cap C \neq \emptyset \neq B \cap C$ is needed. By Axiom P5, we may assume that there are A', $B' \in \mathsf{p}\{X_i : i \in F\}$ such that $A \subset A'$, $B \subset B'$. Let us decompose the index set F into four parts as follows:

$$\begin{array}{cc} A \cup B \subset X_i & (i \in F_0); \\ A \subset X_i,\ B \subset X_i^\tau & (i \in F_1); \\ A \subset X_i^\tau,\ B \subset X_i & (i \in F_2); \\ A \cup B \subset X_i^\tau & (i \in F_3). \end{array}$$

By the accordance, $\mathsf{M}_i|A \cup B$ is the same merotopy compatible with $\delta|A \cup B$ for each $i \in F_0$, and $(\bigcap_{i \in F_0})(\mathsf{c}_i|A \cup B)$ belongs to it, so we can choose $C_i \in \mathsf{c}_i$ $(i \in F_0)$ such that

$$A \cap \bigcap_{i \in F_0} C_i \neq \emptyset \neq B \cap \bigcap_{i \in F_0} C_i. \tag{1}$$

Fix now points x and y from the left hand side, respectively the right hand side of (1); in case $F_0 = \emptyset$, assume only that $x \in A$, $y \in B$. For $i \in F_1$, pick $C_i \in \mathsf{c}_i$ with $x \in C_i$; similarly, for $i \in F_2$, let $y \in C_i \in \mathsf{c}_i$. For $i \in F_3$, take an arbitrary set $C_i \in \mathsf{c}_i$. With $C = \bigcap_{i \in F} C_i^0 \in \mathsf{c}$ we have $x \in A \cap C$, $y \in B \cap C$.

3° *$\mathsf{M}^0|X_i$ is finer than M_i*, since for any $\mathsf{c}_i \in \mathsf{M}_i$, $\mathsf{c}_i^0|X_i = \mathsf{c}_i$, and $\mathsf{c}_i^0 \in \mathsf{M}^0$.

4° *$\mathsf{M}^0|X_i$ is coarser than M_i.* By Lemma 5.3 c), $\mathsf{c}_{A,B}|X_i \in \mathsf{M}^0(\delta_i)$, so Lemma 5.3 b) implies that it belongs to M_i. $\mathsf{c}_j^0|X_i \in \mathsf{M}_i$ follows from the accordance: Taking a $\mathsf{c}_i \in \mathsf{M}_i$ with $\mathsf{c}_i|X_{ij} = \mathsf{c}_j|X_{ij}$, c_i will refine $\mathsf{c}_j^0|X_j$, since if $C_i \in \mathsf{c}_i$ then $C_i \cap X_{ij} = C_j \cap X_{ij} = C_j^0 \cap X_{ij}$ for some $C_j \in \mathsf{c}_j$, and $C_i \subset (C_j^0 \cap X_{ij}) \cup (X_i \setminus X_{ij}) = C_j^0 \cap X_i$.

5° *M^0 is the coarsest extension.* Let M be another extension. $\mathsf{c}_{A,B} \in \mathsf{M}$ by Lemma 5.3 b). For $\mathsf{c}_i \in \mathsf{M}_i$, take a $\mathsf{c} \in \mathsf{M}$ with $\mathsf{c}_i = \mathsf{c}|X_i$; now c refines c_i^0, thus $\mathsf{c}_i^0 \in \mathsf{M}$, too. Hence $\mathsf{M}^0 \subset \mathsf{M}$. ◊

5.5 Theorem. *A family of merotopies in a proximity space has a finest exension iff* $\mathsf{c}(\cap)c'$ *is a* δ*-cover whenever* c *and* c' *are* δ*-covers with traces belonging to* M_i $(i \in I)$. *If so then these covers make up the finest extension* M^1.

Proof. Any cover belonginng to an extension is a δ-cover with traces in M_i, so if the system of these covers is closed for the operation $(\cap)$ then they constitute a merotopy finer than each extension, and this merotopy is an extension by Lemma 5.1 and Theorem 5.4.

Conversely, assume that there exists a finest extension M^1. If $\mathsf{c} \in \mathsf{M}^1$ then c is a δ-cover by Lemma 5.1; $\mathsf{c}|X_i \in \mathsf{M}_i$ is evident. If d is a δ-cover and $\mathsf{d}|X_i \in \mathsf{M}_i$ $(i \in I)$ then $\mathsf{M}^0 \cup \{\mathsf{d}\}$ is a subbase for an extension M. $\mathsf{M}|X_i = \mathsf{M}_i$ is clear; M is compatible, as $\mathsf{M}^0 \subset \mathsf{M}$ and the elements of M are δ-covers; the last statement can be proved using the argument from 2° of the proof of Theorem 5.4, with the changement that $\mathsf{d}|A \cup B$ has to be added to the covers $\mathsf{c}_i|A \cup B$ $(i \in F_0)$, thus $\mathsf{d} \in \mathsf{M}^1$. Hence M^1 consists of the δ-covers with traces in M_i. ◇

5.6 For a non-empty family of merotopies in a proximity space, we have

$$(1) \qquad \mathsf{M}^0 = \sup_{i \in I} \mathsf{M}^0(\delta, \{\mathsf{M}_i\}) = \sup\{\mathsf{M}^0(\delta), \sup_{i \in I} \mathsf{M}^{00}[i]\},$$

where $\mathsf{M}^{00}[i]$ is the coarsest merotopy M on X for which $\mathsf{M}|X_i = \mathsf{M}_i$, i.e. $\{\mathsf{c}_i^0 : \mathsf{c}_i \in \mathsf{M}_i\}$ is a base for $\mathsf{M}^{00}[i]$. (1) follows from 2.2 a), but can also be easily seen from Definition 5.3. (Recall that for merotopies $\mathsf{M}[i]$ $(i \in I \neq \emptyset)$ on X, $\bigcup_{i \in I} \mathsf{M}[i]$ is a subbase for $\sup_{i \in I} \mathsf{M}[i]$.)

5.7 A part of Theorem 3.1 can be deduced in two steps from Theorems 1.2 and 5.4: given a family of merotopies in a symmetric closure space, extend first the induced proximities, and then take the merotopy $\mathsf{M}(\delta^0, \mathsf{M}_i)$; this merotopy is the coarsest extension in (X, c): if M is another extension then $\delta(\mathsf{M})$ is an extension of the proximities $\delta(M_i)$, thus it is finer then δ^0; now

$$\mathsf{M}^0(\delta^0, \mathsf{M}_i) \subset \mathsf{M}^0(\delta(\mathsf{M}), \mathsf{M}_i) \subset \mathsf{M}$$

(the first inclusion can be seen from Definition 5.3, the second one follows from Theorem 5.4, since M is an extension in the proximity space $(X, \delta(\mathsf{M}))$. Therefore:

(1) $$\mathsf{M}^0(c,\mathsf{M}_i)=\mathsf{M}^0(\delta^0(c,\delta(\mathsf{M}_i)),\mathsf{M}_i).$$

If we only want to prove the *existence* of an extension of a family of merotopies in a closure space then δ^1 can also be used instead on δ^0, but $\mathsf{M}^0(\delta^1,\mathsf{M}_i)$ is in general different from $\mathsf{M}^1(c,\mathsf{M}_i)$. It is, however, true that $\mathsf{M}^1(c,\mathsf{M}_i)$ is the finest extension of the merotopies in (X,δ^1) (because it is an extension in (X,c) finer than $\mathsf{M}^0(\delta^1,\mathsf{M}_i)$, so it induces a proximitiy δ' finer that δ^1; δ' is an extension of the proximities $\delta(M_i)$, so it is also coarser than δ^1; thus $\mathsf{M}^1(c,\mathsf{M}_i)$ is indeed an extension in (X,δ^1), and it is the finest one in a larger class of merotopies, namely the extensions in (X,c)). Therefore:

(2) $$\mathsf{M}^1(c,\mathsf{M}_i)=\mathsf{M}^1(\delta^1(c,\delta(\mathsf{M}_i)),\mathsf{M}_i).$$

But there arises a difficulty if we try to deduce the part of Theorem 3.1 concerning finest extensions: it has to be shown somehow that Theorem 5.5 applies to δ^1.

5.8 Conversely, it is also possible to base the proof of Theorems 1.1 and 1.2 on Theorem 3.1 and Lemma 5.3:

Let a family of proximities be given in a symmetric closure space. By Lemma 5.3 b) and c), $\{\mathsf{M}^0(\delta_i) : i\in I\}$ is a family of merotopies in (X,c); Theorem 3.1 furnishes the coarsest, respectively the finest extension M^0 and M^1 of this family. Now $\delta(\mathsf{M}^0)$ and $\delta(\mathsf{M}^1)$ are clearly extensions of the family of proximities. If δ is an extension of the same proximities then $\mathsf{M}^0(\delta)$ is an extension of the merotopies $\mathsf{M}^0(\delta_i)$ (again by Lemma 5.3 c)), thus $\mathsf{M}^0\subset\mathsf{M}^0(\delta)\subset\mathsf{M}^1$, implying $\delta(\mathsf{M}^0)\supset\delta\supset$ $\supset\delta(\mathsf{M}^1)$. So $\delta(M^0)$ and $\delta(M^1)$ are coarsest, respectively finest. Therefore we have:

(1) $$\delta^k(c,\delta_i)=\delta(\mathsf{M}^k(c,\mathsf{M}^0(\delta_i)))\qquad (k=0,1).$$

(Compare these formulas with 4.1 (1).)

B. RIESZ MEROTOPIES IN A PROXIMITY SPACE

5.9 Theorem. *A family of merotopies in a proximity space has a Riesz extension iff the proximity is Riesz and the trace filters are Cauchy; if so then* M^0 *is the coarsest Riesz extension.*

Proof. The conditions are clearly necessary. Conversely, if they are satisfied then M^0 is Riesz (so it is the coarsest Riesz extension by Theorem 5.4):

Let $x \in X$ and $\mathsf{c} \in \mathsf{B}$ (see Definition 5.3) be fixed; we need a $C \in \mathsf{c}$ with $x \in \operatorname{int} C$. If $\mathsf{c} = \mathsf{c}_{A,B}$, $A\bar{\delta}B$ then $x \notin c(A)$ or $x \notin c(B)$ (as δ is Riesz), thus $x \in \operatorname{int} A^r$, $A^r \in \mathsf{c}$, or $x \in \operatorname{int} B^r$, $B^r \in \mathsf{c}$. If $\mathsf{c} = \mathsf{c}_i^0$, $i \in I$, $\mathsf{c}_i \in \mathsf{M}_i$ then there is a $C_i \in \mathsf{c}_i \cap \mathsf{s}_i(x)$ (as the trace filters are Cauchy), thus $C_i^0 \in \mathsf{v}(x)$, i.e. $x \in \operatorname{int} C_i^0$, $C_i^0 \in \mathsf{c}$. ◊

5.10 Theorem. *A family of merotopies in a proximity space has a finest Riesz extension iff δ is Riesz, the trace filters are Cauchy, and $\mathsf{c}(\cap)\mathsf{c}'$ is a δ-cover whenever c and c' are δ-covers with traces belonging to M_i ($i \in I$) such that* $\operatorname{int}\mathsf{c}$ *and* $\operatorname{int}\mathsf{c}'$ *are covers. If so then these covers make up the finest Riesz extension M_R^1.*

Proof. If M_R^1 exists then M^0 is Riesz by Theorem 5.9. Now assuming in the proof of Theorem 5.5 that $\operatorname{int}\mathsf{d}$ is a cover, the extension M defined there is Riesz, thus $\mathsf{d} \in \mathsf{M}_R^1$. ◊

If the conditions of Theorem 5.9 are fullfilled and there exists a finest extension M^1 then so does M_R^1 (take those $\mathsf{c} \in \mathsf{M}^1$ for which $\operatorname{int}\mathsf{c}$ is a cover), but not conversely, not even for $I = \emptyset$:

Example. Take $X = [-1, 1]$ with the Euclidean proximity δ. Let

$$\mathsf{c} = \{[-1,0], [0,1]\} \cup \{\{p,q\} : 0 < -p < q < 1\},$$

and d defined analogously, with $0 < q < -p < 1$. c and d are δ-covers, but $\mathsf{c}(\cap)\mathsf{d}$ is not a δ-cover, so (as in Example 5.3) there is no finest compatible merotopy. But there exists a finest compatible Riesz merotopy, namely the one for which all the open covers form a base. ◊

5.11 It can also occur that M^1 and M_R^1 both exist but differ: let δ be the indiscrete proximity on a three-point set. A better example, with δ separated:

Example. Let X be infinite, $z \in X$, and u a free ultrafilter on X. Take the topology c on X for which $\big\{\{z\} \cup S : S \in \mathsf{u}\big\}$ is the neighbourhood filter of z, and the other points are isolated. Now with $\delta = \delta^1(c) = \delta_R^1(c)$, we have $\mathsf{M}^1(\delta) = \mathsf{M}^1(c)$, and the cover c consisting of all the finite subsets of X belongs to $\mathsf{M}^1(\delta) \backslash \mathsf{M}_R^1(\delta)$. ($\mathsf{c}$ is a δ-cover, so $\mathsf{c} \in \mathsf{M}^1(\delta)$ by Theorem 5.5. $\mathsf{c} \notin \mathsf{M}_R^1$, because $z \notin \bigcup \operatorname{int}\mathsf{c}$). ◊

5.12 Similarly to 5.7 and 5.8, it is possible to deduce from each other Theorem 1.5 and the part of Theorem 3.2 concerning coarsest extensions. (Make use of Lemma 5.3 d).) In addition to the formulas given in 5.7 and 5.8, we have (for a family of merotopies, repsectively proximities, in a weakly separated closure space, with Cauchy, respectively compressed, trace filters):

(1) $$\mathsf{M}^1_R(c, \mathsf{M}_i) = \mathsf{M}^1_R(\delta^1_R(c, \delta(\mathsf{M}_i)), \mathsf{M}_i);$$

(2) $$\delta^1_R(c, \delta_i) = \delta(M^1_R(c, \mathsf{M}^0(\delta_i))).$$

C. LODATO MEROTOPIES IN A PROXIMITY SPACE

5.13 If a family of merotopies in a proximity sapce has a Lodato extension then the proximity and the merotopies are Lodato, the trace filters are Cauchy, and 3.6 (1) holds, since an extension in (X, δ) is necessarily an extension in (X, c). These conditions are not sufficient, not even for a single open subset:

Example. Take X, X_1 and M_1 from Example 3.8, and let δ be the Euclidean proximity on X. Now M_1 and δ are Lodato, M_1 is compatible with $\delta|X_1$, the trace filters are Cauchy, 3.6 (1) is evident (cf. Corollary 3.7), both $\mathcal{U}(\mathsf{M}_1)$ and $\Gamma(\mathsf{M}_1)$ have Lodato extensions, but M_1 does not have one:

Assume indirectly that N is a Lodato extension. Then $\mathsf{c}_1(1)^0 \in \mathsf{N}$, and so $\mathsf{d} = \operatorname{int} \mathsf{c}_1(1)^0 \in \mathsf{N}$; now $\mathsf{d}|X_1^r$ consists of singletons, implying that $\delta|X_1^r$ is discrete, a contradiction. ◊

5.14 Definition. For a family of Lodato merotopies in a Lodato proximity space with Cauchy trace filters, let $\{\operatorname{int} \mathsf{c} : \mathsf{c} \in \mathsf{B}\}$ be a subbase for M^0_L (with B from Definition 5.3). ◊

In other words, $\{\operatorname{int} \mathsf{c} : \mathsf{c} \in \mathsf{M}^0\}$ is a base for M^0_L. ($\operatorname{int} \mathsf{c}$ is a cover by Theorem 5.9.) $\operatorname{int} \mathsf{c}_{A,B} = \mathsf{c}_{c(A),c(B)}$, so the following covers form a subbase B_L for M^0_L:

(1) $\mathsf{c}_{A,B}$ $\quad (A\bar{\delta}B$, A and B are c-closed);

(2) $\operatorname{int} \mathsf{c}^0_i$ $\quad (i \in I$, $\mathsf{c}_i \in \mathsf{M}_i$, c_i is c_i-open).

The covers in this subbase are clearly open in c. M^0_L is finer than the compatible merotopy M^0. On the other hand, the c-openess of the covers implies that $c(\mathsf{M}^0_L)$ is coarser than c; therefore:

Lemma. *Under the assumptions of the definition, M^0_L is a Lodato merotopy compatible with c.* ◊

M_L^0 is not necessarily compatible with δ, see Example 5.13. We shall also see that $\mathsf{M}_L^0|X_i$ can be different from M_i (see Examples 5.19).

5.15 Lemma. *If δ is a Lodato proximity then $\mathsf{M}^0(\delta) = \mathsf{M}_L^0(\delta)$ is the coarsest Lodato merotopy compatible with δ.*
Proof. $\mathsf{M}^0 \subset \mathsf{M}_L^0$ always holds, while the converse follows for $I = \emptyset$ from $\mathsf{B}_L \subset \mathsf{B}$. Now Lemma 5.14 and Theorem 5.4 can be applied. $\diamondsuit$

5.16 Lemma. *Under the assumptions of Definition 5.14, M_L^0 is the coarsest one among those Lodato merotopies M compatible with c that induce a proximity finer than δ, and for which $\mathsf{M}|X_i$ is finer than M_i $(i \in I)$.*
Proof. $\delta(\mathsf{M}_L^0)$ is finer than δ, because M_L^0 is finer than $\mathsf{M}_L^0(\delta)$, and the latter is compatible by Lemma 5.15. $\mathsf{M}_L^0|X_i \supset \mathsf{M}_i$, because if $\mathsf{c}_i \in \mathsf{M}_i$ is c_i-open then $\mathsf{c}_i = (\operatorname{int} \mathsf{c}_i^0)|X_i$. M_L^0 is Lodato and $c(\mathsf{M}_L^0) = c$ (Lemma 5.14).

Let M be a merotopy satisfying the conditions of the lemma; we have to show that $\mathsf{B}_L \subset \mathsf{M}$.

If $A\bar{\delta}B$ then $A\overline{\delta(\mathsf{M})}B$, so $\mathsf{c}_{A,B} \in \mathsf{M}^0(\delta(\mathsf{M})) \subset \mathsf{M}$ by Theorem 5.4. $\mathsf{M}|X_i \supset \mathsf{M}_i$ implies that for any c_i-open cover $\mathsf{c}_i \in \mathsf{M}_i$ there is a $\mathsf{c} \in \mathsf{M}$ with $\mathsf{c}|X_i = \mathsf{c}_i$; $\operatorname{int} \mathsf{c} \in \mathsf{M}$ (as M is Lodato, and it is compatible with c); now $\operatorname{int} \mathsf{c}$ refines $\operatorname{int} \mathsf{c}_i^0$, thus $\operatorname{int} \mathsf{c}_i^0 \in \mathsf{M}$, too. $\diamondsuit$

It has to be assumed in the lemma that M is compatible with c:
Example. On $X = \mathbb{N}^2$, let $A\bar{\delta}B$ iff their projections on the first coordinate are disjoint. Take the discrete merotopy M_0 on $X_0 = \mathbb{N} \times \times \{1\}$, and let M be the merotopy for which $\mathsf{M}^0(\delta') \cup \{\mathsf{c}_0^0\}$ constitutes a subbase, where δ' is the discrete proximity on X, and c_0 consists of the singletons in X_0. Now M is not compatible with c, but the other conditions of the theorem are satisfied. M_L^0 is not coarser than M, because $\mathsf{M}|X_0^\tau$ is contigual, while $(\operatorname{int} \mathsf{c}_0^0)|X_0^\tau \in \mathsf{M}_L^0|X_0^\tau$ cannot be refined by a finite cover. $\diamondsuit$

5.17 Lemma. *A family of merotopies in a proximity space has a Lodato extension iff*

(i) *the proximity and the merotopies are Lodato;*

(ii) $(\bigcap_{i \in F})\operatorname{int} \mathsf{c}_i^0$ *is a δ-cover whenever $\emptyset \neq F \subset I$ is finite, and* $\mathsf{c}_i \in \mathsf{M}_i$ $(i \in F)$;

(iii) $(\operatorname{int} \mathsf{c}_i^0)|X_j \in \mathsf{M}_j$ $(i, j \in I, \mathsf{c}_i \in \mathsf{M}_i)$.

If these conditions are satisfied then M_L^0 *is the coarsest Lodato extension.*

Remarks. a) It is not necessary to assume that the trace filters are Cauchy, since this follows from (ii). (Recall that the trace filters are Cauchy iff each $\operatorname{int} \mathsf{c}_i^0$ is a cover.)

b) It is enough to know (ii) and (iii) for elements of bases for M_i, e.g. for open covers.

c) The cover in (ii) can also be written as $\operatorname{int}(\bigcap_{i\in F})\mathsf{c}_i^0$.

Proof. 1° *Necessity.* It was already mentioned in 5.13 that (i) and (iii) are necessary. If there is a Lodato extension then M_L^0 is an extension by Lemma 5.16. The covers in (ii) belong to M_L^0, so Lemma 5.1 implies that they are δ-covers.

2° *Sufficiency.* The assumptions of Definition 5.14 are fullfilled, see Remark a). $\delta(\mathsf{M}_L^0) \subset \delta$ and $\mathsf{M}_L^0|X_i \supset \mathsf{M}_i$ by Lemma 5.16. Conversely, $\delta(M_L^0) \supset \delta$ follows from Lemma 5.1, since the elements of the base generated by B_L are δ-covers by (ii) and Lemmas 5.3 a) and 5.2; $\mathsf{M}_L^0|X_i \subset \mathsf{M}_i$ follows from (iii) and Lemma 5.2 b) and c). Thus M_L^0 is an extension, Lodato by Lemma 5.14. ◇

Corollary. *A single Lodato merotopy* M_0 *in a Lodato proximity space has a Lodato extension iff* $\operatorname{int} \mathsf{c}_0^0$ *is a* δ*-cover for each (*c_0*-open)* $\mathsf{c}_0 \in \mathsf{M}_0$*; if so then* M_L^0 *is the coarsest extension.* ◇

It can occur that a single merotopy in a proximity space has a Lodato extension, but $\mathsf{M}_L^0 \neq \mathsf{M}^0$ (we have seen in Lemma 5.15 that this is impossible for $I = \emptyset$):

Example. Let $X = \mathbb{R} \times [0, \to[$, with the Euclidean proximity δ, $X_0 = \mathbb{R} \times]0, \to[$, M_0 the Euclidean merotopy on X_0. Now M_0 has a Lodato extension (the Euclidean merotopy on X, which is in fact equal to M_L^0), but $\mathsf{M}_L^0 \neq \mathsf{M}^0$, since $\mathsf{M}^0|X_0^r$ is contigual, while $\mathsf{M}_L^0|X_0^r$ is not contigual. ◇

5.18 $\operatorname{int} \mathsf{c}_0^0$ clearly satisfies the condition in Definition 5.1 for $A, B \subset \subset X_0$, so, in view of Axiom P5, it is enough to assume this condition in Corollary 5.17 for $A \subset X_0^r$ and for B satisfying $B \subset X_0$ or $B \subset X_0^r$. Thus the assumption in Corollary 5.17 splits into two parts:

(a) if $A, B \subset X_0^r$, $A\delta B$ and $\mathsf{c}_0 \in \mathsf{M}_0$ (is open) then there are $C_0 \in \mathsf{c}_0$, $x \in A$ and $y \in B$ such that $C_0 \in \mathsf{s}_0(x) \cap \mathsf{s}_0(y)$;

(b) if $A \subset X_0^r$, $B \subset X_0$, $A\delta B$ and $\mathsf{c}_0 \in \mathsf{M}_0$ (is open) then there are $C_0 \in \mathsf{c}_0$ and $x \in A$ such that $C_0 \in \mathsf{s}_0(x)$ and $C_0 \cap B \neq \emptyset$.

Either of these conditions implies that the trace filters are Cauchy. (For $x \in c(X_0)\setminus X_0$, take $A = \{x\}$ and either $B = \{x\}$ or $B = X_0$.) The next examples show that neither is sufficient in itself for the existence of a Lodato extension.

Examples. a) Modify Example 5.13, replacing each $\mathsf{c}_1(\varepsilon)$ by

$$\{C_1 \cup D_1 : C_1,\ D_1 \in \mathsf{c}_1(\varepsilon),\ C_1 \cap D_1 \neq \emptyset\}.$$

Now (b) holds, but there is no Lodato extension, for the same reason as in 5.13.

b) Let $X = \mathbb{N} \times [0,\to[$, $X_0 = \mathbb{N}\times]0,\to[$. Take the Euclidean proximity δ on X, and let the following covers ($n \in \mathbb{N}$) constitute a base for M_0 on X_0:

$$\big\{\{k\}\times]y, y+\tfrac{1}{n}[: k \in \mathbb{N}, y > 0\big\} \cup \big\{\{k\}\times]0, \tfrac{1}{\max\{k,n\}}[: k \in \mathbb{N}\big\}.$$

Now (a) holds, $\mathcal{U}(\mathsf{M}_0)$ and $\Gamma(\mathsf{M}_0)$ have Lodato extensions in (X,δ) (observe that $\mathcal{U}(\mathsf{M}_0) = \mathcal{U}(\mathsf{N}_0)$ and $\Gamma(\mathsf{M}_0) = \Gamma(\mathsf{N}_0)$, where N_0 is the Euclidean merotopy on X_0), but M_0 does not have one, since (b) fails for $A = X_0^r$ and $B = \{(k, 1/k) : k \in \mathbb{N}\}$. $\diamondsuit$

5.19 Condition (iii) is not superfluous in Lemma 5.17:

Examples. a) Let $X, X_0, X_1, \mathsf{M}_0, \mathsf{M}_1$ be as in Example 3.8, with the following modification: replace $\mathsf{c}_1(\varepsilon)$ by

$$\mathsf{d}_1(\varepsilon) = \mathsf{c}_1(\varepsilon) \cup \big\{(\{1/m, 1/n\}\times]0,\varepsilon[) \cap X_1 : m,n \in \mathbb{N}, m,n > 1/\varepsilon\big\}.$$

Let δ be the Euclidean proximity on X. 5.17 (i) is clearly staisfied.

$\operatorname{int} \mathsf{d}_1(\varepsilon)^0$ is a δ-cover (the modification was needed, because otherwise neither 5.18 (a) nor 5.18 (b) would hold). For $\mathsf{c}_0 \in \mathsf{M}_0$, $\operatorname{int} \mathsf{c}_0^0$ is evidently a δ-cover, since X_0 is closed. M_0 is contigual, so $\operatorname{int} \mathsf{c}_0^0$ is finite for c_0 taken from a base. Hence (ii) holds by Lemma 5.2. The induced semi-uniformities as well as the induced contiguities have an extension (similarly to 3.8, the Euclidean uniformity, respectively the Euclidean contiguity). But M_0 and M_1 do not have a Lodato extension, not even in (X, c), since (iii) is not satisfied for $i = 1$, $j = 0$, $\mathsf{c}_i = \mathsf{d}_1(1)$.

b) There is a much simpler example if we do not insist that the induced semi-uniformities should have a Lodato extension (essentially the same as Example 2.10):

Let $X, X_0, \delta, \mathsf{M}_0$ be as in Example 5.17, $X_1 = X_0^r$, Γ_1 the Euclidean contiguity on X_1, $\mathsf{M}_1 = \mathsf{M}^0(\Gamma_1)$ (cf. 4.1). $\diamondsuit$

5.20 Condition (ii) of Lemma 5.17 cannot be replaced by the weaker assumption that each $\operatorname{int} \mathsf{c}_i^0$ is a δ-cover:

Example. Let $T = \{-1/n, 1/n : n \in \mathbb{N}\}$, $X = T \times \mathbb{R}$, $X_0 = T \times]\leftarrow, 0[$, $X_1 = T \times]0, \rightarrow[$. Let δ be the Euclidean proximity on X, and $\{\mathsf{c}_i(\varepsilon) : \varepsilon > 0\}$ a base for M_i on X_i, where

$$\begin{aligned} \mathsf{c}_1(\varepsilon) = \big\{ (]p, p+\varepsilon[\times]q, q+\varepsilon[) \cap X_1 : (p \in \mathbb{R}, q > 0) \text{ or} \\ (0 \notin]p, p+\varepsilon[, q = 0) \big\} \cup \big\{ \{-1/k, 1/n\} \times]0, \varepsilon[: k > n > 1/\varepsilon \big\}, \\ \mathsf{c}_0(\varepsilon) = \big\{ (]p, p+\varepsilon[\times]q-\varepsilon, q[) \cap X_0 : (p \in \mathbb{R}, q < 0) \text{ or} \\ (0 \notin]p, p+\varepsilon[, q = 0) \big\} \cup \big\{ \{-1/k, 1/n\} \times]-\varepsilon, 0[: n > k > 1/\varepsilon \big\}, \end{aligned}$$

(i) and (iii) are fullfilled, the latter because, for $i \neq j$, $\operatorname{int} \mathsf{c}_i^0 | X_j = \{X_j\}$. The weaker form of (ii) holds, but not (ii) itself, since $\operatorname{int} \mathsf{c}_0(1)^0 (\cap) (\cap) \operatorname{int} \mathsf{c}_1(1)^0$ is not a δ-cover: consider $A = \{1/n : n \in \mathbb{N}\} \times \{0\}$ and $B = \{-1/n : n \in \mathbb{N}\} \times \{0\}$. $\diamondsuit$

5.21 In the extension problems we have discussed up to now, a family of structures could be extended iff each subfamily of cardinality ≤ 2 had an extension. We do not know whether this holds for Lodato extensions of merotopies in a proximity space.

5.22 Theorem. *A family of Lodato merotopies given on closed subsets in a Lodato proximity space has Lodato extensions;* $\mathsf{M}^0 = \mathsf{M}_L^0$ *is the coarsest one.*

Proof. M^0 is the coarsest extension by Theorem 5.4. M^0 is Lodato, since c_i^0 is refined by $(\operatorname{int}_i \mathsf{c}_i)^0 \in \mathsf{M}^0$, which is an open cover, and $\mathsf{c}_{A,B}$ is refined by the open cover $\mathsf{c}_{c(A),c(B)} \in \mathsf{M}^0$, thus M^0 has a subbase consisting of open covers. $\mathsf{M}^0 = \mathsf{M}_L^0$ is also clear from this reasoning. $\diamondsuit$

If the subsets are not closed then it is possible that there exist Lodato extensions, but M_L^0 (by Lemma 5.16, the coarsest one) is strictly finer than M^0:

Example. Take $S = \{1/n : n \in \mathbb{N}\}$, $X = S \times (\{0\} \cup S)$, $X_1 = S^2$. Let δ be the Euclidean proximity on X, and $\{\mathsf{f}_1(k) : k \in \mathbb{N}\}$ a subbase for M_1 on X_1, with $\mathsf{f}_1(k)$ from Example 4.5. Now M_L^0 is a Lodato extension, and $\operatorname{int} \mathsf{f}_1(1)^0 \in \mathsf{M}_L^0 \setminus \mathsf{M}_0$. $\diamondsuit$

5.23 Lemma. *If a family of merotopies in a proximity space has a Lodato extension, and the open δ-covers* c *for which* $\mathsf{c}|X_i \in \mathsf{M}_i$ $(i \in I)$ *form a base for a merotopy* M_L^1 *then* M_L^1 *is the finest Lodato extension.* $\diamondsuit$

It follows from this lemma and Theorem 5.10 that if a family of merotopies has a Lodato extension as well as a finest Riesz extension then it has a finest Lodato extension, too; the converse is not true:

Example. With X, P, Q from Example 5.2, let c denote the topological sum of the cofinite topologies on P and Q. Define $A\delta B$ iff either $c(A)\cap \cap c(B) \neq \emptyset$, or $A \cap P$ and $B \cap Q$ are infinite, or $A \cap Q$ and $B \cap P$ are infinite. δ is a Lodato proximity compatible with c. An open cover c is a δ-cover iff there is a $C \in \mathsf{c}$ with C^r finite; if the open covers c and d have this property then so has $\mathsf{c}(\cap)\mathsf{d}$, thus the open δ-covers constitute a base for a merotopy, which is, according to the lemma, the finest compatible Lodato merotopy.

There is, however, no finest compatible Riesz merotopy, because, by Theorem 5.10, such a merotopy would contain c and d from Example 5.2; but $\mathsf{c}(\cap)\mathsf{d}$ is clearly not a δ-cover, a contradiction. $\diamondsuit$

Problem. Assume that there exists a finest Lodato extension; is it necessarily of the form given in Lemma? (The answer is yes if each X_i is closed: repeat the reasoning from the second paragraph of the proof of Theorem 5.5, considering only c-open, respectively c_i-open covers; if c_i is c_i-open and X_i is closed then c_i^0 is c-open.)

5.24 We need a measurable cardinal in the construction of a proximity space in which the finest compatible Lodato and Riesz merotopies exist but differ (compare with the very simple examples in 5.11):

Example. Let Y be the set of the rationals, Z a set of measurable cardinality, $Y \cap Z = \emptyset$, $X = Y \cup Z$, u a free ultrafilter on Z such that $\cap \mathsf{v} \in \mathsf{u}$ whenever $\mathsf{v} \subset \mathsf{u}$ is countable (see e.g. [4] 12.2). Let c denote the sum of the Euclidean topology on Y and the discrete one on Z. Define $A\delta B$ iff either $c(A) \cap c(B) \neq \emptyset$, or $A \cap Y$ is infinite and $B \cap Z \in \mathsf{u}$, or $B \cap Y$ is infinite and $A \cap Z \in \mathsf{u}$. δ is a Lodato proximity compatible with c. Let c and d be δ-covers for which $\operatorname{int}\mathsf{c}$ and $\operatorname{int}\mathsf{d}$ are covers. Evidently, $\operatorname{int}(\mathsf{c}(\cap)\mathsf{d})$ is also a cover. We are going to show that $\mathsf{c}(\cap)\mathsf{d}$ is a δ-cover; then Theorem 5.10 yields that there exists the finest compatible Riesz merotopy $\mathsf{M}_R^1(\delta)$, implying the existence of the finest compatible Lodato merotopy $\mathsf{M}_L^1(\delta)$.

Given near sets A and B, we need $C \in \mathsf{c}$ and $D \in \mathsf{d}$ such that

$$(1) \qquad A \cap C \cap D \neq \emptyset \neq B \cap C \cap D.$$

If there is a point $x \in c(A) \cap c(B)$ then, as $\operatorname{int}\mathsf{c}$ and $\operatorname{int}\mathsf{d}$ are covers, C and D can be chosen such that $x \in \operatorname{int} C \cap \operatorname{int} D$, and then (1) clearly holds. So we may assume without loss of generality that $A \subset Y$ and $B \subset Z$, A is infinite and $B \in \mathsf{u}$.

We shall define by recursion sets $A_n \subset A_1 = A$, $B_n \subset B_1 = B$ satisfying $A_n \delta B_n$, and points $x_n \in A$ ($n \in \mathbb{N}$). If A_n and B_n are defined then consider the sets $B_n \setminus \mathrm{St}(x, \mathsf{c})$ for $x \in A_n$. If all these sets belonged to u then we would have $E = B_n \setminus \mathrm{St}(A_n, \mathsf{c}) \in \mathsf{u}$; now $A_n \delta E$, contradicting the assumption that c is a δ-cover. Hence there is an $x_n \in A_n$ such that $Z \cap \mathrm{St}(x_n, c) \in \mathsf{u}$; define now $A_{n+1} = A_n \setminus \{x_n\}$ and $B_{n+1} = B_n \cap \mathrm{St}(x_n, \mathsf{c})$; clearly, $A_{n+1} \delta B_{n+1}$, and the points x_n are different. Take $H = \{x_n : n \in \mathbb{N}\}$ and $K = \bigcap_{n \in \mathbb{N}} B_n$; then $H \delta K$, and

$$\mathrm{St}(y, \mathsf{c}) \supset K \qquad (y \in H). \tag{2}$$

d being a δ-cover, there is a $D \in \mathsf{d}$ such that $D \cap H \neq \emptyset \neq D \cap K$. Taking points $y \in D \cap H$ and $z \in D \cap K$, (2) implies that $y, z \in C$ for some $C \in \mathsf{c}$, i.e. (1) holds indeed.

Consider the cover

$$\mathsf{e} = \{Y\} \cup \big\{\{y\} \cup Z : y \in Y\big\}.$$

$\mathrm{int}\, \mathsf{e} = \{Y, Z\}$ is a cover, and e is a δ-cover, thus $\mathsf{e} \in \mathsf{M}^1_R(\delta)$ by Theorem 5.10. But $\mathsf{e} \notin \mathsf{M}^1_L(\delta)$, since $\mathrm{int}\, \mathsf{e}$ is not a δ-cover. Hence $\mathsf{M}^1_R(\delta) \neq$ $\neq \mathsf{M}^1_L(\delta)$. ◇

Problem. Is there a similar example in ZFC, or at least in a consistent model of ZFC? (Perhaps there exists such an example only with $I \neq \emptyset$.)

5.25 It follows easily from the definition that under the conditions of Definition 5.14,

$$\mathsf{M}^0_L = \sup_{i \in I} \mathsf{M}^0_L(\delta, \{\mathsf{M}_i\}) \tag{1}$$

holds for $I \neq \emptyset$. (1) cannot be deduced from 2.2 a) 1° in such generality, since it holds only for $p = q = 1$ that M^0_L is always a pq-overextension (see the last paragraph in 5.14), but it is not the coarsest one (Example 5.16). We can, however, generalize 2.2 a) 1° to meet the present situation (with $p = q = 1$; cf. Lemma 5.16): let us require in the definition of a pq-overextension that d should satisfy a property inherited by suprema of non-empty collections. (The **C**-structure on X is allowed to figure in the property.)

5.26 Statements similar to those in 5.7 and 5.8 hold for Lodato extensions, too. It should be mentioned that extending a family of merotopies in a closure space in two steps is now even more problematic (because Lodato merotopies behave badly in a proximity space), e.g. Corollary 3.8 can be obtained this way only for closed subsets, and not for open ones.

6. Extending a family of contiguities in a proximity space

A. WITHOUT SEPARATION AXIOMS

6.1 A family of contiguities in a proximity space always has extensions; this will be deduced from the corresponding result for merotopies, using the method of § 4. We shall utilize the facts mentioned in the second paragraph of 4.1. Only the coarsest extension can be obtained this way, although there exists a finest one, too; its existence can be proved easily: take the supremum Γ of all the extensions (i.e. their union is a subbase for Γ); now Γ is compatible by the lemma below, and $\Gamma|X_i = \Gamma_i$ is evident. This proof is, however, superfluous, since we shall construct the finest extension.

Lemma. *For a contiguity Γ on X, $\delta(\Gamma)$ is coarser than δ iff every $\mathsf{f} \in \Gamma$ is a δ-cover iff Γ has a subbase consisting of δ-covers.*

Proof. The statement on subbases follows from Lemma 5.2. ◇

6.2 Definition. For a family of contiguities in a proximity space,

a) Let Γ^0 be the contiguity for which the following covers form a subbase: f_i^0 ($i \in I$, $\mathsf{f}_i \in \Gamma_i$) and $\mathsf{c}_{A,B}$ ($A\bar{\delta}B$).

b) Let Γ^1 consist of those finite δ-covers f for which $\mathsf{f}|X_i \in \Gamma_i$ ($i \in I$). ◇

Clearly, $\Gamma^0 = \Gamma(\mathsf{M}^0(\delta, \mathsf{M}^0(\Gamma_i)))$.

Theorem. *A family of contiguities in a proximity space always has extensions. Γ^0 is the coarsest, and Γ^1 the finest extension.*

Remark. A direct proof not making use of Theorem 5.4 would be much simpler than the proof of that theorem, since, the covers being finite, the argument in 5.4 2° can be replaced by applying Lemma 5.2 (or 6.1).

Proof. Γ^0 is an extension by Theorem 5.4. If Γ is another extension then $\mathsf{M}^0(\Gamma)$ is an extension of the merotopies $\mathsf{M}^0(\Gamma_i)$, hence $\mathsf{M}^0(\delta, \mathsf{M}^0(\Gamma_i)) \subset \mathsf{M}^0(\Gamma)$, thus $\Gamma^0 \subset \Gamma$. If $\mathsf{f} \in \Gamma$ then it satisfies the conditions in Part b) of the definition, so $\mathsf{f} \in \Gamma^1$, and therefore $\Gamma \subset \Gamma^1$; in particular, $\Gamma^0 \subset \Gamma^1$, implying that $c(\Gamma^1)$ is finer than c and $\Gamma_i \subset \Gamma^1|X_i$. Conversely, $c(\Gamma^1)$ is coarser than c by Lemma 6.1, and $\Gamma^1|X_i \subset \Gamma_i$ is evident from the definition. Thus Γ^1 is indeed the finest extension. ◇

6.3 Γ^0 and Γ^1 are different in general: let $|X| = 3$, $I = \emptyset$ and δ the indiscrete proximity on X. Γ^0 and Γ^1 can, in fact, coincide only under very strong assumptions: $\Gamma^0(\delta) = \Gamma^1(\delta)$ iff each δ-compressed filter is the intersection of at most two ultrafilters; this will be proved in [3], along with the following results: all the δ-covers of cardinality ≤ 3 form a subbase for $\Gamma^1(\delta)$; if proximities $\delta[i]$ ($i \in I \neq \emptyset$) are given on the same set then

$$\sup_{i\in I} \Gamma^1(\delta[i]) = \Gamma^1(\sup_{i\in I} \delta[i]).$$

6.4 The analogue for contiguities of 5.6 (1) and a similar formula for Γ^1 follow easily from 2.2 a).

Statements corresponding to 5.7 and 5.8 are also valid; things are simplified by the existence of a finest extension. Only one point is worth going into: the formulas

$$(1) \qquad \delta^k(c, \delta_i) = \delta(\Gamma^k(c, \Gamma^0(\delta_i))) \qquad (k = 0, 1)$$

remain valid if we substitute $\Gamma^1(\delta_i)$ for $\Gamma^0(\delta_i)$. The formulas make sense, because the contiguities $\Gamma^1(\delta_i)$ are accordant. It follows from (1) that $\delta(\Gamma^1(c, \Gamma^1(\delta_i)))$ is finer than $\delta^1(c, \delta_i)$, so they are the same, as the latter is the finest extension, and the former is an extension, too. Concerning the case $k = 0$, observe that $\Gamma^0(c, \Gamma^1(\delta_i)) \subset \Gamma^1(\delta^0(c, \delta_i))$, since (see Definition 4.1 a)) $\mathsf{c}_{x,B}$ belongs to any contiguity compatible with c, while if f_j is a finite δ_j-cover then f_j^0 is a finite $\delta^0(c, \delta_i)$-cover; hence

$$\delta(\Gamma^0(c, \Gamma^1(\delta_i))) \supset \delta^0(c, \delta_i) = \delta(\Gamma^0(c, \Gamma^0(\delta_i))) \supset \delta(\Gamma^0(c, \Gamma^1(\delta_i))).$$

B. RIESZ CONTIGUITIES IN A PROXIMITY SPACE

6.5 Definition. For a family of contiguities in a proximity space, let

$$\Gamma^1_R = \{\mathsf{f} \in \Gamma^1 : \operatorname{int}\mathsf{f} \text{ is a cover of } X\}. \ \diamondsuit$$

(The same definition was used in a closure space, with a different meaning of Γ^1, of course, see 4.2.)

Theorem. *A family of contiguities in a proximity space has a Riesz extension iff the proximity is Riesz and the trace filters are Cauchy; if so then Γ^0 is the coarsest and Γ^1_R the finest Riesz extension.*

Proof. In view of Theorem 5.9, it is enough to show that Γ^1_R is the finest Riesz extension. If Γ is a Riesz extension then $\Gamma \subset \Gamma^1$ by Theorem 6.2, so $\Gamma \subset \Gamma^1_R$ follows from the definition. In particular, $\Gamma^0 \subset \Gamma^1_R$; on the other hand, $\Gamma^1_R \subset \Gamma^1$ is evident, thus Γ^1_R is an extension by Theorem 6.2. Γ^1_R is clearly Riesz, and we have already seen that it is finer than any other Riesz extension. $\diamondsuit$

Γ^0, Γ^1_R and Γ^1 can be different:

Example. Let δ be the Euclidean proximity on $X = \mathrm{R}\setminus\{0\}$. Denote by Q and D the set of the rationals, respectively dyadic rationals, in X. Now

$$\mathsf{f} = \{Q, D^r, D \cup Q^r\} \in \Gamma^1(\delta)\setminus\Gamma^1_R(\delta),$$
$$\mathsf{f}' = \{]\leftarrow, 0[,]0, \rightarrow[\} \cup \mathsf{f} \in \Gamma^1_R(\delta)\setminus\Gamma^0(\delta). \ \diamondsuit$$

6.6 It follows from 2.2 a) 3° and 4° that, under the assumptions of Theorem 6.5,

(1) $$\Gamma^1_R = \inf_{i\in I} \Gamma^1_R(\delta, \{\Gamma_i\}) = \inf\{\Gamma^1_R(\delta), \inf_{i\in I} \Gamma^{11}[i]\},$$

where $\Gamma^{11}[i]$ is the finest contiguity (= the finest Riesz contiguity) Γ on X for which $\Gamma|X_i = \Gamma_i$, i.e. $\Gamma^{11}[i]$ consists of all those finite covers f of X for which $\mathsf{f}|X_i \in \Gamma_i$. ($\Gamma^{11}[i]$ is Riesz because $\operatorname{int}'A = (A\setminus X_i) \cup \operatorname{int}_i(A \cap \cap X_i)$, where int' is to be understood in $c(\Gamma^{11}[i])$.) (1) is in fact obtained with inf taken in the category of Riesz contiguities, but this coincides with inf in the category of contiguities, assuming that there exists a coarsest one among the closures induced by the contiguities considered. (And observe that $\delta(\Gamma^{11}[i]) \subset \delta$, implying that $c(\Gamma^{11}[i])$ is finer than $c = c(\Gamma^1_R(\delta))$.)

6.7 The finest Riesz extension of a family of contiguities in a closure space can be obtained in two steps, cf. 5.12 (1) (but now the existence of a finest extension can in fact be *proved* in two steps):

(1) $$\Gamma^1_R(c, \Gamma_i) = \Gamma^1_R(\delta^1_R(c, \delta(\Gamma_i)), \Gamma_i).$$

Conversely, if we have a family of proximities in a weakly separated closure space such that the trace filters are compressed then it follows from 5.12 (2) that

$$\delta_R^1(c,\delta_i) = \delta(\Gamma_R^1(c,\Gamma^0(\delta_i))). \tag{2}$$

If we try to replace here $\Gamma^0(\delta_i)$ by $\Gamma_R^1(\delta_i)$ (cf. 6.4) then the trace filters are not necessarily Cauchy, thus $\Gamma_R^1(c,\Gamma_R^1(\delta_i))$ is not a Riesz extension (in fact, not an extension at all, see the example below); all the same, (2) remains valid even with $\Gamma^1(\delta_i)$, since $\Gamma_R^1(c,\Gamma^0(\delta_i))$ and $\Gamma_R^1(c,\Gamma^1(\delta_i))$ induce the same proximity (using Definition 4.2, check that if $\mathsf{f} \in \Gamma_R^1(c,\Gamma^1(\delta_i))$ and $|\mathsf{f}| = 2$ then $\mathsf{f} \in \Gamma_R^1(c,\Gamma^0(\delta_i))$); and $\Gamma_R^1(c,\Gamma_R^1(\delta_i)) = \Gamma_R^1(c,\Gamma^1(\delta_i))$.

Example. Let $X = \mathbb{N}$, $X_0 = \{1\}^r$, $S \in \mathsf{v}(1)$ iff $1 \in S$ and S^r is finite, and let the other points be isolated in c. For disjoint $A, B \subset X_0$, define $A\delta_0 B$ iff A and B are infinite. Take disjoint infinite sets $A, B, C \subset$ $\subset X_0$. Now $\mathsf{f}_0 = \{X_0 \setminus A, X_0 \setminus B, X_0 \setminus C\} \in \Gamma_R^1(\delta_0)$, so the δ_0-compressed filter $\mathsf{s}_0(1)$ is not $\Gamma_R^1(\delta_0)$-Cauchy, because $\mathsf{s}_0(1) \cap \mathsf{f}_0 = \emptyset$. Moreover, $\Gamma_R^1(c,\Gamma_R^1(\delta_0))$ is not an extension, since if f belongs to it then $1 \in \bigcup \operatorname{int} \mathsf{f}$ implies that $\mathsf{f}|X_0$ contains a cofinite set, i.e. $\mathsf{f}|X_0 \neq \mathsf{f}_0$. ◊

C. LODATO CONTIGUITIES IN A PROXIMITY SPACE

6.8 Definition. For a family of contiguities in a proximity space,

a) Let $\Gamma_L^1 = \{\mathsf{f} \in \Gamma^1 : \operatorname{int} \mathsf{f} \in \Gamma^1\}$.

b) Assuming that the proximity and the contiguities are Lodato and the trace filters are Cauchy, let Γ_L^0 be the contiguity on X for which $\{\operatorname{int} \mathsf{f} : \mathsf{f} \in \Gamma^0\}$ is a base. ◊

Observe that $\Gamma_L^0 = \Gamma(\mathsf{M}_L^0(\delta, \mathsf{M}^0(\Gamma_i)))$. A subbase for Γ_L^0 can be described similarly to 5.14 (1) – (2). If c is a topology then the c-open covers in Γ^1 form a base for Γ_L^1.

Lemma. *A family of contiguities in a proximity space has a Lodato extension iff*

(i) *the proximity and the contiguities are Lodato;*

(ii) $\operatorname{int} \mathsf{f}_i^0$ *is a* δ*-cover* $(i \in I, \mathsf{f}_i \in \Gamma_i)$;

(iii) $(\operatorname{int} \mathsf{f}_i^0)|X_j \in \Gamma_j$ $(i, j \in I, \mathsf{f}_i \in \Gamma_i)$.

If these conditions are satisfied then Γ_L^0 *is the coarsest and* Γ_L^1 *the finest extension.*

Proof. It follows from Lemmas 5.2, 5.17 and 5.16 that the conditions are necessary and sufficient, and Γ^0_L is the coarsest extension.

Assume that Γ is a Lodato extension, and $\mathfrak{f} \in \Gamma$. Then $\operatorname{int} \mathfrak{f} \in \Gamma$, so $\operatorname{int} \mathfrak{f} \in \Gamma^1$ by Theorem 6.2, therefore $\mathfrak{f} \in \Gamma^1_L$, i.e. $\Gamma \subset \Gamma^1_L$. This means that if Γ^1_L is a Lodato extension then it can only be the finest one. Taking $\Gamma = \Gamma^0_L$, we have $\Gamma^0_L \subset \Gamma^1_L$, and $\Gamma^1_L \subset \Gamma^1$ by the definition; hence Γ^1_L is an extension, and, being compatible, it is clearly Lodato. ♢

Condition (iii) is not superfluous: take the contiguities from Example 4.5, with the Euclidean proximity on X. Condition (ii) can be, similarly to 5.18 (a) and (b), decomposed into two parts, neither of which is sufficient in itself (although either implies that the trace filters are Cauchy, see in 5.18):

Examples. a) Taking X, X_1 from Example 4.5, with the Euclidean proximity on X, we modify Γ_1 by interchanging the role of the coordinates, and adding one more member to the covers in the subbase: let $\{\mathfrak{f}_1(k) : k \in \mathbb{N}\}$ be a subbase for Γ_1, where

$$\begin{aligned}\mathfrak{f}_1(k) = \{\{(1/m, 1/n) : m, n \geq k, n \not\equiv \mu(\text{mod}3)\} : \mu = 0,1,2\} \cup \\ \cup\{\{(1/m,1/n) : n \geq k\} : m < k\} \cup \{\{(1/m,1/n) : m \geq k\} : n < k\} \cup \\ \cup\{\{(1/m,1/n)\} : m,n < k\} \cup \{\{(1/m,1/n) : n > m \geq k\}\}.\end{aligned}$$

Now the last member in the definition of $\mathfrak{f}_1(k)$ guarantees that the condition analogous to 5.18 (a) is satisfied. But (b) fails: take $\mathfrak{c}_1 = \mathfrak{f}_1(1)$, $A = X_1^\tau$ and $B = \{(1/n, 1/n) : n \in \mathbb{N}\}$.

b) Let $X = (\mathbb{R}\setminus\{0\}) \times \mathbb{R}$, $X_0 = (\mathbb{R}\setminus\{0\})^2$, δ the Euclidean proximity on X, $S_1 =]\leftarrow, 0[$, $S_2 =]0, \rightarrow[$,

$$\mathfrak{e}_0 = \{X_0\setminus(S_u \times S_v) : u = 1,2,\ v = 1,2\},$$

and $\Gamma^0(\delta_0) \cup \{\mathfrak{e}_0\}$ a subbase for Γ_0. $\Gamma^0(\delta_0)$ is compatible and Lodato (Lemma 5.15), and $\mathfrak{e}_0$ is a c_0-open δ_0-cover, so Γ_0 is a compatible Lodato contiguity by Lemma 6.1. Now $\mathfrak{e}_0$, $S_1 \times \{0\}$ and $S_2 \times \{0\}$ show that (a) is not fullfilled. But (b) holds:

We may assume (by Axiom P5, and for reasons of symmetry) that $A \subset S_2 \times \{0\}$ and $B \subset (S_1 \cup S_2) \times S_2$. Take $\mathfrak{f}_0 \in \Gamma^0(\delta_0)$ such that $\mathfrak{f}_0(\cap)\mathfrak{e}_0$ refines the prescribed $\mathfrak{c}_0 \in \Gamma_0$. As $\Gamma^0(\delta)$ is a Lodato extension of $\Gamma^0(\delta_0)$, (b) holds with $\mathfrak{f}_0$ instead of $\mathfrak{c}_0$, thus we can pick $F_0 \in \mathfrak{f}_0$ and $x \in A$ such that $F_0 \in \mathfrak{s}_0(x)$ and $F_0 \cap B \neq \emptyset$. Now with $C_0 = F_0 \cap (X_0\setminus S_1^2) \in \mathfrak{f}_0(\cap)\mathfrak{e}_0$ we have $C_0 \in \mathfrak{s}_0(x)$ and $C_0 \cap B \neq \emptyset$ (since $B \subset (X_0\setminus S_1^2)$); hence (b) holds with $\mathfrak{f}_0(\cap)\mathfrak{e}_0$, therefore also with $\mathfrak{c}_0$. ♢

These examples could also have been used in 5.18, had we not made in § 5C a point of requiring that the induced contiguities and semi-uniformities should have Lodato extensions whenever possible.
Corollary. *A family of contiguities in a Lodato proximity space has a Lodato extension iff* $\{\Gamma_i, \Gamma_j\}$ *has a Lodato extension for any* $i, j \in I$. ◇
Compare this corollary with 5.21.

6.9 Corollary. *A family of contiguities in a Lodato proximity space has a Lodato extension iff it has a Lodato extension in* (X, c) *and each* $\{\Gamma_i\}$ *has a Lodato extension in* (X, δ).
Proof. Lemma 6.8 and Theorem 4.3. ◇

6.10 Lemma. *Under the assumptions of Definition 6.8 b), a family of contiguities in a proximity space has a Lodato extension iff* $\Gamma^0_L \subset \Gamma^1_L$.
Proof. The necessity follows from the last statement in Lemma 6.8. Conversely, assume that $\Gamma^0_L \subset \Gamma^1_L$. It is clear from the definitions that $\Gamma^1_L \subset \Gamma^0_L$ and $\Gamma^1_L \subset \Gamma^1$, hence Γ^1_L is an extension by Theorem 6.2; Γ^1_L is Lodato, because c is a topology. ◇

6.11 Theorem. *A family of Lodato contiguities given on closed subsets in a Lodato proximity space has Lodato extensions;* $\Gamma^0 = \Gamma^0_L$ *is the coarsest and* Γ^1_L *the finest Lodato extension.*
Proof. Theorem 5.22 and Lemma 6.8. ◇

Γ^0 and Γ^0_L can be different if the subsets are not closed: take X, X_1 and Γ_1 from Example 4.5, with the Euclidean proximity of X (cf. Example 5.22). $(\Gamma^0(\delta) =)\Gamma^0_L(\delta) \neq \Gamma^1_L(\delta)$ for δ from Example 5.2: if A, B, $C \subset X$ are disjoint infinite sets then $\mathfrak{f} = \{A^r, B^r, C^r\}$ is clearly a finite open δ-cover, so $\mathfrak{f} \in \Gamma^1_L(\delta)$; but $\mathfrak{f} \notin \Gamma^0(\delta)$, since each cover $\mathfrak{c}_{P,Q}$ $(P\bar{\delta}Q)$, and so each element of $\Gamma^0(\delta)$, contains at least one cofinite set. (The result cited in 6.3 could also be used, since c is discrete, and so $\Gamma^1_L(\delta) = \Gamma^1(\delta)$.) In Example 6.5, $\mathfrak{f}' \in \Gamma^1_R(\delta)\backslash\Gamma^1_L(\delta)$; $\Gamma^1_R(\delta)$ and $\Gamma^1(\delta)$ were different in the same example.

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ON REFLEXIVE SHEAVES WITH LOW SECTIONAL GENERA ON THREEFOLDS

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Abstract: Here we define the sectional genus $g(F)$ of a reflexive sheaf F over a projective variety V. Here we classify (char 0) all such pairs (V,F) with $\dim(V)=3$, V smooth, F not locally free but curvilinear, F ample and spanned, and $2g(F)-2\leq c_3 F$; we have $V\cong\mathbf{P}^3$, all such F are described explicitely and the set of such F is parametrized by $\mathbf{P}^3$.

We work over an algebraically closed field $\mathbf{K}$ with $\mathrm{char}(\mathbf{K})=0$. Fix a complete variety V; set $n=\dim(V)$; here we need only the case $n=3$. Recall that a coherent sheaf F on V is called reflexive if the natural map from F to its double dual F^{**} is an isomorphism; this is the case for instance if F is locally free, but it happens in several other interesting cases: see [6] for the background, motivation (i.e. their link with space curves) and the general theory of reflexive sheaves. We fix a rank$-n(n-1)$ reflexive sheaf E on V. We say (as usual) that E is ample if the tautological line bundle $O_{\mathbf{P}}(E)(1)$ on $\mathbf{P}(E)$ is ample. From now on we will assume that E is spanned by its global sections and that the set S of points of V at which E is not locally free is finite (if $n=3$ the last condition is automatically statisfied ([6], 1.4)). Then (since char(**K**)= 0 and S is finite) a standard form of Bertini theorem (same proof as for $n = 3$ ([6]), which in turn is essentially a standard form of Bertini for spanned vector bundles on $V\backslash S$) gives that for a general $s \in H^0(V, E)$, its zero-locus $C := (s)_0$ is a pure-dimensional curve which is smooth outside S (and C must contain S, as shown in [6], Th. 4.1); if V is smooth around S, C is locally Cohen-Macaulay; since C is generically reduced by the finiteness of S, C is reduced; set $g := p_a(C)$. We will call g the sectional genus of E (see the introduction of [1] for a discussionn of why (in the locally free case) among many other competing ones, this is a very natural and useful definition). But we do not claim or use that if E is ample the general such C is integral of at least connected; in particular we will consider also the case "$g < 0$" (which often will be easily shown to lead to a contradiction). From now on in this paper we assume V smooth. In particular $K_0(V) \cong K^0(V)$ and the Chern classes are defined for all coherent sheaves on V. Set $L := c_1(E)$; L is a line bundle; note also that $C = (s)_0$ represents $c_{n-1}(E)$ (exactly as in the locally free case, to which, when S is finite, it could be easily reduced). From now on we assume $n = 3$. The aim of this paper is the proof of Theorem 0.1 below. To clarify its statement we need the following "adjunction formula" ([6], th. 4.1) valid for any section s with $C := (s)_0$ of dimension 1 and arithmetic genus g:

$$2g - 2 = (K_V + L)C + c_3(E) \tag{1}$$

It is known (see [6], Prop. 2.6) that $c_3(E) \geq 0$, $c_3(E) > 0$ if and only if $S \neq \emptyset$, and that indeed $c_3(E)$ is a very good measure of the "number" of singularities of E. A reflexive sheaf E on V is called curvilinear if for each $P \in S$ there are formal parameters x, y, z of the completion A of $O_{V,P}$ such that the completion of the stalk of E at P is isomorphic to Coker(j), where $j : A \to 3A$ is defined by: $j(u) = (xu, yu, zy)$; by [6], 4.1.1, if there is $s \in H^0(E)$ with as $(s)_0$ a smooth curve, then E is curvilinear; by [3], Prop. 4, if E is curvilinear and spanned, a general $s \in H^0(E)$ has $(s)_0$ smooth.

Theorem 0.1. *Let V be a smooth complete variety with* $\dim(V) = 3$, *and E an ample spanned* rank $- 2$ *reflexive sheaf on V with sectional genus g; set $c_i = c_i(E)$. Assume $2g - 2 \leq c_3$ and E not locally free. Then there are exactly* 4 *families of (V, E):*

(i) $V \cong \mathbf{P}^3$, $L \cong O(3)$, $g = 0$, $c_3 = 1$ *and E is described in the following way.* Fix$P \in \mathbf{P}^3$ *and consider homogeneous coordinates $x_0, \ldots, x_3$, such*

that $P = (1;0;0;0)$. Let E be the cokernel of the map $j : O_V \to 3O_V(1)$ given by $j(c) = (cx_1, cx_2, cx_3)$; then E is a solution; any solution with these invariants differs from E by the action of an element of $\mathrm{Aut}(\mathbf{P}^3)$*; any two solutions are isomorphic if and only if they have the same singular set, P.*
(ii) $V \cong \mathbf{P}^3$, $L \cong O(4)$, $c_2 = 7$, $c_3 = 2g - 2 = 8$; *furthermore $h^0(E(-1)) \neq 0$, and, for general E, a general $m \in H^0(E(-1))$ has as $(m)_0$ the complete intersection of 2 quadrics.*
(iii) $V \cong \mathbf{P}^3$, $L \cong O(4)$, $c_2 = 6$, $c_3 = 2g - 2 = 4$; *furthermore $h^0(E(-1)) \neq 0$ and, for general E, a general $m \in H^0(E(-1))$, has as $(m)_0$ a rational normal curve.*
(iv) *V is a smooth quadric in $\mathbf{P}^4$, $L \cong O(3)$, $c_2 = 6$, $c_3 = 2g - 2 = 2$; furthermore $h^0(E(-1)) \neq 0$ and a general $m \in H^0(E(-1))$ has as $(m)_0$ a plane conic.*

In particular the space of solutions in case (i) is parametrized by $\mathbf{P}^3$. We will give some more informations on the possible sheaves E's in cases (ii), (iii) and (iv) in §2, repsectively in (β) case 5) (β), case 5), and (γ); here suffice to say that by the general recipe in [6], Th. 4.1, the datum $(m)_0$ (plus a suitable divisor on $(m)_0$) is sufficient to reconstruct E.
At the end of the paper we discuss briefly the case E not "curvilinear".

The proof of 0.1 depends heavily on [1] (which in turn depends on [11], hence on Mori's theory and its applications ([8], plus the classifications of Fano threefolds due to Iskovskih and Mori-Mukai); both [1] and the present paper we inspired by [11].

The paper is dedicated to the memory of Giorgio Gamberni.

1. Fix a smooth, irreducible, complete variety V and a rank-2 ample reflexive sheaf E on V; assume that E is spanned by its global sections. Let $S \subset V$ be the set of points of V at which E is not locally free. We will always assume $S \neq \emptyset$ (i.e. $c_3(E) > 0$). Set $L := c_1(E)$ and $c_i = c_i(E)$. All these notations will be always assumed, even if not explicitly stated. We write O and K instead of O_V and K_V; for a closed subscheme A of V, I_A will denote its ideal sheaf. For any sheaf G on V, we write $H^i(A)$ and $h^i(A)$ instead of $H^i(V, A)$ and $h^i(V, A)$. Fix a general $s \in H^0(E)$; set $C := (s)_0$. By the assumptions (since $\mathrm{char}(\mathbf{K}) = 0$) C is a reduced pure dimensional curve which is smooth outside $S \cap C$. Furthermore (e.g. [6], proof of Th. 4.1) $S \subset C$. The choice of s induces an exact sequence:

$$0 \to O \to E \to L \otimes I_C \to 0. \tag{2}$$

C will always denote $(s)_0$ with $s \in H^0(E)$, s general enough; thus s will give (2) and C will satisfy (2).

Remark 1.1. *Let T be a closed subvariety of V and $\pi : T' \to T$ a finite morphism. Then every quotient sheaf of $\pi^*((E|T))$ is ample.* Proof. By definition of ampleness and Grothendieck's definition of P we are reduced to the known case in which E is a line bundle. ◇

Lemma 1.2. *Fix an integral curve $T \subset V$ and let $\pi : T' \to T$ be its normalization. Then $\deg(\pi^*(E/T) \geq 2$ and we have equality if and only if $T \cong P^1$, $T \cap S = \emptyset$ and E/T is the direct sum of two line bundles of degree 1.*

Proof. The "if" part is obvious. Set $F := \pi^*(E|T)$. Since T' is smooth, F is the direct sum of a rank-2 locally free sheaf F' and a torsion sheaf F'', with $F'' = 0$ if $T \cap S = \emptyset$. By 1.1 and the fact that π is finite, we get that F' is ample. By construction F' is spanned. Thus if $p_a(T') > 0$, $\deg(F) \geq \deg(F') \geq 3$. Assume $p_a(T') = 0$. By [11], 3.2.1, $\deg(F') \geq 2$ and $\deg(F') = 2$ if and only if F' is the direct sum of two line bundles of degree one. Note that if $T \cap S \neq \emptyset$, we have $F'' \neq \emptyset$ (hence $\deg(F) > \deg(F') \geq 2$) because a coherent sheaf on a reduced variety is locally free if its fibers have constant dimension. Thus we may assume also $T \cap S = \emptyset$ i.e. E locally free near T. Then the proof of [11], 3.2.1, gives that T is smooth. ◇

Remark 1.3. *Under the assumption of* 1.2 *and with the notations* (2), *assume* $\dim(T \cap C) = 0$*; then LT* (i.e. $\deg(\pi^*(L))$) *is at least* $1+$ $+\mathrm{card}(C \cap T)$. *Furthermore if T is smooth around $S \cap T$, $LT \geq 1+$* $+\mathrm{length}(C \cap T)$. Proof. Look at (2) and note that $S \subset C$. Restrict (2) to T and pull it back by π^*; the first map in the corresponding sequence (2)' is again injective because $O_{T'}$ has no torsion. There is a map $j : \pi^*(I_{C,V} \otimes L \otimes O_T) \to O_{T'}$ whose image defines the ideal sheaf of a non-negative divisor $\mathbf{a}$, $\mathrm{supp}(\mathbf{a})$ must contain every point of $\pi^{-1}(C \cap T)$. Since by (2)' and 1.1 $\pi^*(L|T)(-\mathbf{a})$ is ample, we get the first part of first inequality. The last inequality follows from the fact that $\mathbf{a} = T \cap C$ as schemes, 1.2 and (2)'. ◇

Remark 1.4. *If $T \subset V$ is a curve, then $LT \geq 2$; indeed if $T \cap S = \emptyset$, this is* [1], 1.1; *if $T \cap S \neq \emptyset$, then $T \cap C \neq \emptyset$ and* 1.4 *follows from* 1.3.

Remark 1.5. *By* 1.4 *(V,L) is its own reduction in the sense of* [8], 0.11.

To prove the amplenesss of the last 3 families of sheaves in the statement of 0.1, we need a lemma.

Lemma 1.6. *Assume only that E is spanned. E is ample if and only if for every integral curve $T \subset V$, E/T is ample; the last condition is equivalent (if $\pi : T' \to T$ is the normalization) to the ampleness of the locally free part of $\pi^*(E/T)$.*

Proof. A similar crieterion is true for every spanned line bundle on every variety. Since the restriction of the tautological line bundle of $\mathbf{P}(E)$ to every fiber of $P(E) \to V$ is ample, we get the first part. To get second one, note that we may check the ampleness of the tautological line bundle of $\mathbf{P}(E|T)$ after the base-change by π (e.g. see [4], prop. 2.1). $\diamondsuit$

2. We use the notations introduced in §1; we will use heavily the proofs in [1]; thus the reader need a copy of [1] nearby. We assume that E is ample and spanned (unless otherwise stated).

First assume that $K + L$ is semi-ample. Then there is an integer $m > 0$ such that $m(K + L)$ is spanned. Since $(K + L)C \leq 0$ for every C as in (2) and we may find such a C through a general point of V (e.g. count the dimensions and use the spannedness of E and that $c_2(E) \neq \emptyset$) we get $m(K + L) \cong O$. Thus $-K$ is ample by Kleiman numerical critertion of ampleness. Thus V is a Fano 3-fold. Therefore $\mathrm{Pic}(V)$ has no torsion; hence $L = -K$.

First we assume $b_2(V) = 1$, i.e. (for Fano 3-folds) $\mathrm{Pic}(V) \cong \mathbf{Z}$. Let r be the index of V. By 1.4 and [11], 2.3, we have $r \geq 2$. Such V are classified, and we have to check all the possible V as was done in [1].

(α) (case (1) in [1], §1) Now assume $(V, L) = (\mathbf{P}^3, O(3))$. By 1.2 for every line A with $A \cap S = \emptyset$, we have $E|A \cong O_A(2) \oplus O_A(1)$, while for every line D with $D \cap S \neq \emptyset$, $(E|D)/\mathrm{Tors}(E|D) \cong 2\,O(1)$ and $\mathrm{Tors}(E|D)$ has length 1. By the proof of 1.2 we have $\mathrm{card}(S) = 1$; set $\{P\} := S$. Fix a two-dimensional linear subspace M with $M \cap S = \emptyset$. By [10] either $E|M \cong T\mathbf{P}^2$ or $E|M = O(2) \oplus O(1)$. In the second case we get $c_2(E) = 2$, i.e. C is a conic; thus by (2) $h^0(E(-2)) \neq 0$; but every section of $E(-2)$ must vanishes identically on every line trough P, contradiction. Thus we may assume $E|M \cong T\mathbf{P}^2$ for every plane M with $P \notin M$. Thus $c_2(E) = 3$, i.e. $\deg(C) = 3$; since C has no trisecant line by (2) and 1.3, $g \leq 0$; by (1) we have $g = 0$ and $c_3 = 1$. This implies that E has a very mild singularity at P (it is called "convenable" or "suitable": see [7]); suffice to say that this implies "curvilinear" and thus that we may take C smooth without assuming a priori that E is curvilinear). Thus C is a rational normal curve in $\mathbf{P}^3$. Since all the possible configurations of such pairs (C, P) are projectively equivalent, we get the uniqueness of E, up to the action of $\mathrm{Aut}(\mathbf{P}^3)$; the sheaf given in 0.1 is a solution. For this sheaf we have $h^0(E) = 11$, $h^0(E(-1)) = 2$. Thus we see that given 5 general points $P_i \in V$, there is $s \in H^0(E)$ with $\{P, P_1, \ldots, P_5\} \subset (s)_0$; since six points of $\mathbf{P}^3$ in linear general position are contained in a unique rational normal curves, we see that for E all C can occur. Thus we see that E is uniquely determined by $\mathbf{P}^3$. In particular the set of solutions is parametrized by $\mathbf{P}^3$.

(β) Now we assume $(V, L) = (\mathbf{P}^3, O_V(4))$ (hence $c_3(E) = 2g - 2$); fix (2). By (1) we have $2g - 2 = c_3 > 0$. By 1.3 C has no line D, D no component of C, with $\mathrm{length}(D \cap C) \geq 4$ (hence no quadrisecant in the sense of [5]; in particular, since $g > 1$, C spans $\mathbf{P}^3$.

First assume C does not contain a line as irreducible component. We will show that this case not only gives solutions (ii) and (iii) in the statement of 0.1 but also gives in natural way a few (known) classes of interesting (from our point of view) reflexive sheaves. By [5], Prop. 2.4, the fact that there is no quadrisecant line implies that $[(d-2)(d-3)^2(d-4)/12] = [g(d^2 - 7d + 13 - g)/2]$. Since $g > 1$, we get easily (e.g. using some bounds for the arithmetic genus of (reducible) curves) that (d, g) has one of the following values: (5,2), (6,3), (6,4), (7,5), (9,10), (9,21); the last case cannot occur since there C has no plane component of degre ≥ 4, hence its genus cannot be so large. Note that if $h^0(I_C(2)) \neq 0$, we get infinitely many quadrisecant lines, except maybe if $(d, g) = (6, 4)$ or (5,2). If $h^0(I_C(3)) \neq 0$ and $h^0(I_C(2)) = 0$, by (2) we have $h^0(E(-1)) \neq 0$ and $h^0(E(-2)) = 0$; thus there is $t \in H^0(E(-1))$ with $\dim((t)_0) = 1$. Set $B := (t)_0$; B may be unreduced; we get an exact sequence

$$0 \to O(1) \to E \to I_B(3) \to 0 \tag{3}$$

Thus we see that E is not ample if B has a trisecant line not contained in B. Note that if $h^0(I_C(3)) > 1$, we have $h^0(I_B(2)) \neq 0$, hence B has a trisecant line (not contained in B!) if $\deg(B) \geq 5$, except maybe if B is union of (multiple) lines on a quadric cone (if $\deg(B) \geq 6$

this case will be checked in cases 3) and 4)). By [6], 2.2 and 4.1, $\deg(B) = c_2(E(-1)) = c_2(E) - c_1(E) + 1^2 = c_2(E) - 3 = \deg(C) - 3$, and $p_a(B)$ is given (in term of (d, g)). Now we check separately each case. We assume always C connected, leaving for part ($\beta 2$) the discussion of what happens in the disconnnected case.

1): $(d, g) = (5, 2)$. By Riemann-Roch C is contained in a quadric. By (2) $h^0(E(-2)) \neq 0$ and $h^0(E(-3)) = 0$. Thus there is $m \in H^0(E)$ with $\dim((m)_0) = 1$. By [6], Cor. 2.2, we have

$$0 \to O(2) \to E \to I_D(2) \to 0 \tag{4}$$

We want to show that $E|D$ is not ample. By [6], Th. 4.1, (4) corresponds to a choice (up to a constant) of $h \in H^0(h, O_T(2))$ i.e. to a degree 2 positive divisor $\mathbf{a}$ on T. Two possibilities: $\mathbf{a}$ is reduced or not. Furthermore, up to the action of $\text{Aut}(\mathbf{P}^3)$, these are the only possibilities for $(D, \mathbf{a})$ hence for $E(-2)$. Thus we see that there are two families of reflexive sheaves, such that for any two elements E and E' of each family, there is $g \in \text{Aut}(\mathbf{P}^3)$ with $E' \cong g^*(E)$; furthermore, since the unreduced divisor is the limit of a flat family of reduced ones, we see that the second family is a limit of the first one. By (2) every sheaf E with $E(-2)$ given by (4) is spanned and its restriction $E|T$ to any curve $T \neq D$ is ample (see the proof of case 5) below). We want to check that $E|D$ is not ample. Assume the contrary. By (4) $E|D$ has a factor $O_D(2)$. To obtain a contradiction, it is sufficient (for degree reason) to check that $E|D$ has a torsion part $\text{Tors}(E|D)$ with $\text{length}(\text{Tors}E|T)) \geq 2$. Indeed this length is exactly 2 and the torsion is isomorphic to $\mathbf{a}$; however by semicontinuity to obtain the inequality it is sufficient to check the case "$\mathbf{a}$ reduced" and show in that case that there is some torsion at each of the points in the support of $\mathbf{a}$. This is obtained tensoring (4) by O_D and make a local homological calculation.

2): $(d, g) = (6, 4)$. C must be a canonical curve, complete intersection of a cubic and a quadric; thus we have again (4) with, now, $\deg(D) = 2$. Thus D has (many) secant lines (even if it is not reduced) and each of them is an obstruction (by (4)) to the ampleness of E.

3): $(d, g) = (9, 10)$. By Riemann-Roch $h^0(I_C(3)) \geq 2$; thus we have (3) with $h^0(I_B(2)) \neq 0$, $\deg(B) = 6$. We claim that B has infinitely many trisecant lines, and in particular a trisecant line not in B, hence an obstruction to the ampleness of E. The claim is obvious except if the quadric A containing B is a quadric cone and B is a union (may be unreduced) of lines. But in this case, looking at the minimal desingularization $F_2 \to A$ of A, we see that B is the intersection of A with a cubic surface (hence all the lines of A are trisecant to B). If E is "convenable" in the sense of [7] (sometime translated as "suitable"), i.e. card$(S) = c_3$), there is another proof for this case.

4): $(d,g) = (7,8)$. Since $h^1(O_C(2)) \leq 1$, we would have $h^0(I_C(2)) \neq 0$. Look at the proof of case 3); if now B is a union of lines in a quadric cone A, B contains the complete intersection of A with a cubic surface; hence there is no such E.

5): $(d,g) = (7,5)$. By Riemann-Roch we have $h^0(I_C(3)) \geq 3$ thus there is (3) with $h^0(I_B(2)) \geq 2$; if there is E, B has no plane component of degree ≥ 3; since $p_a(B) = 1$ by (1) (i.e. [6], 2.2 and 4.1), we get that B must be the complete intersection of 2 quadrics. Vieceversa, starting with such B and $m \in H^0(\omega_B(2)) \cong H^0(O_B(2))$, m vanishing only at finitely many points, by [6], Th. 4.1, we get a reflexive sheaf $E(-1)$ with E given by (3). We get an irreducible family of such bundles and the choice of B and (3) give that any such E is spanned. We check that they are ample, at least for general B, m; we assume that B is irreducible. Fix an integral curve $T \neq B$. Fix a general quadric with $B \subset A$, with T not in A. Note that $T \cap B$ (as scheme) is contained in the scheme $T \cap A$ which is a Cartier divisor on T with degree 2deg(T); by (3) and 1.6, $E|T$ is ample. We have to check the ampleness of $E|B$. Given B, we get E, hence C with $h^0(I_C(3)) \geq 3$; hence given E we may find B' instead of B giving the same E; since we know that $E|B'$ is ample, we know that $E|B$ is ample, too.

6): $(d,g) = (6,3)$. Exactly the same proof as in case (5) shows how to get the family claimed by 0.1. To get the sheaf for simplicity start from an irreducible B, i.e. from a rational normal curve. We note only that, for C smooth, a necessary and sufficient condition for the spannedness of the corresponding E is that $h^0(I_C(2) = 0$; this is known to be equivalent to the fact that C is not hyperelliptic.

(β2) Now we assume that C is not connected; if C contains no line, the quotation of [5] works again and can reduce very much the possible cases. But it is easier to consider all the case simultaneously. If $h^0(V, I_{C,V}(2)) \neq 0$ and $d = 5$, again we have (3) and conclude. Thus, since $g > 1$, we may assume $d > 5$. Again we do not have quadrisecant lines (hence the plane components have low degrees). The trick is to fix one or more components which together have a 1-dimensional family of trisecant lines (one can take 3 disjoint lines or two disjoint conics or an irreducible curve of degree d' and genus g' with $(d',g') \neq (3,0)$ and $(4,1)$, or ...). Then look at the intersection of the other components with the surface union of these trisecant lines. This work (details left to the reader) unless there are exactly two irreducible components both with (degree, genus)=(3,0) or (4,1). But by (1) this implies $c_3 \leq 0$, contradiction.

(γ) (case (2) in [1], §2). Assume that V is a smooth quadric $Q \subset \mathbf{P}^4$ and $L = \mathcal{O}(3)$ (hence $c_3(E) = 2g - 2$ by (1)). By (2) E is not ample if C has a trisecant line. Note that every trisecant line to C in $\mathbf{P}^4$ is contained in V. First assume that C is contained in a hyperplane. By (2) this means $h^0(E(-2)) \neq 0$; the morphism $\mathcal{O}(2) \to E$ is an obstruction to the ampleness of E (much easier that case 1) in (β)). Thus we will assume that C spans $\mathbf{P}^4$.

Assume C connected. The smoothness of C, $g > 1$, and all page 533 in [2] give that either $g = 2$, $\deg(C) = 6$, or $g = 5$, $\deg(C) = 8$ (and C is the complete intersection of 3 quadrics in $\mathbf{P}^4$ in the latter case). In both cases C is projectively normal and $h^0(\mathcal{I}_{C,V}(2)) \geq 2$, $h^0(\mathcal{I}_{C,V}(1)) = 0$; thus by (2) we get $h^0(E(-1)) \geq 2$ and $h^0(E(-2)) = 0$; fix $t \in H^0(E(-1))$ with $\dim((t)_0) = 1$. Set $B := (t)_0$; by [6], 2.2, $\deg(B) = \deg(C) - 4$. We get the following exact sequence:

$$0 \to \mathcal{O}(1) \to E \to \mathcal{I}_B(2) \to 0. \tag{5}$$

By [6], Th. 4.1 (i.e. by (1)) if $d = 6$ we have $p_a(B) = 0$, while if $d = 8$ we have $p_a(B) = 1$. In both cases any sheaf E fitting in (5) is spanned. Since $h^0(E(-1)) > 1$, $h^0(\mathcal{I}_B(1)) \neq 0$; we get that if $d = 8$ there are lines $D \subset V$ with length$(B \cap D) \geq 2$; by (5) the line D prevents the ampleness of E for $d = 8$. Now we assume $d = 6$, hence $h^0(\mathcal{I}_{C,V}(2)) = 3$ and $h^0(\mathcal{I}_B(1)) = 2$. Thus B is the intersection of V with a plane Π. As in (β), case 5), we get from (5) the ampleness of E. By [6], Th. 4.1, E is uniquely determined when we fix B and a degree 2 positive divisor $\mathbf{a}$ on B with support S. A dimensional count shows that the orthogonal group Aut(V) acts transitively on the pairs $(B, \mathbf{a})$ with B smooth conic and $\mathbf{a}$ reduced positive divisor of degree 2 on B, and on the pairs $(B, \mathbf{a})$ with B smooth conic and $\mathbf{a}$ double point on it. Thus we get exactly two irreducible families of solutions (since [6], Th. 4.1, gives an equivalence between (E, s) and $(B, \mathbf{a})$), the second one being a specialization of the first one. Furthermore two sheaves in the same family differ by an element of the orthogonal group Aut(V). Since $h^0(E(-1)) > 1$ a dimensional count shows also that for each E there is a subgroup $G \subset$Aut(V) with dim(G) $= 1$ and such that $g^*(E) \cong E$ for every $g \in G$. The two irreducible families are distinguished exactly by the condition: "card(S) = 2" or "card(S) = 1". In the case "card(S) = 2" (i.e. card(S) $= c_3$), we get a priori only convenable sheaves in the sense of [7] (hence curvilinear sheaves, without making a priori this assumption).

To handle the other cases and get further we need a lemma.

Lemma 2.1. *With the usual notations, V is not covered by a flat family of smooth rational curves $\{T\}$ with $LT = 2$.*

Proof. Fix $P \in S$. Assume by contradiction there is $T \cong \mathbf{P}^1$ with $P \in T$ and fix a general C. If length($C \cap T$) ≥ 2, the contradiction comes from 1.3. Assume length($C \cap T$) $= 1$ (and in particular C has embedding dimension at most 2 at $\mathbf{P}$). A local calculation shows that the torsion part of $I_C \otimes O_T$ has length 1. From the restriction of (2) to T we get $h^0(T, E|T)=4$, contradicting 1.2. ◊

By 2.1 we get at once all the cases in [1], §3 (i.e. the cases with $b_2(V) \geq 2$) and cases (3), (4), (5) (since the case left was done in the "safe" §3), and (8) of [1], §2. Now we will check how 2.1 gives cases (6) and (7) of [1], §2; in these cases V is respectively the intersection of 2 quadrics in $\mathbf{P}^5$ and a cubic hypersurface in $\mathbf{P}^4$ and $L = O(2)$; by 12.1, since $S \neq \emptyset$, it is sufficient to check that every point of V is contained in a line contained in V; a general hyperplane section contains a line (by the explicit theory of Del Pezzo surfaces); thus V contains a two-dimensional family of lines; they cover all the points of V by the properness of the Hilbert scheme (here of the Grassmannians).

Now look at case (10) of [1], §2; again we find $h(C)$ a line; now there is no contradiction to the spannedness of E, but, as in the remark just after that case we get $g = 1$ by the Riemann-Hurwitz formula, contradicting (1). The proof of 0.1 is over.

Now we want to spend a few lines for the case "E not curvilinear". The reduction (as in [1]) to the very few cases considered in §2 does not use the curvilinear assumption. To handle the single cases, however more care than I have is needed. At some point (in particular in (α)) we stressed that we never used the curvilinear assumption. Care for case (9) of [1], §2; but this is not a big problem. Care with the search for the quadrisecant line in (β); however [5] works for singular reducible curves with no line as component; essentially the reducible case looks easy (as was the disconnected one) and reduces the problem to subcases (1),...,(6). In (γ) we used heavily the curvilinear assumption when we used [1], p. 533; there it was used in an essential way the enumerative formula for the number of trisecant lines to a curve in $\mathbf{P}^4$; this formula is proved in [9] only for smooth curves. Summary: we do not claim, even for (β), to have checked all possible configurations, and we do not claim that in the cases giving the families (ii), (iii), and (iv) of the statements of 0.1 the non curvilinear sheaves arise only as limit of curvilinear solutions. But there is a case in which both problems about multisecant lines could be answered very easily, showing that no new solution can arise; and this happens exactly if the singularities of E are bad i.e. there is $P \in S$ such that even for general $s \in H^0(E)$, $(s)_0$ has embedded dimension 3 at P; for instance this is the case if $(s)_0$ is not locally a complete intersection at P and by (2) this condition means that the fiber of E at P has dimension > 3. Assume that E has such a bad point P; and consider cases (β) or (γ); every line T in V through P intersects C at P in a scheme length ≥ 2; in (γ) take as T a line through P and another point of C (the only trouble arises if C is union of lines through P); in (β) to find the quadrisecant line it should be sufficient to project from P and apply the genus formula for plane curves and one of the available (even a very weak one) bound for reducible space curves whose plane components have low degrees (but we have not made all the numerical checkings). Exactly for the same reasons it should be very easy to handle the case in which E is assumed to be not curvilinear at two different points (or more).

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MEROMORPHIC FUNCTIONS SHARING THREE VALUES

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Abstract: A well known theorem of R. Nevanlinna [5] states that there are at most two distinct meromorphic functions sharing three distinct values CM (counting multiplicities). This does not hold, if one only demands sharing two values CM and one IM (ignoring multiplicities). But in this case we are able to show that there are at most three distinct functions. This result is sharp in the sense that its conclusion does not hold, if one only demands sharing one value CM and two IM. Besides, we will present some other extensions of Nevanlinna's theorem dealing with the case that there are only few zeros and poles of the functions, or that there exists a nonempty set, which is "shared" by the functions.

1. Introduction

Given $n \geq 2$ meromorphic functions $f_1, \ldots, f_n$ on $\mathbb{C}$. We say that

$f_1, \dots, f_n$ *share* a value $c \in \mathbb{C} \cup \{\infty\}$ if all the sets $C_j = \{z \in \mathbb{C} : f_j(z) = c\}$, $j = 1, \dots, n$ are equal. In the following it will be helpful to make a distinction between sharing a value CM (counting multiplicities) and IM (ignoring multiplicities). In the first case we have a k-fold c-point of f_i exactly at the same points of the complex plane where f_j takes a k-fold c-point for $i, j = 1, \dots, n$. Particularly, if c is a value not taken by $f_1, \dots, f_n$, this value is CM-shared. In the second case we allow the multiplicities of the c-points to be different.

We assume that the reader is familiar with the notations and standard results of Nevanlinna theory (see e.g. [3], [4]).

In this paper, $S(r, f)$ denotes a quantity which is $o(T(r, f))$ as $r \to \infty$, possibly outside a set of finite Lebesgue measure. A frequently used lemma is the following: *If two nonconstant meromorphic functions f and g share three values IM, then* $S(r, f) = S(r, g)$. (One can easily prove this using the second fundamental theorem, see e.g. [4], p. 72.)

Beside the standard notations we will use the following: $\overline{N}(r, f)$ denotes the counting function of the poles, where each pole will be counted only once without regard to multiplicity. $N_0(r, c, f, g, h)$ is the counting function of the only once counted common c-points of f, g and h, again without regard to multiplicity. $N_1(r, f)$ counts the multiple poles of f, that is, $N_1(r, f) := N(r, f) - \overline{N}(r, f)$. $\mathcal{E}$ is the set of all $E \subset [0, \infty)$ of finite Lebesgue measure.

2. Results

The following is a well known result of R. Nevanlinna ([5], p. 125).

Theorem 1. *If three nonconstant meromorphic functions f, g, h share three distinct values CM, then at least two of them are equal.*

It is well-known, too, that the functions in general need not be Möbius transformations of each other. Theorem 1 is sharp in the sense that it is not correct for sharing two values CM and one value IM. To see this, consider an entire function β and define

$$f = \frac{e^{3\beta}}{e^{\beta}+1-e^{2\beta}}; \; g = \frac{e^{\beta}}{e^{\beta}+1-e^{2\beta}}; \; h = \frac{e^{-\beta}}{e^{\beta}+1-e^{2\beta}}.$$

Since $e^z \neq 0, \infty$ for all complex z it is obvious that there are no zeros of the three functions, which means, they share the 0-points CM, and they share the poles CM, since the denominator is exactly the same for all of them. A short calculation gives:

$$\begin{aligned} f=1 &\Leftrightarrow (e^\beta-1)(e^\beta+1)^2=0;\\ g=1 &\Leftrightarrow (e^\beta-1)(e^\beta+1)=0;\\ h=1 &\Leftrightarrow (e^\beta-1)^2(e^\beta+1)=0. \end{aligned}$$

This means f, g, h share 1 IM (not CM). (It is easily seen that none of the three functions is a Möbius transformation of another.)

Another direction to get results of the above kind without the assumption of sharing three values gives the following theorem (whose proof uses some ideas due to G. Brosch [1]).

Theorem 2. *If there are three distinct meromorphic nonconstant functions f, g and h with*

$$\overline{N}(r,f),\ \overline{N}(r,1/f)=S(r,f); \tag{2.1}$$

$$\overline{N}(r,g),\ \overline{N}(r,1/g)=S(r,g); \tag{2.2}$$

$$\overline{N}(r,h),\ \overline{N}(r,1/h)=S(r,h); \tag{2.3}$$

then there exists a set $E \in \mathcal{E}$ such that

$$\tau := \limsup_{r\to\infty, r\notin E} \frac{N_0(r,1,f,g,h)}{T(r,f)+T(r,g)+T(r,h)} \le 1/4.$$

An immediate consequence is the following result.

Corollary 3. *If there are three nonconstant meromorphic functions f, g and h that share 1-points IM and for which* (2.1), (2.2), (2.3) *hold, then at least two of them are equal.*

The inequality in Theorem 2 is sharp. This means that there are three functions for which (2.1), (2.2), (2.3) and $\tau = 1/4$ hold. Put for example

$$f(z)=e^z;\ g(z)=e^{-z};\ h(z)=e^{2z}.$$

After an easy calculation one obtains

$$\begin{gathered} T(r,f)=r/\pi+O(1)=T(r,g);\ T(r,h)=2r/\pi+O(1);\\ f=1 \Leftrightarrow e^z=1 \Leftrightarrow g=1;\ h=1 \Leftrightarrow e^z=\pm 1. \end{gathered}$$

Since there do not exist zeros or poles of the three functions, it follows by the second fundamental theorem of R. Nevanlinna that τ equals $1/4$.

With the Corollary 3 we are able to prove the following

Theorem 4. *If there are four nonconstant meromorphic functions f, g, h and k that share three distinct values of $\mathbb{C}\cup\{\infty\}$, two of them CM and one IM, then at least two of them are equal.*

Theorem 4 is sharp in the same sense mentioned after Theorem 1. To see this, let α be an entire function and define

$$f = 2e^{\alpha} - 1;\ g = e^{-\alpha}(2e^{\alpha} - 1);\ h = e^{-2\alpha}(2e^{\alpha} - 1);\ k = (2e^{\alpha} - 1)^2.$$

Similarly to the above reasoning it is seen that this four functions have no poles, i.e., they share ∞ CM, and that they share 0 IM (not CM). An easy computation gives:

$$\begin{aligned} f = 1 &\Leftrightarrow 2(e^{\alpha} - 1) &= 0;\\ g = 1 &\Leftrightarrow (e^{\alpha} - 1) &= 0;\\ h = 1 &\Leftrightarrow (e^{\alpha} - 1)^2 &= 0;\\ k = 1 &\Leftrightarrow 4e^{\alpha}(e^{\alpha} - 1) &= 0. \end{aligned}$$

This shows that the functions share 1-points IM (not CM).

For further details about the construction of such functions see [6]. Other conditions on the functions were given by F. Gross and C.F. Osgood [2].

Definition 5. (*Preimage sharing*) Let M be a finite, nonempty set in $\mathbb{C} \cup \{\infty\}$. Two meromorphic functions f, g *share the set* M if it follows from $f(z) \in M$ that $g(z) \in M$ and vice versa, with regard to multiplicity.

With this definition they gave

Theorem 6. *If there are two nonconstant entire functions f, g of finite order, which share 0 CM and the set $\{-1, +1\}$, one of the following equalities holds: $f \equiv \pm g$ or $fg \equiv \pm 1$.*

It was shown independently by G. Brosch ([1], p. 48) and K. Tohge ([7], p.251) that this result remains true for functions of infinite order. They proved that this even holds if f and g are meromorphic functions which share ∞ CM. We strengthen their result as follows.

Theorem 7. *If there are two nonconstant meromorphic functions f, g sharing ∞ IM, 0 CM and the set $\{-1, +1\}$, one of the following equalities holds: $f \equiv \pm g$ or $fg \equiv \pm 1$.*

For further results concerning unicity problems of meromorphic functions see [6].

3. Proofs

Proof of Theorem 2. Given three distinct nonconstant meromorphic functions f, g, h with (2.1) – (2.3), we have to show that $\tau \leq 1/4$. Let us define auxiliary functions

$$\begin{aligned}\alpha_1 &= \left(\tfrac{f''}{f'} - 2\tfrac{f'}{f-1}\right) - \left(\tfrac{g''}{g'} - 2\tfrac{g'}{g-1}\right);\\ \alpha_2 &= \left(\tfrac{g''}{g'} - 2\tfrac{g'}{g-1}\right) - \left(\tfrac{h''}{h'} - 2\tfrac{h'}{h-1}\right);\\ \alpha_3 &= \left(\tfrac{h''}{h'} - 2\tfrac{h'}{h-1}\right) - \left(\tfrac{f''}{f'} - 2\tfrac{f'}{f-1}\right);\\ \beta_1 &= f'/f;\ \beta_2 = g'/g;\ \beta_3 = h'/h.\end{aligned}$$

It follows from the lemma of the logarithmic derivative (see e.g. [4], p. 65) that

$$\begin{aligned}m(r, \alpha_1) &= S(r, f) + S(r, g);\\ m(r, \alpha_2) &= S(r, g) + S(r, h);\\ m(r, \alpha_3) &= S(r, h) + S(r, f).\end{aligned} \tag{3.1}$$

Because of (2.1) – (2.3) and the lemma of the logarithmic derivative it follows that

$$T(r, \beta_1) = S(r, f);\ T(r, \beta_2) = S(r, g);\ T(r, \beta_3) = S(r, h). \tag{3.2}$$

Since f, g, h are nonconstant we have $\beta_i \not\equiv 0$ for $i = 1, 2, 3$.
Let z_0 be a zero of f' but not of f. With $\overline{N}(r, 1/f) = S(r, f)$ and (3.2) we get

$$\begin{aligned}\overline{N}(r, 1/f') &= \overline{N}(r, 1/f') - \overline{N}(r, 1/f) + S(r, f)\\ &\leq N(r, 1/\beta_1) + S(r, f)\\ &\leq T(r, \beta_1) + S(r, f) = S(r, f).\end{aligned}$$

This and a similar argument leads to

$$\overline{N}(r, 1/f') = S(r, f);\ \overline{N}(r, 1/g') = S(r, g);\ \overline{N}(r, 1/h') = S(r, h). \tag{3.3}$$

Noting that β_1 vanishes in multiple 1-points of f and using a similar argument for 1-points of g and h, we get

$$N_1(r, \tfrac{1}{f-1}) = S(r, f);\ N_1(r, \tfrac{1}{g-1}) = S(r, g);\ N_1(r, \tfrac{1}{h-1}) = S(r, h). \tag{3.4}$$

This shows that "most" of the 1-points of the functions are simple ones.

By expanding the α_i into their Laurent series it is easily shown that

(3.5) α_1 (α_2 *and* α_3 *respectively*) *vanishes at simple common 1-points of* f, g (g, h *and* h, f *respectively*).

If $\alpha_1 \not\equiv 0$, then (3.2) – (3.5), together with the first fundamental theorem and the assumptions (2.1) and (2.2), give

$$
\begin{aligned}
N_0(r,1,f,g,h) &\le N(r,1/\alpha_1)+S(r,f)+S(r,g)\\
&\le T(r,\alpha_1)+S(r,f)+S(r,g)\\
&= m(r,\alpha_1)+N(r,\alpha_1)+S(r,f)+S(r,g)\\
&\le N(r,1/(f-1))-N_0(r,1,f,g,h)+\overline{N}(r,f)+\\
&\quad +\overline{N}(r,1/f')+N(r,1/(g-1))-N_0(r,1,f,g,h)+\\
&\quad +\overline{N}(r,g)+\overline{N}(r,1/g')+S(r,f)+S(r,g)\\
&\le T(r,f)+T(r,g)+S(r,f)+S(r,g)-2N_0(r,1,f,g,h).
\end{aligned}
$$

Thus we have proved (with a similar argument for $\alpha_1, \alpha_2 \not\equiv 0$)

$$
\begin{aligned}
&3N_0(r,1,f,g,h) \le (1+o(1))(T(r,f)+T(r,g)) \text{ for } \alpha_1 \not\equiv 0;\\
&3N_0(r,1,f,g,h) \le (1+o(1))(T(r,g)+T(r,h)) \text{ for } \alpha_2 \not\equiv 0;\\
&3N_0(r,1,f,g,h) \le (1+o(1))(T(r,h)+T(r,f)) \text{ for } \alpha_3 \not\equiv 0.
\end{aligned}
\tag{3.6}
$$

Now we distinguish the following four cases.

Case 1. $\alpha_1, \alpha_2, \alpha_3 \not\equiv 0$;

Case 2. $\alpha_1 \equiv 0$; $\alpha_2, \alpha_3 \not\equiv 0$;

Case 3. $\alpha_1, \alpha_2 \equiv 0$; $\alpha_3 \not\equiv 0$;

Case 4. $\alpha_1, \alpha_2, \alpha_3 \equiv 0$.

We proceed to obtain $\tau \le 1/4$ in each one of them.

Case 1. In this case (3.6) leads to

$$9N_0(r,1,f,g,h) \le (2+o(1))(T(r,f)+T(r,g)+T(r,h)).$$

Hence we get $\tau \le 2/9$. This shows $\tau \le 1/4$.

Case 2. An easy calculation shows that $\alpha_1 \equiv 0$ is equivalent to $f = L\circ g$ with a Möbius transformation L. Because of (2.1) and (2.2) we get $f \equiv g$ or $f \equiv 1/g$. Since $f \equiv g$ gives a contradiction, we assume that $f \equiv 1/g$. This means $T(r,g) = T(r,f) + S(r,f)$. $N_0(r,1,f,g,h) = S(r,f) + S(r,g) + S(r,h)$ means $\tau = 0$, which gives (2.4). Now let $N_0(r,1,f,g,h) \ne S(r,f) + S(r,g) + S(r,h)$. This, together with (3.6) for $\alpha_3 \not\equiv 0$ and $N(r,1/(f-1)) \ge N_0(r,1,f,g,h)$, yields

$$
\begin{aligned}
\tau &= \limsup_{r\to\infty, r\notin E} \frac{N_0(r,1,f,g,h)}{T(r,f)+T(r,g)+T(r,h)}\\
&= \limsup_{r\to\infty, r\notin E} \frac{N_0(r,1,f,g,h)}{2T(r,f)+T(r,h)+S(r,f)}\\
&\le \limsup_{r\to\infty, r\notin E} \frac{N_0(r,1,f,g,h)}{T(r,f)+3N_0(r,1,f,g,h)+S(r,f)+S(r,h)}\\
&\le \limsup_{r\to\infty, r\notin E} \frac{N_0(r,1,f,g,h)}{4N_0(r,1,f,g,h)+S(r,f)+S(r,h)}\\
&= 1/4.
\end{aligned}
$$

This gives $\tau \le 1/4$.
Cases 3/4. $\alpha_1, \alpha_2 \equiv 0$ yields

$$g \equiv h \text{ or } g \equiv 1/h$$

and

$$g \equiv f \text{ or } g \equiv 1/f,$$

which is a contradiction to our assumption that the three functions are pairwise distinct. ◇

Proof of Corollary 3. Assume there are three such functions f, g, h. The second fundamental theorem in the $\overline{N}$-version gives

$$\begin{aligned} T(r,f) &\le \overline{N}(r,1/f) + \overline{N}(r,1/(f-1)) + \overline{N}(r,f) + S(r,f) \\ &= \overline{N}(r,1/(f-1)) + S(r,f) \\ &\le T(r,f) + S(r,f). \end{aligned}$$

This means $\overline{N}(r,1/(f-1)) = T(r,f) + S(r,f)$. In the same way $\overline{N}(r,1/(g-1)) = T(r,g)+S(r,g)$ and $\overline{N}(r,1/(h-1)) = T(r,h)+S(r,h)$ hold. Because of sharing 1-points, we conclude $\tau = 1/3$. Therefore Theorem 2 gives a contradiction. ◇

Proof of Theorem 4. General assumption: There are four nonconstant, distinct functions f, g, h, k sharing $0, \infty$ CM and 1 IM. Since the functions share three values the equations $S(r,f) = S(r,g) = S(r,h) = = S(r,k) =: S(r)$ hold. It is convenient to define

$$\overline{N}(r,0) = \overline{N}(r,1/f); \; \overline{N}(r,1) = \overline{N}(r,1/(f-1)).$$

Here it is of no interest whether these counting functions are defined with f or g, because the functions f and g share the zeros and 1-points. Without loss of generality we can suppose that

$$\overline{N}(r,0), \; \overline{N}(r,1) \ne S(r). \tag{3.7}$$

This is valid because of Corollary 3.
We define the entire functions α, β, γ by

$$f/g = e^{\alpha}; \; f/h = e^{\beta}; \; f/k = e^{\gamma}. \tag{3.8}$$

Since there are 1-points, we get

$$\alpha, \beta, \gamma, \alpha - \beta, \; \alpha - \gamma \not\equiv \textit{constant}. \tag{3.9}$$

Further we define the meromorphic functions A, B, C by

$$\tfrac{f-1}{g-1} = A; \; \tfrac{f-1}{h-1} = B; \; \tfrac{f-1}{k-1} = C. \tag{3.10}$$

Since there are zeros, we get

$$A, B, C, A/B, A/C, B/C \not\equiv \textit{constant}. \tag{3.11}$$

We get the following representations for the function g:

$$g = \frac{A-1}{A-e^{\alpha}}; \tag{3.12}$$

$$g = \frac{B/A-1}{B/A-e^{\beta-\alpha}} e^{\beta-\alpha}; \tag{3.13}$$

$$g = \frac{C/A-1}{C/A-e^{\gamma-\alpha}} e^{\gamma-\alpha}. \tag{3.14}$$

We equate (3.12) with (3.13) and (3.12) with (3.14) and get

$$e^{\alpha-\beta} = \frac{e^{\alpha}(B-A)+(A-AB)}{B-AB}; \tag{3.15}$$

$$e^{\alpha-\gamma} = \frac{e^{\alpha}(C-A)+(A-AC)}{C-AC}. \tag{3.16}$$

A short computation gives the following equivalent representations:

$$e^{\beta-\alpha} = \frac{e^{\beta}(A-B)+(B-AB)}{A-AB}; \tag{3.15'}$$

$$e^{\gamma-\alpha} = \frac{e^{\gamma}(A-C)+(C-AC)}{A-AC}. \tag{3.16'}$$

From (3.15) and (3.15') we conclude that e^{α} and e^{β} share the 1-points. So we have three functions which have neither poles nor zeros, sharing 1-points, and they are nonconstant and distinct because of (3.9). Corollary 3 shows that this can not be true. So two of the exponential functions have to be equal and this means with (3.7) that two of the functions f, g, h, k have to be equal. This yields the expected contradiction and completes the proof. ◇

Proof of Theorem 7. Since f, g share ∞ IM, 0 CM and the set $\{-1, 1\}$, the functions $F := f^2$, $g := g^2$ share 0,1 CM and ∞ IM. Now it is easily seen that

$$S(r, F) = S(r, G) = S(r, f) = S(r, g) =: S(r). \tag{3.17}$$

Case 1. $\overline{N}(r, f) \neq S(r)$.

At any pole of f, F and G have a multiple pole. Define $N_2^{F,G}(r)$ as the counting function of the common multiple poles of F and G counted only once. Thus we get

$$N_2^{F,G}(r) \neq S(r). \tag{3.18}$$

Now we consider the following auxiliary function

$$\beta := \left(\frac{F'}{F} - \frac{F'}{F-1}\right) - \left(\frac{G'}{G} - \frac{G'}{G-1}\right) = -\frac{F'}{F(F-1)} + \frac{G'}{G(G-1)}.$$

Because of sharing 0,1 CM and ∞ IM we conculde from (3.17) and the lemma of the logarithmic derivative

$$T(r,\beta) = S(r). \tag{3.19}$$

The second representation of β shows that it vanishes at common multiple poles, so we get from (3.18) and (3.19) for $\beta \not\equiv 0$

$$N_2^{F,G}(r) \leq N(r, 1/\beta) \leq T(r,\beta) + S(r) = S(r).$$

This is a contradiction and therefore we conclude $\beta \equiv 0$. An easy calculation shows that this is equivalent to $F \equiv G$. This yields $f \equiv \pm g$.
Case 2. $\overline{N}(r,f) = S(r)$.
Since ∞ is shared by the two functions we have $\overline{N}(r,g) = S(r)$.
Subcase 2.a. $\overline{N}(r,1/f) \neq S(r)$.
At any zero of f, F and G have a multiple zero. So in this case there are many multiple zeros of F and G. An analogous consideration as in Case 1, here with the auxiliary function

$$\gamma := \frac{F'}{F-1} - \frac{G'}{G-1},$$

yields $\gamma \equiv 0$, which is in this case equivalent to $f \equiv \pm g$.
Subcase 2.b. $\overline{N}(r,1/f) = S(r)$.
Hence

$$\overline{N}(r,F), \overline{N}(r,1/F), \overline{N}(r,G), \overline{N}(r,1/G) = S(r) \tag{3.20}$$

is valid. From (3.20) we get $F \equiv G$ or $FG \equiv 1$.
Otherwise assume that $F \not\equiv G$ and $FG \not\equiv 1$. Now define the function $H := 1/G$. It is clear that F, G, H share 1-points and with (3.20) we get

$$\overline{N}(r,H), \overline{N}(r,1/H) = S(r). \tag{3.21}$$

Since F, G, H share 1-points and because of (3.20) and (3.21) Corollary 3 shows that two of the functions have to be equal. This gives a contradiction to the above assumption that $F \not\equiv G$ and $FG \not\equiv 1$ or to the assumption that f, g are nonconstant. So we have shown $F \equiv G$ or $FG \equiv 1$ and this leads to $f \equiv \pm g$ or $fg \equiv \pm 1$ and therefore our proof is complete. $\Diamond$

Remark. K. Tohge has strengthened Theorem 6 in the following way:

If f, g share $0, \infty$ CM and the set $M = \{c \in \mathbb{C} : c^n = 1\}$ for a given integer $n \geq 2$, then $f \equiv d_1 g$ or $fg \equiv d_2$ with complex constants d_1, d_2 such that $d_1^n = 1$ and $d_2^n = 1$ holds.

In a similar way as in the proof of Theorem 7 with $F := f^n$ and $G := g^n$ instead of $F := f^2$ and $G := g^2$ one can easily prove that Tohge's result still remains true for sharing the poles IM instead of CM. It seems that this can not be obtained by Tohge's method.

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D-CONHARMONIC CHANGE IN A SPECIAL PARA-SASAKIAN MANIFOLD

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Abstract: The notion of D-conformal change in a para-Sasakian and a special para-Sasakian manifold is introduced by G. Chūman [4]. The D-concircular change is a special kind of D-conformal change in a special para-Sasakian manifold. It is introduced and studied in [2].

In this paper, we introduce the D-conharmonic change, an another special kind of D-conformal change in a special para-Sasakian manifold. We obtain the tensor field invariant under this change and discuss the manifolds for which this tensor field vanishes.

1. A special para-Sasakian manifold and the D-conformal change.

Let us consider an n-dimensional differentiable manifold M with a positive definite Riemannian metric g_{ij}. We suppose that M admits a unit covariant vector field η_i satisfying

$$\nabla_j \eta_i = \bar{\varepsilon}(-g_{ij} + \eta_i \eta_j), \qquad \bar{\varepsilon} = \pm 1, \tag{1.1}$$

where ∇_j denotes the covariant differentiation with respect to g_{ij} and indices take the values $1, 2, \dots, n$. If we put

$$\xi^i = g^{ik}\eta_k, \quad \psi^i_j = \nabla_j \xi^i, \quad \psi_{ij} = g_{ik}\psi^k_j,$$

we have

$$(1.2) \qquad \begin{cases} \eta_i\xi^i = 1, \quad \psi^i_j\psi^k_i = \delta^k_j - \eta_j\xi^k, \quad \psi^i_j\xi^j = 0, \quad \eta_i\psi^i_j = 0, \\ g_{ij}\psi^i_p\psi^j_q = g_{pq} - \eta_p\eta_q, \quad \psi_{ij} = \psi_{ji}, \quad \operatorname{rank}(\psi^i_j) = n-1. \end{cases}$$

The relations (1.2) show that M is an almost paracontact Riemannian manifold (ψ, ξ, η, g). Because of (1.1), it is a *special para-Sasakian manifold* [7].

There is in M an (n-1)-dimensional distribution D defined by a Pfaffian equation $\eta = 0$ and called the D-distribution. Assume in M two para-Sasakian structures (ψ, ξ, η, g) and $({}^*\psi, {}^*\xi, {}^*\eta, {}^*g)$ satisfy

$$(1.3) \qquad \begin{cases} {}^*g_{ij} = e^{2\alpha}g_{ij} + (e^{2\sigma} - e^{2\alpha})\eta_i\eta_j \\ {}^*\xi^i = \varepsilon e^{-\sigma}\xi^i, \quad {}^*\psi^i_j = \varepsilon\psi^i_j, \quad {}^*\eta_i = \varepsilon e^{\sigma}\eta_i, \qquad \varepsilon = \pm 1 \end{cases}$$

where α and σ are functions. Then (ψ, ξ, η, g) and $({}^*\psi, {}^*\xi, {}^*\eta, {}^*g)$ have the same D-distribution. The relation (1.3) is called by Chūman [4] a *D-conformal change* of (ψ, ξ, η, g). When the function α is constant, (1.3) is called *a D-homothetic change.* G. Chūman proved [4] that if a para-Sasakian manifold is not special (i.e. $\psi^2 \neq (n-1)^2$), then any D-conformal change is necessarily D-homothetic. That is why non D-homothetic D-conformal change occurs only in a special para-Sasakian manifold.

By the change (1.3), M is also transformed into an almost paracontact Riemannian manifold. Furthermore, if $\psi^i_j = \nabla_j\xi^i$ is invariant under the change (1.3), then a special para-Sasakian M is transformed into a special para-Sasakian manifold. Hereafter, we consider the D-conformal change (1.3) satisfying

$$\psi^i_j = \nabla_j\xi^i, \qquad {}^*\psi^i_j = {}^*\nabla^*_j\, \xi^i,$$

where ${}^*\nabla$ is covariant differentiation with respect to ${}^*g_{ij}$ in a special para-Sasakian manifold M. By [4] we have

$$(1.4) \qquad \sigma_i = \sigma_p\xi^p\eta_i, \quad \bar{\varepsilon}\alpha_p\xi^p = 1 - e^{\sigma}, \quad \sigma_i = \nabla_i\sigma, \quad \alpha_i = \nabla_i\alpha.$$

From (1.3) we get

$$(1.5) \qquad {}^*g^{ji} = e^{-2\alpha}g^{ij} + (e^{-2\sigma} - e^{-2\alpha})\xi^i\xi^j \qquad ({}^*g^{ij}\,{}^*g_{jk} = \delta^i_k).$$

Thus, in a special para-Sasakian manifold, we have the following relation between $^*\{ {}^{k}_{ij} \}$ and $\{ {}^{k}_{ij} \}$ (cf. [4]):

$$(1.6)\quad ^*\{ {}^{h}_{ij} \} = \{ {}^{h}_{ij} \} + \alpha_j(\delta^h_i - \eta_i\xi^h) + \alpha_i(\delta^h_j - \eta_j\xi^h) - \alpha^h(g_{ij} - \eta_i\eta_j) + \\ + \bar{\varepsilon}(e^{2\alpha-\sigma} - e^{\sigma})(g_{ij} - \eta_i\eta_j)\xi^h + \sigma_j\eta_i\xi^h.$$

Let $R_{kji}{}^h$, R_{ij} and R denote the Riemannian curvature tensor, the Ricci tensor and the scalar curvature of the manifold M respectively. Then the tensor field

$$(1.7)\quad B_{kji}{}^h = R_{kji}{}^h - \tfrac{R+2}{(n-2)(n-3)}(g_{ki}\delta^h_j - g_{ji}\delta^h_k) + \\ + \tfrac{1}{n-3}[R_{ki}(\delta^h_j - \eta_j\xi^h) - R_{ji}(\delta^h_k - \eta_k\xi^h) + (g_{ki} - \eta_k\eta_i)R^h_j - \\ -(g_{ji} - \eta_j\eta_i)R^h_k] + \tfrac{R+2(n-1)}{(n-2)(n-3)}(g_{ki}\eta_j\xi^h - g_{ji}\eta_k\xi^h + \eta_k\eta_i\delta^h_j - \eta_j\eta_i\delta^h_k)$$

is invariant under any D-conformal change in a special para-Sasakian manifold ($n > 4$) (cf. [4]).

In §4 we shall need the following

Theorem A. ([1]) *Let M be an n-dimensional special para-Sasakian manifold. Then M is transformed into a manifold of constant curvature -1 by a D-conformal change if and only if $B_{kji}{}^h = 0$ ($n > 4$).*

Also, we note that in a special para-Sasakian manifold M, we have the following equations

$$(1.8)\qquad R_{kji}{}^h\eta_h = g_{ki}\eta_j - g_{ji}\eta_k, \qquad R_{ji}\xi^i = -(n-1)\eta_j.$$

2. The D-conharmonic change in a special para-Sasakian manifold.

In a special para-Sasakian manifold M, the Pfaffian equation $\eta = 0$ is completely integrable. The integral manifolds of $\eta = 0$ are called the level surfaces. In a local coordinate system of M, each level surface N is expressed by parametric equations

$$x^h = x^h(u^\lambda).$$

Here and in the sequel the Greek indeces have the range $(1, 2, \dots, n-1)$. Putting $B^h_\lambda = \frac{\partial x^h}{\partial u^\lambda}$ we have

$$(2.1)\qquad \eta_i B^i_\lambda = 0$$

while for the induced Riemannian metric $g_{\nu\mu}$ on N we have

(2.2) $$g_{\nu\mu} = B^i_\nu B^j_\mu g_{ij},$$

(2.3) $$g^{ij} = g^{\mu\nu} B^i_\mu B^j_\nu + \xi^i \xi^j,$$

where $(g^{\mu\nu}) = (g_{\rho\omega})^{-1}$. Also

(2.4) $$\nabla_\mu B^i_\nu = h_{\mu\nu}\xi^i,$$

where ∇_μ is the operator of the covariant differentiation with respect to $g_{\nu\mu}$ and $h_{\mu\nu}$ is the second fundamental tensor of N.

It is easy to see that each level surface N is totally umbilical. In fact, differentiating (2.1) along N and using (1.1), (2.2) and (2.4), we find $h_{\mu\nu} = \bar{\varepsilon} g_{\mu\nu}$. Therefore (2.4) can be written in the form

(2.5) $$\nabla_\mu B^i_\nu = \bar{\varepsilon} g_{\mu\nu}\xi^i.$$

If we put

$$B^\lambda_h = g^{\lambda\omega} g_{hk} B^k_\omega,$$

we have

(2.6) $$B^\nu_i B^j_\nu = \delta^j_i - \eta_i \xi^j, \quad B^\nu_i B^i_\mu = \delta^\nu_\mu, \quad B^\lambda_h \xi^h = 0.$$

The D-conformal change (1.3) induces in N the conformal change

(2.7) $$^*g_{\nu\mu} = e^{2\alpha} g_{\nu\mu},$$

where $^*g_{\nu\mu} = {}^*g_{ij} B^i_\nu B^j_\mu$ and α is now considered as a function of u^λ in N. If this conformal change satisfies $\alpha_{\nu\mu} = \varphi g_{\nu\mu}$, where φ is a function of u^λ and

$$\alpha_{\nu\mu} = \nabla_\nu \alpha_\mu - \alpha_\nu \alpha_\mu + \tfrac{1}{2}\alpha_\omega \alpha^\omega g_{\nu\mu}, \quad \alpha_\nu = \nabla_\nu \alpha, \quad \alpha^\omega = g^{\omega\nu}\alpha_\nu$$

then (2.7) is the concircular transformation (cf. [8]). Using this T. Adati and G. Chūman in [2] defined and studied D-concircular transformations.

In this paper we suppose that the conformal change (2.7) is conharmonic one (cf. [6]), i.e. we suppose that the function α in (2.7) satisifies

(2.8) $$\alpha_{\nu\mu} g^{\nu\mu} = 0.$$

Using this, we shall define the D-conharmonic change in M.

From $\alpha_\mu = B^i_\mu \alpha_i$ and (2.5), we have (cf. [1])

$$\nabla_\nu \alpha_\mu = B^j_\nu B^i_\mu (\nabla_j \alpha_i + \bar{\varepsilon} \alpha_p \xi^p g_{ij}).$$

On the other hand, using (2.3), we find

$$\alpha_\omega \alpha^\omega = g^{\nu\mu} \alpha_\nu \alpha_\mu = g^{\nu\mu} B^i_\mu B^j_\nu \alpha_i \alpha_j = (g^{ij} - \xi^i \xi^j) \alpha_i \alpha_j = \\ = \alpha_p \alpha^p - (\alpha_p \xi^p)^2$$

and taking (1.4) into account, we get

$$\alpha_{\nu\mu} = B^j_\nu B^i_\mu [\nabla_j \alpha_i - \alpha_j \alpha_i + \tfrac{1}{2}(\alpha_p \alpha^p - e^{2\sigma} + 1) g_{ji}].$$

Therefore

$$\alpha_{\nu\mu} g^{\nu\mu} = (g^{ij} - \xi^i \xi^j)[\nabla_j \alpha_i - \alpha_j \alpha_i + \tfrac{1}{2}(\alpha_p \alpha^p - e^{2\sigma} + 1) g_{ij}] = 0.$$

Thus, the D-conformal change (1.3) induces conharmonic changes on each level surface if and only if

$$(g^{ij} - \xi^i \xi^j)[\nabla_j \alpha_i - \alpha_j \alpha_i + \tfrac{1}{2}(\alpha_p \alpha^p - e^{2\sigma} + 1) g_{ij}] = 0. \tag{2.9}$$

Definition. The D-conformal change (1.3) satisfying (2.9) is called a *D-conharmonic change.*

The condition (2.9) can be written in the form

$$g^{ij}(\nabla_j \alpha_i - \alpha_i \alpha_j) - \nabla_j \alpha_i \xi^i \xi^j + (\alpha_p \xi^p)^2 + \tfrac{1}{2}(n-1)(\alpha_p \alpha^p - e^{2\sigma} + 1) = 0,$$

or, using (1.4), in the form

$$g^{ij}(\nabla_j \alpha_i - \alpha_i \alpha_j) - \nabla_j \alpha_i \xi^i \xi^j + (1 - e^\sigma)^2 + \\ + \tfrac{1}{2}(n-1)(\alpha_p \alpha^p - e^{2\sigma} + 1) = 0. \tag{2.10}$$

On the other hand, differentiating the second equation (1.4) and using (1.1), we get

$$(\nabla_j \alpha_i)\xi^i - \bar{\varepsilon}\alpha_j + \bar{\varepsilon}\alpha_i \xi^i \eta_j = -\bar{\varepsilon}\sigma_j e^\sigma.$$

Therefore

$$(\nabla_j \alpha_i)\xi^i \xi^j = -\bar{\varepsilon}\sigma_j \xi^j e^\sigma.$$

Substituting this into (2.10), we obtain

$$g^{ij}(\nabla_j \alpha_i - \alpha_i \alpha_j) + \bar{\varepsilon}\sigma_p \xi^p e^\sigma + (1 - e^\sigma)^2 + \\ + \tfrac{1}{2}(n-1)(\alpha_p \alpha^p - e^{2\sigma} + 1) = 0. \tag{2.11}$$

Also, (2.9) can be expressed as follows

$$\nabla_j \alpha_i (g^{ij} - \xi^i \xi^j) + \tfrac{n-3}{2}\alpha_t \alpha^t + (\alpha_t \xi^t)^2 + \tfrac{1}{2}(n-1)(1 - e^{2\sigma}) = 0,$$

from which, taking into account (1.4), we get

$$(2.12) \qquad \nabla_j\alpha_i(g^{ij} - \xi^i\xi^j) + \tfrac{n-3}{2}\alpha_t\alpha^t + \tfrac{n+1}{2} - 2e^\sigma - \tfrac{n-3}{2}e^{2\sigma} = 0.$$

3. The second access to the notion of the D-conharmonic change.

Let us consider a function A in M. It is, in the level surface N, a function of the coordinates u^λ. If this last function satisfies

$$g^{\nu\mu}\,\nabla_\nu A_\mu = 0, \qquad A_\mu = \nabla_\mu A,$$

it is said to be a *harmonic function* in N (cf. [6]). Let us search for the corresponding condition in M. Since

$$A_\mu = B^i_\mu A_i, \qquad A_i = \nabla_i A$$

and (2.5), we have

$$\nabla_\nu A_\mu = B^i_\mu B^j_\nu\,\nabla_j A_i + \bar{\varepsilon} g_{\nu\mu} A_t\xi^t,$$

from which, in view of (2.3), we have

$$g^{\nu\mu}\,\nabla_\nu A_\mu = (g^{ij} - \xi^i\xi^j)\,\nabla_j A_i + \bar{\varepsilon}(n-1)A_t\xi^t.$$

Thus, the function A is the harmonic function on each level surface if and only if

$$(3.1) \qquad (g^{ji} - \xi^j\xi^i)\,\nabla_j A_i + \bar{\varepsilon}(n-1)A_t\xi^t = 0.$$

Now, let us consider in M the function

$${}^*A = e^{2p\alpha}A,$$

where p is a suitable constant and A is a function satisfying (3.1). Let us look for the condition upon α ensuring that

$$(3.2) \qquad ({}^*g^{ji} - {}^*\xi^j\ {}^*\xi^i){}^*\nabla_j\ {}^*A_i + \bar{\varepsilon}(n-1){}^*A_t{}^*\xi^t = 0.$$

We have

$$(3.3) \qquad {}^*A_i = e^{2p\alpha}(2p\alpha_i A + A_i),$$

$$(3.4)\ {}^*\nabla_j\,{}^*A_i = e^{2p\alpha}[(4p^2\alpha_i\alpha_j + 2p\ {}^*\nabla_j\alpha_i)A + 2p(\alpha_j A_i + \alpha_i A_j) + {}^*\nabla_j A_i].$$

Using (1.6), we compute

$${}^*\nabla_j\,\alpha_i = \nabla_j\alpha_i - 2\alpha_j\alpha_i + (\alpha_j\eta_i + \alpha_i\eta_j)\xi^t\alpha_t + (g_{ij} - \eta_i\eta_j)\alpha_t\alpha^t -$$

$$-\bar{\varepsilon}(e^{2\alpha-\sigma}-e^{\sigma})(g_{ij}-\eta_i\eta_j)\xi^t\alpha_t-\sigma_j\eta_i\xi^t\alpha_t$$

and

$$^*\nabla_j A_i=\nabla_j A_i-(\alpha_j A_i+\alpha_i A_j)+(\alpha_j\eta_i+\alpha_i\eta_j)\xi^t A_t+$$
$$+(g_{ij}-\eta_i\eta_j)\alpha^t A_t-\bar{\varepsilon}(e^{2\alpha-\sigma}-e^{\sigma})(g_{ij}-\eta_i\eta_j)\xi^t A_t-\sigma_j\eta_i\xi^t A_t.$$

Substituting this into (3.4), we find

$$^*\nabla_j {}^*A_i=e^{2p\alpha}\{2pA[\nabla_j\alpha_i+2(p-1)\alpha_j\alpha_i+(\alpha_j\eta_i+\alpha_i\eta_j)\xi^t\alpha_t+$$
$$+(g_{ij}-\eta_i\eta_j)\alpha_t\alpha^t-\bar{\varepsilon}(e^{2\alpha-\sigma}-e^{\sigma})(g_{ij}-\eta_i\eta_j)\xi^t\alpha_t-\sigma_j\eta_i\xi^t\alpha_t]+$$
$$+\nabla_j A_i+(2p-1)(\alpha_j A_i+\alpha_i A_j)+(\alpha_j\eta_i+\alpha_i\eta_j)\xi^t A_t+$$
$$+(g_{ij}-\eta_i\eta_j)A_t\alpha^t-\bar{\varepsilon}(e^{2\alpha-\sigma}-e^{\sigma})(g_{ij}-\eta_i\eta_j)\xi^t A_t-\sigma_j\eta_i A_t\xi^t\}.$$

Using (1.3) and (1.5), we have

$$^*g^{ij}-{}^*\xi^i\,{}^*\xi^j=e^{-2\alpha}(g^{ij}-\xi^i\xi^j).$$

Therefore,

$$({}^*g^{ji}-{}^*\xi^j{}^*\xi^i)^*\nabla_j{}^*A_i=$$
$$=e^{2(p-1)\alpha}\{2pA[\nabla_j\alpha_i(g^{ij}-\xi^i\xi^j)+(2p+n-3)\alpha_t\alpha^t-2(p-1)(\xi^t\alpha_t)^2-$$
$$-\bar{\varepsilon}(n-1)(e^{2\alpha-\sigma}-e^{\sigma})\xi^t\alpha_t]+$$
$$+\nabla_j A_i(g^{ij}-\xi^i\xi^j)+(4p+n-3)\alpha^t A_t-2(2p-1)\xi^t\alpha_t\xi^p A_p-$$
$$-\bar{\varepsilon}(n-1)(e^{2\alpha-\sigma}-e^{\sigma})\xi^t A_t\}.$$

Now, if we use (3.3) and take $\varepsilon=+1$ in (1.3), we find

$$\bar{\varepsilon}(n-1)^*A_t{}^*\xi^t=\bar{\varepsilon}(n-1)e^{2p\alpha-\sigma}(2p\alpha_t\xi^t A+A_t\xi^t).$$

Therefore

$$({}^*g^{ji}-{}^*\xi^j{}^*\xi^i)^*\nabla_j{}^*A_i+\bar{\varepsilon}(n-1)^*A_t{}^*\xi^t=$$
$$=2pA\{e^{2(p-1)\alpha}[\nabla_j\alpha_i(g^{ij}-\xi^i\xi^j)+(2p+n-3)\alpha_t\alpha^t-2(p-1)(\xi^t\alpha_t)^2-$$
$$-\bar{\varepsilon}(n-1)(e^{2\alpha-\sigma}-e^{\sigma})\xi^t\alpha_t]+e^{2p\alpha-\sigma}\bar{\varepsilon}(n-1)\xi^t\alpha_t\}+ \tag{3.5}$$
$$+\{e^{(2p-1)\alpha}[\nabla_j A_i(g^{ij}-\xi^i\xi^j)+(4p+n-3)\alpha^t A_t-2(2p-1)\xi^t\alpha_t\xi^p A_p-$$
$$-\bar{\varepsilon}(n-1)(e^{2\alpha-\sigma}-e^{\sigma})\xi^p A_p]+e^{2p\alpha-\sigma}\bar{\varepsilon}(n-1)\xi^p A_p\}.$$

If we choose the constant $p=-\frac{n-3}{4}$ and take into account (1.4), the expression in the second bracket of (3.5) reduces to

$$e^{2(p-1)\alpha}[\nabla_j A_i(g^{ij}-\xi^i\xi^j)+\bar{\varepsilon}(n-1)\xi^t A_t],$$

and so, (3.5) becomes

$$({}^*g^{ji}-{}^*\xi^j{}^*\xi^i)^*\nabla_j{}^*A_i+\bar{\varepsilon}(n-1)^*A_p{}^*\xi^p=$$
$$=2pA\{e^{2(p-1)\alpha}[\nabla_j\alpha_i(g^{ji}-\xi^j\xi^i)+\tfrac{n-3}{2}\alpha_t\alpha^t+\tfrac{n+1}{2}(\xi^t\alpha_t)^2-$$
$$-\bar{\varepsilon}(n-1)(e^{2\alpha-\sigma}-e^{\sigma})\xi^t\alpha_t]+\bar{\varepsilon}(n-1)e^{2p\alpha-\sigma}\xi^t\alpha_t\}+$$
$$+e^{2(p-1)\alpha}[\nabla_j A_i(g^{ij}-\xi^j\xi^i)+\bar{\varepsilon}(n-1)\xi^t A_t],$$

from which we find that (3.2) follows from (3.1) if and only if

$$e^{2(p-1)\alpha}[\nabla_j\alpha_i(g^{ji}-\xi^j\xi^i)+\tfrac{n-3}{2}\alpha_t\alpha^t]+ \\ +\tfrac{n+1}{2}e^{2(p-1)\alpha}(\xi^t\alpha_t)^2-\bar{\varepsilon}(n-1)e^{2(p-1)\alpha}(e^{2\alpha-\sigma}-e^{\sigma})\xi^t\alpha_t+ \\ +\bar{\varepsilon}(n-1)e^{2p\alpha-\sigma}\xi^t\alpha_t=0.$$

In view of (1.4), the last condition can be written as follows

$$e^{(2p-1)\alpha}[\nabla_j\alpha_i(g^{ji}-\xi^j\xi^i)+\tfrac{n-3}{2}\alpha_t\alpha^t]+ \\ +\tfrac{n+1}{2}e^{2(p-1)\alpha}(1-e^{\sigma})^2-(n-1)(e^{2p\alpha-\sigma}-e^{2(p-1)\alpha+\sigma)})(1-e^{\sigma})+ \\ +(n-1)e^{2p\alpha-\sigma}(1-e^{\sigma})=0,$$

from which we get (2.12). Thus, we have

Theorem. *Let A be a function in M and let $^*A=e^{-\frac{n-3}{2}\alpha}A$. Then the conditions* (3.1) *and* (3.2) *are equivalent if and only if the D-conformal change* (1.3) *with $\varepsilon=+1$ is D-conharmonic.*

4. D-conharmonic curvature tensor

Let us denote by $^*R_{kji}{}^h$, $^*R_{ji}$ and $^*R$ the Riemannian curvature tensor, the Ricci tensor and the scalar curvature of metric $^*g$ respectively. Then we have ([1], (3.18) and (3.19)):

$$(4.1)\quad {}^*R_{kji}{}^h-{}^*g_{ki}\delta_j^h+{}^*g_{ji}\delta_k^h=R_{kji}{}^h-g_{ki}\delta_j^h+g_{ji}\delta_k^h+ \\ +\alpha_{ki}(\delta_j^h-\eta_j\xi^h)-\alpha_{ji}(\delta_k^h-\eta_k\xi^h)+(g_{ki}-\eta_k\eta_i)\alpha_j^h-(g_{ji}-\eta_j\eta_i)\alpha_k^h,$$

$$(4.2)\quad {}^*R_{ji}+(n-1){}^*g_{ji}=R_{ji}+(n-1)g_{ji}-(n-3)\alpha_{ji}-\alpha_t{}^t(g_{ji}-\eta_j\eta_i),$$

where

$$(4.3)\quad \alpha_{ji}=\nabla_j\alpha_i-\alpha_j\alpha_i-\bar{\varepsilon}e^{\sigma}(\alpha_j\eta_i+\alpha_i\eta_j)+ \\ +\tfrac{1}{2}(\alpha_p\alpha^p-e^{2\sigma}+1)(g_{ji}-\eta_j\eta_i)+(\bar{\varepsilon}e^{\sigma}\sigma_p\xi^p-e^{2\sigma}+1)\eta_j\eta_i.$$

From (4.2) we get

$$(4.4)\quad \alpha_t{}^t=\frac{1}{2(n-2)}\{R+n(n-1)-e^{2\sigma}[{}^*R+n(n-1)]\}.$$

But from (4.3) we find

$$\alpha_t{}^t=\alpha_{ji}g^{ji}=g^{ji}(\nabla_j\alpha_i-\alpha_i\alpha_j)-2\bar{\varepsilon}e^{\sigma}\xi^i\alpha_i+ \\ +\tfrac{1}{2}(n-1)(\alpha_p\alpha^p-e^{2\sigma}+1)+(\bar{\varepsilon}e^{\sigma}\sigma_p\xi^p-e^{2\sigma}+1),$$

or, using the second equation (1.4),

$$\alpha_t{}^t = g^{ji}(\nabla_j \alpha_i - \alpha_j \alpha_i) + (e^\sigma - 1)^2 + \tfrac{1}{2}(n-1)(\alpha_p \alpha^p - e^{2\sigma} + 1) + \bar{\varepsilon} e^\sigma \sigma_p \xi^p.$$

Thus, for a D-conharmonic change, according to (2.11) we have $\alpha_t{}^t = 0$, and so by (4.2)

$$\alpha_{ji} = \tfrac{1}{n-3}[R_{ji} + (n-1)g_{ji}] - \tfrac{1}{n-3}[{}^*R_{ji} + (n-1){}^*g_{ji}].$$

Substituting this into (4.1) we find

$$\begin{aligned}
&{}^*R_{kji}{}^h - {}^*g_{ki}\delta^h_j + {}^*g_{ji}\delta^h_k + \tfrac{1}{n-3}\{[{}^*R_{ki} + (n-1){}^*g_{ki}](\delta^h_j - {}^*\eta_j {}^*\xi^h) - \\
&\quad -[{}^*R_{ji} + (n-1){}^*g_{ji}](\delta^h_k - {}^*\eta_k {}^*\xi^h) + \\
&+({}^*g_{ki} - {}^*\eta_k {}^*\eta_i)[{}^*R^h_j + (n-1)\delta^h_j] - ({}^*g_{ji} - {}^*\eta_j {}^*\eta_i)[{}^*R^h_k + (n-1)\delta^h_k]\} = \\
&= R_{kji}{}^h - g_{ki}\delta^h_j + g_{ji}\delta^h_k + \\
&+\tfrac{1}{n-3}\{[R_{ki} + (n-1)g_{ki}](\delta^h_j - \eta_j \xi^h) - [R_{ji} + (n-1)g_{ji}](\delta^h_k - \eta_k \xi^h) + \\
&+(g_{ki} - \eta_k \eta_i)[R^h{}_j + (n-1)\delta^h_j] - (g_{ji} - \eta_j \eta_i)[R^h{}_k + (n-1)\delta^h_k].
\end{aligned}$$

Let us put

$$\begin{aligned}
E_{kji}{}^h &= R_{kji}{}^h - g_{ki}\delta^h_j + g_{ji}\delta^h_k + \\
&+\tfrac{1}{n-3}\{[R_{ki} + (n-1)g_{ki}](\delta^h_j - \eta_i \xi^h) - [R_{ji} + (n-1)g_{ji}](\delta^h_k - \eta_k \xi^h) + \\
&+[R^h{}_j + (n-1)\delta^h_j](g_{ki} - \eta_k \eta_i) - [R^h{}_k + (n-1)\delta^h_k](g_{ji} - \eta_j \eta_i)\},
\end{aligned}$$

or

$$\begin{aligned}
(4.5)\qquad E_{kji}{}^h &= R_{kji}{}^h + \tfrac{n+1}{n-3}(g_{ki}\delta^h_j - g_{ji}\delta^h_k) - \\
&-\tfrac{n-1}{n-3}(g_{ki}\eta_j \xi^h - g_{ji}\eta_k \xi^h + \delta^h_j \eta_k \eta_i - \delta^h_k \eta_j \eta_i) + \\
&+\tfrac{1}{n-3}[R_{ki}(\delta^h_j - \eta_j \xi^h) - R_{ji}(\delta^h_k - \eta_k \xi^h) + R^h_j(g_{ki} - \eta_k \eta_i) - R^h_k(g_{ji} - \eta_j \eta_i)].
\end{aligned}$$

Then we have the

Theorem. *Let M be an n-dimensional special para-Sasakian manifold and $n \geq 4$. Then the tensor field $E_{kji}{}^h$ is invariant under a D-conharmonic change.*

The tensor field $E_{kji}{}^h$ is called *the D-conharmonic curvature tensor field* in a special para-Sasakian manifold $n > 3$.

Let us suppose that M is a manifold of constant curvature -1, i.e.

$$R_{kji}{}^h = g_{ki}\delta^h_j - g_{ji}\delta^h_k.$$

Then

$$R_{ji} = -(n-1)g_{ji}.$$

Substituting this into (4.5), we find $E_{kji}{}^h = 0$. Thus, if the special para-Sasakian manifold is a manifold of constant curvature -1, then its D-conharmonic curvature tensor vanishes.

Now, we shall investigate the reverse case. To do that, we note first that contracting (4.5) with respect to h and k, we find

$$E_{pji}{}^{p} = E_{ji} = -\tfrac{1}{n-3}[n(n-1)+R](g_{ji}-\eta_j\eta_i). \tag{4.6}$$

Using (4.5) and (4.6), we can express the D-conformal curvature tensor (1.7), as follows

$$\begin{aligned} B_{kji}{}^{h} &= E_{kji}{}^{h} + \tfrac{1}{n-2}(E_{ki}\delta_j^h - E_{ji}\delta_k^h) + \\ &+ \tfrac{1}{(n-2)(n-3)}[n(n-1)+R](g_{ki}\eta_j\xi^h - g_{ji}\eta_k\xi^h). \end{aligned} \tag{4.7}$$

Now, suppose that $E_{kji}{}^{h} = 0$. Then $E_{ji} = 0$ too, and from (4.6) we get

$$n(n-1) + R = 0. \tag{4.8}$$

Therefore, $B_{kji}{}^{h} = 0$, from which follows, according Theorem A, that M can be transformed into a manifold $^*M$ of constant curvature -1 by a D-conformal change (1.3). This change is conharmonic one. In fact, for $^*M$ we have

$$^*R_{kji}{}^{h} = {}^*g_{ki}\delta_j^h - {}^*g_{ji}\delta_k^h,$$

from which

$$^*R_{ji} = -(n-1)^*g_{ji} \qquad \text{and} \qquad ^*R = -n(n-1).$$

Substituting this and (4.8) into (4.4), we find $\alpha_t{}^t = 0$. But the D-conformal change (1.3) satisfying $\alpha_t{}^t = 0$ is also D-conharmonic. Thus, we have

Theorem. *A necessary and sufficient condition that an n-dimensional* $(n > 4)$ *special para-Sasakian manifold may be transformed into a manifold of constant curvature* -1 *by a suitable conharmonic transformation, is* $E_{kji}{}^{h} = 0$.

Let us denote by $K_{\omega\nu\mu}{}^{\lambda}$ and $K_{\nu\mu}$ the curvature tensor and Ricci tensor of N respectively. Between tensors of M and N, the following relations are known (cf. [3]):

$$B_k^\omega B_j^\nu B_i^\mu B_\lambda^h K_{\omega\nu\mu}{}^{\lambda} = R_{kji}{}^{h} - g_{ki}\delta_j^h + g_{ji}\delta_k^h, \tag{4.9}$$

$$B_j^\nu B_i^\mu K_{\nu\mu} = R_{ji} + (n-1)g_{ji}. \tag{4.10}$$

Also, in view of (2.2), (2.3) and (2.6), we have

$$g^{\mu\nu} = g^{ij}B_i^\mu B_j^\nu, \quad g_{\nu\mu}B_i^\nu B_j^\mu = g_{ij} - \eta_i\eta_j \tag{4.11}$$

and

(4.12) $$K^{\lambda}{}_{\mu} B^{\mu}_{i} B^{h}_{\lambda} = R^{h}_{i} + (n-1)\delta^{h}_{i}.$$

Now, using (2.6), (4.9), (4.10), (4.11) and (4.12), it is easily proved that

(4.13) $$Z_{\omega\nu\mu}{}^{\lambda} B^{\omega}_{k} B^{\nu}_{j} B^{\mu}_{i} B^{h}_{\lambda} = E_{kji}{}^{h},$$

where $Z_{\omega\nu\mu}{}^{\lambda}$ is the conharmonic curvature tensor of level surface N, i.e. (cf. [6])

$$Z_{\omega\nu\mu}{}^{\lambda} = K_{\omega\nu\mu}{}^{\lambda} + \frac{1}{n-3}(g_{\omega\nu} K^{\lambda}{}_{\nu} - g_{\nu\mu} K^{\lambda}{}_{\omega} + \delta^{\lambda}_{\nu} K_{\omega\mu} - \delta^{\lambda}_{\omega} K_{\nu\mu}).$$

From (4.13), we have

Theorem. *The tensor field $E_{kji}{}^{h}$ of a special para-Sasakian manifold vanishes if and only if $Z_{\omega\nu\mu}{}^{\lambda} = 0$ in every level surface, i.e. if and only if each level surface is conharmonicly flat.*

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DER HOMOGENE PARALLEL-TRIEB-MECHANISMUS

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Zusammenfassung: Gegenstand der vorliegenden Arbeit ist ein räumlicher Mechanismus, bestehend aus vier starren Körpern, die über vier Parallelogrammgelenke so miteinander verbunden sind, daß eine geschlossene kinematische Kette entsteht. Ein Parallelogrammgelenk ist ein spezielles Viergelenk, das als Einzelgelenk aufgefaßt wird und die Kurzbezeichnung Π erhält. Ziel der Arbeit ist die Berechnung der drei Übertragungsfunktionen des homogenen 4Π-Paralleltrieb-Mechanismus, der durch gleiche Gelenkabmessungen und gleiche Lageparameter der Gelenke in den Gliedkörpern gekennzeichnet ist.

1. Einleitung

Wegen der grundlegenden Bedeutung für die Roboter- bzw. die Manipulator-Technik waren die räumlichen Mechanismen (nullter Ordnung) in den letzten Jahren häufig Gegenstand wissenschaftlicher Untersuchungen. Mit der Veröffentlichung der Arbeit von H. Y. Lee und Ch. G. Liang [2] haben diese Untersuchungen, soweit sie die Kinematik der räumlichen Mechanismen betreffen, einen vorläufigen Höhepunkt erreicht. Diesen beiden Autoren ist es erstmals gelungen, die Hauptübertragungsfunktion des allgemeinen, aus sieben Gliedern und sieben

Drehgelenken bestehenden 7R Mechanismus nullter Ordnung in der endgültigen Form anzugeben: Die gleich Null gesetzte Determinante einer 8×8 Matrix führt auf eine in den Tangens des halben Eingangs- bzw. des Ausgangswinkels algebraische Gleichung 16-ter Ordnung.

In den bekannt gewordenen Arbeiten über räumliche Mechanismen werden ausschließlich Mechanismen mit Drehgelenken (R), Prismengelenken (P), Zylindergelenken (C), Kugelgelenken (G) und Schraubgelenken (H) analytisch behandelt. In der Robotertechnik kommen die R und die P-Gelenke am häufigsten zur Anwendung. Bei den ebenen Manipulatoren (z.B. den Hebebühnen) werden auch sogenannte Parallelogrammgelenke verwendet [4]. Ein Parallelogrammgelenk ist ein Viergelenk mit besonderen Abmessungen. Ein solches Viergelenk kann auch als Einzelgelenk aufgefaßt werden, das zwei starre Körper mit einem relativen Freiheitsgrad aneinander bindet. Die Relativbahnen der Punkte von zwei so verbundenen Körpern sind kongruente Kreise, die Relativbewegung ist translatorisch. Wir werden dieses Einzelgelenk Parallelogramm-Gelenk nennen und mit der Kurzbezeichnung Π versehen. Π-Gelenke vermögen P-Gelenke bei entsprechender Auslegung überall zu ersetzen, und haben gegenüber diesen technologisch gesehen den Vorteil, daß die Führungsbahn nicht ungeschützt ist. Abgesehen davon aber ist es jedenfalls von theoretischem Interesse, Mechanismen die teilweise oder ausschließlich Π-Gelenke enthalten, zu untersuchen.

Es ist zu erwarten, daß der Ersatz eines P-Gelenkes in einem Raummechanismus durch ein Π-Gelenk, die Ordnung der algebraisierten Übertragungsfunktion beträchtlich erhöht; so kann z.B. vermutet werden, daß der 5RΠ-Mechanismus auf eine Übertragungsfunktion mindestens 32.ter Ordnung führt. Diese Fragen sollen in einigen Folgearbeiten abgehandelt werden.

In der vorliegenden Arbeit wird der einfachste Mechanismus mit nur Π-Gelenken untersucht, das ist der homogene 4Π-Mechanismus, dessen Gelenke gleiche Abmessungen und gleiche Lageparameter in den Gliedkörpern besitzen.

2. Das Parallelogramm-Gelenk

Die Abb. 1 zeigt das Parallelogramm-Gelenk und die relevanten Parameter. Die festen Systemparameter a_i, α_i, s_i, Θ_i, h_i und der (variable) Lageparameter σ_i bestimmen die Lage der im Gliedkörper K_i fixierten Vektorbasis $(O_i, \vec{e}_\alpha^{(i)})$ in der Vektorbasis $(O_{i-1}, \vec{e}_\alpha^{(i-1)})$ in K_{i-1}.

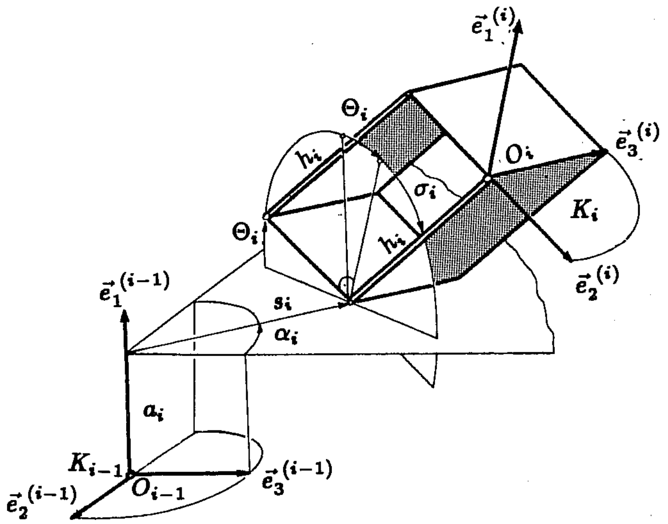


Abb.1

Das Parallelogramm-Gelenk II

Die orthonormierten Vektoren $\vec{e}_1^{(i)}, \vec{e}_2^{(i)}, \vec{e}_3^{(i)}$ in K_i können als Linearkombinationen der Basisvektoren $\vec{e}_1^{(i-1)}, \vec{e}_2^{(i-1)}, \vec{e}_3^{(i-1)}$ in K_{i-1} dargestellt werden. Mit den drei Matrizen

(1) $\underset{\sim}{e}^{(i)} :=$

$\vec{e}_1^{(i)}$
$\vec{e}_2^{(i)}$
$\vec{e}_3^{(i)}$

$\underset{\sim}{e}^{(i-1)} :=$

$\vec{e}_1^{(i-1)}$
$\vec{e}_2^{(i-1)}$
$\vec{e}_3^{(i-1)}$

$\underset{\sim}{A}_i :=$

$\cos\Theta_i$	$\sin\Theta_i \cos\alpha_i$	$\sin\Theta_i \sin\alpha_i$
$-\sin\Theta_i$	$\cos\Theta_i \cos\alpha_i$	$\cos\Theta_i \sin\alpha_i$
Θ	$-\sin\alpha_i$	$\cos\alpha_i$

läßt sich die gegenseitige Abhängigkeit der Basisvektoren in der folgenden Form anschreiben:

(2) $$\underset{\sim}{e}^{(i)} = \underset{\sim}{A}_i \underset{\sim}{e}^{(i-1)} \quad \text{bzw.} \quad \underset{\sim}{e}^{(i-1)} = \underset{\sim}{A}_i^T \underset{\sim}{e}^{(i)}$$

Die Matrix $\underset{\sim}{A}_i$ ist eine orthogonale Matrix d.h. es gilt $\underset{\sim}{A}_i^{-1} = \underset{\sim}{A}_i^T$.

Mit dem Drehwinkel φ_i und dem Einheitsvektor $\vec{e}_i$ in der Drehachse kann der Drehtensor $\mathbf{R}_i$, der die Einheitsvektoren $\vec{e}_\alpha^{(i-1)}$ in die

Einheitsvetkoren $\vec{e}_\alpha^{(i)}$ überführt, in der folgenden Weise dargestellt werden [1]:

(3) $\mathbf{R}_i = \underset{\sim}{\vec{e}}^{(i-1)T} \underset{\sim}{R}_i \underset{\sim}{\vec{e}}^{(i-1)} = \underset{\sim}{\vec{e}}^{(i)T} \underset{\sim}{R}_i \underset{\sim}{\vec{e}}^{(i)}$
$= \exp(\varphi_i \vec{n}_i \times) = \cos\varphi_i \mathbf{I} + \sin\varphi_i(\vec{n}_i\times) + (1-\cos\varphi_i)(\vec{n}_i\vec{n}_i).$

Aus $\mathbf{R}_i \circ \underset{\sim}{\vec{e}}^{(i-1)T} = \underset{\sim}{\vec{e}}^{(i)T} = \underset{\sim}{\vec{e}}^{(i-1)T} \underset{\sim}{R}_i \underset{\sim}{\vec{e}}^{(i-1)} \circ \underset{\sim}{\vec{e}}^{(i-1)T} = \underset{\sim}{\vec{e}}^{(i-1)T} \underset{\sim}{R}_i \Rightarrow$
$\Rightarrow \underset{\sim}{\vec{e}}^{(i)} = \underset{\sim}{R}_i{}^T \underset{\sim}{\vec{e}}^{(i-1)} = \underset{\sim}{A}_i \underset{\sim}{\vec{e}}^{(i-1)}$ folgt

$$\underset{\sim}{R}_i{}^T = \underset{\sim}{A}_i. \tag{4}$$

Der Vergleich der Koordinaten von $\underset{\sim}{A}_i$ und $\underset{\sim}{R}_i{}^T$ liefert für die Koordinaten $n_\alpha^{(i)}$ des Einheitsvektors $\vec{n}_i$ sowohl in der Basis $(O_i, \vec{e}_\alpha^{(i)})$ als auch in der Basis $(O_{i-1}, \vec{e}_\alpha^{(i-1)})$ zunächst

$$\begin{aligned} n_1^{(i)} \sin\varphi_i &= (1/2)\sin\alpha_i(1+\cos\Theta_i) \\ n_2^{(i)} \sin\varphi_i &= -(1/2)\sin\alpha_i \sin\Theta_i \\ n_3^{(i)} \sin\varphi_i &= (1/2)\sin\Theta_i(1+\cos\alpha_i). \end{aligned} \tag{5}$$

Für den Drehwinkel φ_i erhält man aus $\mathrm{tr}(\underset{\sim}{R}_i{}^T) = \mathrm{tr}(\underset{\sim}{A}_i)$:

$$\varphi_i = \pm \arccos[(1+\cos\alpha_i)(1+\cos\Theta_i)/2 - 1]. \tag{6}$$

Die Wahl des Vorzeichens von φ_i ist belanglos, weil sich mit dem Vorzeichen von φ_i auch das Vorzeichen des Einheitsvektors $\vec{n}_i$ ändert.

3. Der homogene Paralleltrieb-Mechanismus 4Π

Für kinematische Ketten, bestehend aus n Gliedkörpern die über g Gelenke verbunden sind gilt die Zwanglaufbedingung

$$F = 1 = \Sigma f_i + k(\Omega + 1).$$

Hierin bezeichnen $\Omega = g - n$ die Ordnung, k den Grundfreiheitsgrad aller Gliedkörper und Σf_i die Summe der relativen Freiheitsgrade der Gliedkörper die die, sie verbindenden Gelenke zulassen. Für kinematische Ketten nullter Ordnung mit Gelenken die nur einen relativen Freiheitsgrad zulassen gilt demnach: $\Sigma f_i = n = g$. In einem Paralleltrieb-Mechanismus sind sämtliche Gelenke Π-Gelenke, die Gliedkörper können sich daher nur noch ohne Richtungsänderungen, d.h. nur translatorisch bewegen, ihr Grundfreiheitsgrad ist daher gleich 3. Damit erhält man für den räumlichen Paralleltrieb-Mechanismus nullter Ordnung: $n = 4$. Unter einem homogenen 4Π-Mechanismus wird hier ein Paralleltrieb-Mechanismus mit gleichen Gelenken verstanden. Demnach gilt für diesen speziellen Mechanismus

$$(7) \qquad a_i = a, \quad \alpha_i = \alpha, \quad s_i = s, \quad \Theta_i = \Theta, \quad h_i = h \qquad i = 1 \div 4.$$

Die Koordinaten des Drehtensors und des Einheitsvektors in der Drehachse und der Drehwinkel selbst, sind dann in allen vier Koordinatensystemen $(O_i, \vec{e}_\alpha^{(i)})$ gleich groß:

$$(8) \qquad \underset{\sim}{R}_i = \underset{\sim}{R}, \quad n_\alpha^{(i)} = n_\alpha, \quad \varphi_i = \varphi \qquad i = 1 \div 4.$$

Die vier aufeinanderfolgenden Drehungen mit dem Drehtensor $\mathbf{R}$ führen die Basisvektoren in sich selbst zurück. Aus

$$\mathbf{R} \circ \mathbf{R} \circ \mathbf{R} \circ \mathbf{R} \circ \underset{\sim}{\vec{e}}^{(i-1)T} = \underset{\sim}{\vec{e}}^{(i-1)T} \text{ folgt}$$

$$(9) \qquad \mathbf{R}^4 = \mathbf{I} = \exp(4\varphi\vec{n}\times) \Rightarrow 4\varphi = 2\pi \Rightarrow \varphi = \pi/2.$$

Die Winkel α und Θ können beim homogenen 4Π-Mechanismus nicht unabhängig von einander gewählt werden. Denn mit $\varphi = \pi/2$ erhält man aus (6) den Zusammenhang

$$(10) \qquad (1 + \cos\alpha)(1 + \cos\Theta) = 2.$$

Diese Bedingungsgleichung beschränkt die Werte für α und für Θ auf den Bereich

$$-\pi/2 \leq \alpha, \ \Theta \leq \pi/2.$$

Unter Berücksichtigung von (10) erhält man für die in den vier Koordinatensystemen $(O_i, \vec{n}_\alpha^{(i)})$ gleichen Koordinaten des Richtungsvektors der Drehachse

$$(11) \quad n_1 = \tan(\alpha/2), \quad n_2 = -\tan(\alpha/2) \cdot \tan(\Theta/2), \quad n_3 = \tan(\Theta/2).$$

Nur eine der Koordinaten von $\vec{n}$ kann also frei zwischen +1 und −1 gewählt werden. Zwischen den Koordinaten von $\vec{n}$ bestehen die Beziehungen: $n_2 = -n_1 n_3$, $n_1^2(1 + n_3^2) = 1 - n_3^2$, $n_3(1 + n_1^2) = 1 - n_1^2$.

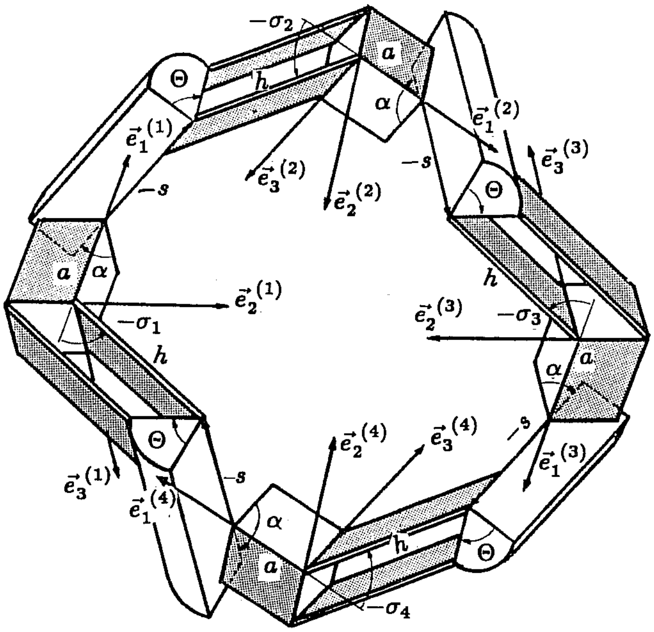


Abb. 2.
Der homogene 4Π-Mechanismus (Liniensymmetrische Bauart, Blickrichtung von $\vec{n}$)

4. Das Grundgleichungssystem

Von noch zu besprechenden Sonderfällen abgesehen, ist der homogene 4Π-Mechanismus zwangläufig. Die drei Übertragungsfunktionen sind aus drei voneinander unabhängigen Gleichungen, die die Lagewinkel σ_1, σ_2, σ_3 und σ_4 enthalten, zu berechnen. Diese Gleichungen liefert die vektorielle Schießbedingung. Diese lautet:

$$\vec{s} := [\vec{e}_1^{(1)} + \vec{e}_1^{(2)} + \vec{e}_1^{(3)} + \vec{e}_1^{(4)}]a + [\vec{e}_3^{(1)} + \vec{e}_3^{(2)} + \vec{e}_3^{(3)} + \vec{e}_3^{(4)}]s +$$
$$+h[\vec{e}_1^{(1)} \cos\sigma_1 + \vec{e}_2^{(1)} \sin\sigma_1] + h[\vec{e}_1^{(2)} \cos\sigma_2 + \vec{e}_2^{(2)} \sin\sigma_2] +$$
$$+h[\vec{e}_1^{(3)} \cos\sigma_3 + \vec{e}_2^{(3)} \sin\sigma_3] + h[\vec{e}_1^{(4)} \cos\sigma_4 + \vec{e}_2^{(4)} \sin\sigma_4] = \vec{0}.$$

Mit $\vec{e}_1^{(1)} + \vec{e}_1^{(2)} + \vec{e}_1^{(3)} + \vec{e}_1^{(4)} = (\mathbf{I} + \mathbf{R} + \mathbf{R}^2 + \mathbf{R}^3) \circ \vec{e}_1^{(1)} =$
$$= 4\vec{n}\vec{n} \circ \vec{n}_1^{(1)} = 4n_1\vec{n}$$

und $\vec{e}_3^{(1)} + \vec{e}_3^{(2)} + \vec{e}_3^{(3)} + \vec{e}_3^{(4)} = (\mathbf{I} + \mathbf{R} + \mathbf{R}^2 + \mathbf{R}^3) \circ \vec{e}_3^{(1)} =$

$$= 4\vec{n}\vec{n} \circ \vec{n}_3^{(1)} = 4n_3\vec{n}$$

sowie der Abkürzung

(12) $$\vec{r}_i = \vec{e}_1^{\,(i)} \cos\sigma_i + \vec{e}_2^{\,(i)} \cos\sigma_i$$

kann die Schießbedingung in der folgenden Form angeschrieben werden:

(13) $$\vec{s} := 4(n_1 \tfrac{a}{h} + n_3 \tfrac{s}{h})\vec{n} + \vec{r}_1 + \vec{r}_2 + \vec{r}_3 + \vec{r}_4 = \vec{0}.$$

Um die vorhandenen Symmetrien möglichst in Rechnung zu stellen, soll der Vektor $\vec{s}$ nach den drei Richtungen zerlegt werden, die die orthonormalen Vektoren

(14) $$\vec{n},\ \vec{f}_1 = (\vec{e}_1^{\,(1)} - n_1\vec{n})/\sqrt{1-n_1^2},\ \vec{f}_2 = \mathbf{R} \circ \vec{f}_1 = \\ = \vec{n} \times \vec{f}_1 = (\vec{e}_1^{\,(2)} - n_1\vec{n})/\sqrt{1-n_1^2}$$

bestimmen. Die in (13) vorkommenden Einheitsvektoren können als Linearkombinationen der Vektoren $\vec{n}$, $\vec{f}_1$, $\vec{f}_2$ wie folgt dargestellt werden (Abb. 3):

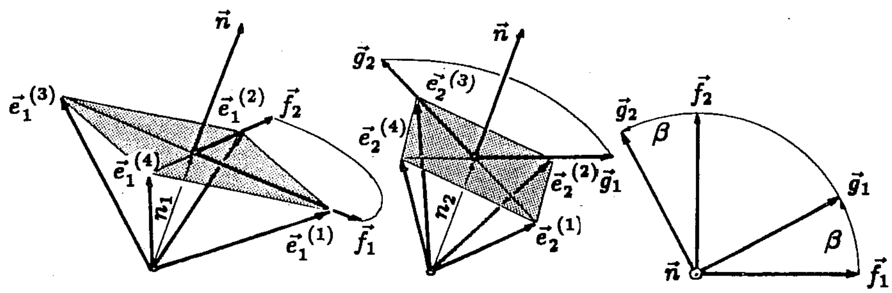


Abb.3

Darstellung der Vektoren $\vec{e}_1^{\,(i)}$ und $\vec{e}_2^{\,(i)}$ in der Vektorbasis $(\vec{n}, \vec{f}_1, \vec{f}_2)$

(15)
$$\begin{aligned}
\vec{e}_1^{\,(1)} &= n_1\vec{n} + \sqrt{1-n_1^2}\vec{f}_1,\ \vec{e}_2^{\,(1)} = n_2\vec{n} + \sqrt{1-n_2^2}\vec{g}_1 = \\
&= n_2\vec{n} + \sqrt{1-n_2^2}(\cos\beta\vec{f}_1 + \sin\beta\vec{f}_2) \\
\vec{e}_1^{\,(2)} &= n_1\vec{n} + \sqrt{1-n_1^2}\vec{f}_2,\ \vec{e}_2^{\,(2)} = n_2\vec{n} + \sqrt{1-n_2^2}\vec{g}_2 = \\
&= n_2\vec{n} + \sqrt{1-n_2^2}(-\sin\beta\vec{f}_1 + \cos\beta\vec{f}_2) \\
\vec{e}_1^{\,(3)} &= n_1\vec{n} - \sqrt{1-n_1^2}\vec{f}_1,\ \vec{e}_2^{\,(3)} = n_2\vec{n} - \sqrt{1-n_2^2}\vec{g}_1 = \\
&= n_2\vec{n} - \sqrt{1-n_2^2}(\cos\beta\vec{f}_1 + \sin\beta\vec{f}_2)
\end{aligned}$$

$$\vec{e}_1^{(4)} = n_1\vec{n} - \sqrt{1-n_1^2}\vec{f}_2,\ \vec{e}_2^{(4)} = n_2\vec{n} - \sqrt{1-n_2^2}\vec{g}_2 = \\ = n_2\vec{n} - \sqrt{1-n_2^2}(-\sin\beta\vec{f}_1 + \cos\beta\vec{f}_2)$$

Der hierin auftretende Winkel β kann aus den Koordinaten n_1, n_2 und n_3 berechnet werden:
Aus $\vec{f}_1 \circ \vec{g}_1 = \cos\beta = (\vec{e}_1^{(1)} - n_1\vec{n}) \circ (\vec{e}_1^{(2)} - n_2\vec{n})/\sqrt{(1-n_1^2)(1-n_2^2)}$ folgt $\sqrt{1-n_2^2}\cos\beta = n_1^2 n_3/\sqrt{1-n_1^2}$ und aus $(\vec{f}_1 \times \vec{g}_1)\circ\vec{n} = \sin\beta$ erhält man $\sqrt{1-n_2^2}\sin\beta = n_3/\sqrt{1-n_1^2}$.

Mit Hilfe dieser Formeln kann die Zerlegung des Nullvektors $\vec{s}$ nach den Richtungen $\vec{n}, \vec{f}_1$ und $\vec{f}_2$ durchgeführt werden. Als Ergebnis erhält man die folgenden drei Gleichungen:

$$(16)\quad 4[n_1(a/h) + n_3(s/h)] + n_1(\cos\sigma_1 + \cos\sigma_2 + \cos\sigma_3 + \cos\sigma_4) + \\ + n_2(\sin\sigma_1 + \sin\sigma_2 + \sin\sigma_3 + \sin\sigma_4) = 0$$

$$(1-n_1^2)(\cos\sigma_1 - \cos\sigma_3) + n_3[n_1^2(\sin\sigma_1 - \sin\sigma_3) - (\sin\sigma_2 - \sin\sigma_4)] = 0$$

$$(1-n_1^2)(\cos\sigma_2 - \cos\sigma_4) + n_3[n_1^2(\sin\sigma_2 - \sin\sigma_4) - (\sin\sigma_3 - \sin\sigma_1)] = 0$$

Es erweist sich als vorteilhaft nun anstelle der Lagewinkel σ_i neue Lagewinkel τ_i durch

$$(17)\qquad \tau_i = \sigma_i + \Theta/2$$

einzuführen. Ersetzt man nämlich in (16) σ_i durch $\tau_i - \Theta/2$ und berücksichtigt, daß einerseits $n_3 = \tan(\Theta/2)$ und andererseits die Beziehungen $n_1 n_3 = -n_2$ und $n_1^2(1+n_3^2) = 1 - n_3^2$ gelten, dann erhält man – nach Addition bzw. Subtraktion der letzten beiden Gleichungen – unter Einführung eines neuen Parameters K

$$(18)\qquad K := 4[n_1(a/h) + n_3(s/h)]/(n_1\sqrt{1+n_3^2})$$

den sehr viel durchsichtigeren Gleichungssatz für die Lagewinkel τ_i:

$$(19)\qquad \begin{aligned} &K + \cos\tau_1 + \cos\tau_2 + \cos\tau_3 + \cos\tau_4 = 0 \\ &(\sin\tau_1 - \sin\tau_3) + n_3(\cos\tau_2 - \cos\tau_4) = 0 \\ &(\sin\tau_2 - \sin\tau_4) + n_3(\cos\tau_3 - \cos\tau_1) = 0. \end{aligned}$$

Dieses Grundgleichungssystem, das nur zwei Systemparameter enthält, läßt bereits einiges von der Struktur der zu bestimmenden Übertragungsfunktionen erkennen.

5. Folgerungen aus dem Grundgleichungssystem

Die erste Gleichung (19) zeigt, daß reelle Werte für die Lagewinkel τ_i sich nur ergeben können, wenn der Parameter K der folgenden Bedingung genügt

$$-4 \leq K \leq 4. \tag{20}$$

Die Grenzwerte von K sind den aus (19) ablesbaren Punktlösungen $\tau_i = 0$ bzw. $\tau_i = \pm\pi$ zugeordnet, der entsprechende 4Π-Mechanismus ist nur "infinitesimal" beweglich.

Erfüllen die Lagewinkel τ_i $(i = 1 \div 4)$ das Gleichungssystem (19) mit dem Parameterpaar K, n_3, dann erfüllen die Winkel $\tilde{\tau}_i = \tau_i \pm \pi$ dieses Gleichungssystem für das Parameterpaar $-K$, n_3, und die Winkel $\tilde{\tau}_1 = \tau_1$, $\tilde{\tau}_2 = \tau_4$, $\tilde{\tau}_3 = \tau_3$ und $\tilde{\tau}_4 = \tau_2$ erfüllen es für das Parameterpaar K, $-n_3$. Demnach kann der Variationsbereich von K und n_3 z.B. auf $(0 \div 4) \times (0 \div 1)$ beschränkt werden.

Das Grundgleichungssystem (19) geht bei zyklischer Vertauschung der Lagewinkel τ_i in sich selbst über. Wir bezeichnen die aus (19) durch Elimination von je zwei Winkeln zu ermittelnden Übertragungsfunktionen mit $f(\tau_2, \tau_1) = 0$, $g(\tau_3, \tau_1) = 0$ und $h(\tau_4, \tau_1)$. Aus der zyklischen Vertauschbarkeit der Winkel ist auf

$$f(\tau_2, \tau_1) = f(\tau_1, \tau_4) \equiv h(\tau_4, \tau_1) = 0 \tag{21}$$

zu schließen, d.h. der Graph $\tau_4(\tau_1)$ geht aus $\tau_2(\tau_1)$ durch Spiegelung an der Geraden $\tau_2 = \tau_1$ und der Umbenennung von τ_2 in τ_4 hervor. Ferner zeigt die zyklische Vertauschung

$$g(\tau_3, \tau_1) = g(\tau_1, \tau_3) = 0, \tag{22}$$

daß der Graph $\tau_3(\tau_1)$ symmetrisch ist in bezug auf die Gerade $\tau_3 = \tau_1$. Das Grundgleichungssystem (19) läßt neben der zyklischen Vertauschung aber auch noch eine andere Variablenvertauschung zu, nämlich:

$$\tau_1 \rightarrow -\tau_3, \; \tau_2 \rightarrow -\tau_2, \; \tau_3 \rightarrow -\tau_1, \; \tau_4 \rightarrow -\tau_4.$$

Demnach gelten auch die Beziehungen $f(\tau_2, \tau_1) = f(-\tau_2, -\tau_3) = 0$, $g(\tau_3, \tau_1) = g(-\tau_1, -\tau_3) = 0$ und $h(\tau_4, \tau_1) = h(-\tau_4, -\tau_3) = 0$. Mit den zyklischen Vertauschungen $f(-\tau_2, -\tau_3) = f(-\tau_1, -\tau_2) = 0$ und $0 = h(-\tau_4, -\tau_3) = h(-\tau_1, -\tau_4)$ erhält man dann

$$\begin{gathered} f(\tau_2, \tau_1) = f(-\tau_1, -\tau_2) = 0, \; g(\tau_3, \tau_1) = g(-\tau_1, -\tau_3) = 0, \\ h(\tau_4, \tau_1) = h(-\tau_1, -\tau_4) = 0, \end{gathered} \tag{23}$$

d.h., die Graphen der drei Übertragungsfunktionen sind bezüglich ihrer Querdiagonalen ($\tau_2 = -\tau_1, \tau_3 = -\tau_1, \tau_4 = -\tau_1$) symmetrisch.

6. Nichtzwangsläufige homogene 4Π-Mechanismen

Der ebene homogene 4Π-Mechanismus (Abb. 4) ist gekennzeichnet durch $\Theta = \pi/2$. Die erste Gleichung von (19) kann mit (18) wie folgt angeschrieben werden:

$$4[n_1(a/h)+n_3(s/h)]+n_1\sqrt{1+n_3^2}(\cos\tau_1+\cos\tau_2+\cos\tau_3+\cos\tau_4)=0$$

Mit $n_3 = \tan\Theta/2 = 1$ und $n_1 = 0$ erhält man daraus: $s = 0$. Die vier Winkel τ_i müssen nur den beiden letzten Gleichungen von (19) mit $n_3 = 1$ genügen. Diese sind gleichwertig dem Gleichungssatz

$$\begin{aligned} \cos\sigma_1 - \sin\sigma_2 - \cos\sigma_3 + \sin\sigma_4 &= 0 \\ \sin\sigma_1 + \cos\sigma_2 - \sin\sigma_3 - \cos\sigma_4 &= 0 \end{aligned}$$

der im allgemeinen nur die Lösung $\sigma_3 = \sigma_1$, $\sigma_4 = \sigma_2$ zuläßt, σ_2 ist im allgemeinen unabhängig von σ_1, der ebene 4Π-Mechanismus besitzt zwei Freiheitsgrade. Wählt man allerdings σ_3 = konst und $\sigma_4 = -\pi/2 + \sigma_3$, dann besitzt der Mechanismus immer noch einen Freiheitsgrad wobei $\sigma_2 = \pi/2 + \sigma_1$ wird.

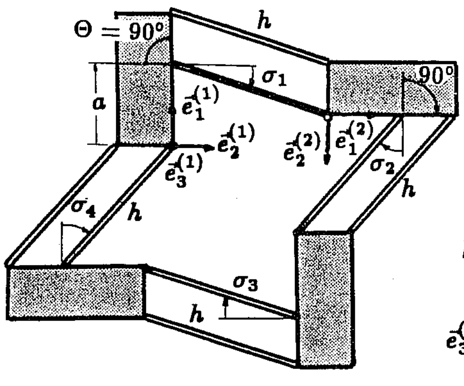


Abb.4
Der ebene homogene
4Π-Mechanismus ($F = 2$)

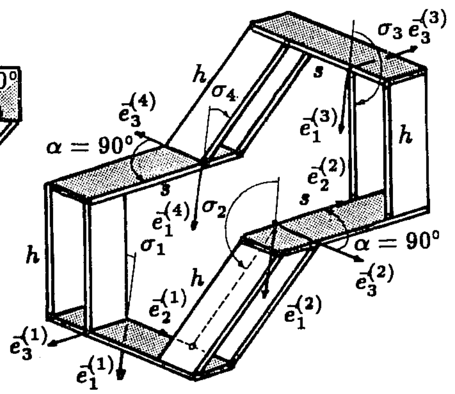


Abb.5
Der räumliche homogene
4Π-Mechanismus ($F = 2$)

Einen zweiten homogenen 4Π-Mechanismus mit zwei Freiheitsgraden (Abb. 5) erhält man mit den Systemparametern $\alpha = \pi/2$ und $a = 0$. In diesem Falle stimmen die Lagewinkel τ_i und σ_i überein $(\Theta = 0)$ und die Gleichungen, die sie zu erfüllen haben, sind die folgenden:

$$\begin{aligned} &\cos\sigma_1 + \cos\sigma_2 + \cos\sigma_3 + \cos\sigma_4 = 0 \\ &\sin\sigma_3 = \sin\sigma_1, \quad \sin\sigma_4 = \sin\sigma_2 \end{aligned} \tag{24}$$

Diese Gleichungen lassen zwei qualitativ sehr verschiedene Lösungen zu, nämlich

$$\sigma = \pm\pi \pm \sigma_1, \quad \sigma_3 = \sigma_1, \quad \sigma_4 = \sigma = \pm\pi \pm \sigma_1 \quad \text{und} \tag{25}$$

$$\sigma_3 = \pm\pi - \sigma_1, \quad \sigma_4 = \pm\pi - \sigma_2 \tag{26}$$

Die erste Lösung entspricht einem zwangsläufigen Mechanismus, und die zweite einem mit zwei Freiheitsgraden. Der durch $(\alpha = 0) \wedge (a = 0)$ charakterisierte homogene 4Π-Mechanismus ist insofern bemerkenswert als er beim Durchgang durch eine besondere Lage $(\sigma_i = \pm\pi/2)$ den Freiheitsgrad ändert.

7. Die Übertragungfunktionen

Bevor an die allgemeine Lösung des Grundgleichungssystems herangegangen wird, soll die enthaltene, explizite angebbare Lösung gesondert betrachtet werden. Es handelt sich dabei um die Übertragungsfunktionen des *liniensymmetrischen* homogenen 4Π-Mechanismus (Abb. 2).

Setzt man in (16) $\tau_3 = \tau_1$ und $\tau_4 = \tau_2$, dann sind die letzten beiden Gleichungen von (16) bereits erfüllt und die erste vereinfacht sich zur unmittelbar auflösbaren Gleichung:

$$(K/2) + \cos\tau_2 + \cos\tau_1 = 0.$$

Die Übertragungsfunktionen des, bezüglich der Drehachse (die der Einheitsvektor $\vec{n}$ richtungsmäßig angibt) liniensymmetrischen homogenen 4Π-Mechanismus ergeben sich auf diese Weise zu:

$$\tau_2(\tau_1) = \pm\arccos[-(\tfrac{K}{2} + \cos\tau_1)], \; \tau_3(\tau_1) = \tau_1, \; \tau_4(\tau_1) = \tau_2(\tau_1). \tag{27}$$

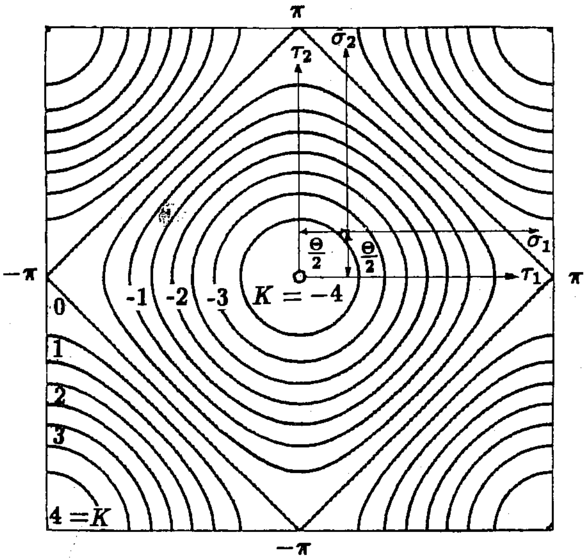


Abb.6

Die Übertragungsfunktion $\tau_2(\tau_1)$ bzw. $\sigma_2(\sigma_1)$ des liniensymmetrischen homogenen 4Π-Mechanismus

Diese Übertragungsfunktionen, dargestellt in den Lagewinkeln τ_i sind unabhängig von dem Systemparameter n_3. Die Übertragungsfunktionen in den ursprünglich gewählten Lageparameter erhält man gemäß (17) durch Verschiebung des Koordinatenursprungs in beiden Richtungen um $\Theta/2 = \arctan n_3$ (Abb. 6).

Die allgemeine Lösung des Grundgleichungssystems (19) und damit die Übertragungsfunktionen der nichtsymmetrischen 4Π-Mechanismen können nur in impliziter Form angegeben werden. Die Elimination von zwei Winkeln aus den drei Gleichungen (19) gelingt in zwei Schritten: Zuerst können aus (19) zwei Gleichungen gewonnen werden, die einen Lagewinkel nicht enthalten und die im Sinus und im Cosinus der restlichen drei Lagewinkel linear sind. Aus diesen beiden Gleichungen kann dann mit Hilfe der Euler-Sylvesterschen Resultanten-Methode [3] der zweite Lagewinkel eliminiert werden.

Das Grundgleichungssystem (19) soll zunächst in der Form einer Matrizengleichung angeschrieben werden. Mit der Abkürzung

$$c = \sqrt{1 - n_3^2} \qquad (28)$$

und den Matrizen

$$\underset{\sim}{e}^T = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

(29)
$$\underset{\sim}{k}_1{}^T = \begin{bmatrix} c & 0 & n_3 \end{bmatrix} \cos\tau_1 + \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \sin\tau_1$$
$$\underset{\sim}{k}_2{}^T = \begin{bmatrix} c & n_3 & 0 \end{bmatrix} \cos\tau_2 + \begin{bmatrix} 0 & 0 & -1 \end{bmatrix} \sin\tau_2$$
$$\underset{\sim}{k}_3{}^T = \begin{bmatrix} c & 0 & -n_3 \end{bmatrix} \cos\tau_3 + \begin{bmatrix} 0 & -1 & 0 \end{bmatrix} \sin\tau_3$$
$$\underset{\sim}{k}_4{}^T = \begin{bmatrix} c & -n_3 & 0 \end{bmatrix} \cos\tau_4 + \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \sin\tau_4$$

nimmt (19) folgende Gestalt an

(30)
$$\underset{\sim}{e}Kc + \underset{\sim}{k}_1 + \underset{\sim}{k}_2 + \underset{\sim}{k}_3 + \underset{\sim}{k}_4 = \underset{\sim}{0}\,.$$

Die hier eingeführten $\underset{\sim}{k}_i$ Matrizen sind (nichtorthogonale) Einheits-Spaltenmatrizen, d.h. es gilt, unabhängig vom Lagenwinkel τ_i der $\underset{\sim}{k}_i$ festlegt:

(31)
$$\underset{\sim}{k}_i{}^T \underset{\sim}{k}_i = 1 \qquad (i = 1 \div 4).$$

Die Matrizen

(32)
$$\underset{\sim}{w}_1{}^T = \begin{bmatrix} -n_3 & 0 & c \end{bmatrix} \quad \underset{\sim}{w}_2{}^T = \begin{bmatrix} -n_3 & c & 0 \end{bmatrix}$$
$$\underset{\sim}{w}_3{}^T = \begin{bmatrix} -n_3 & 0 & -c \end{bmatrix} \quad \underset{\sim}{w}_4{}^T = \begin{bmatrix} -n_3 & -c & 0 \end{bmatrix}$$

sind ebenfalls Einheits-Spaltenmatrizen die außerdem folgende Orthogonalitätsbedingungen:

(33)
$$\underset{\sim}{w}_i{}^T \underset{\sim}{k}_i = \underset{\sim}{0} \qquad (i = 1 \div 4)$$

erfüllen. Mit (31) und (33) kann aus (30) auf zweifache Weise ein Lagenwinkel so eliminiert werden, daß man Gleichungen erhält die im Sinus und Cosinus der verbleibenden Winkel linear sind. Aus

$$\underset{\sim}{w}_3{}^T \underset{\sim}{k}_3 = 0 = -\underset{\sim}{w}_3{}^T(\underset{\sim}{e}Kc + \underset{\sim}{k}_1 + \underset{\sim}{k}_2 + \underset{\sim}{k}_4) \quad \text{und}$$
$$\underset{\sim}{k}_3{}^T \underset{\sim}{k}_3 = 1 = (\underset{\sim}{e}Kc + \underset{\sim}{k}_1 + \underset{\sim}{k}_2 + \underset{\sim}{k}_4)^T(\underset{\sim}{e}Kc + \underset{\sim}{k}_1 + \underset{\sim}{k}_2 + \underset{\sim}{k}_3)$$

ergeben sich die folgenden zwei τ_3-freien Gleichungen

(34) $n_3(K + 2\cos\tau_1) + (n_3\cos\tau_2 - \sin\tau_2) + (n_3\cos\tau_4 + \sin\tau_4) = 0$

(35)
$$1 + c^2K^2/2 + c^2K(\cos\tau_1 + \cos\tau_2) + (c^2\cos\tau_1 + n_3\sin\tau_1)\cos\tau_2 -$$
$$-n_3\cos\tau_1\sin\tau_2 + [c^2K + c^2\cos\tau_1 - n_3\sin\tau_1 + (c^2 - n_3^2)\cos\tau_2]\cos\tau_4 +$$
$$+[n_3\cos\tau_1 - \sin\tau_2]\sin\tau_4 = 0.$$

Mit

$$x_4 = \tan(\tau_4/2) \tag{36}$$

und den Spaltenmatrizen

$$\underset{\sim}{c}_1 = \begin{bmatrix} n_3(K-1) + 2\cos\tau_1 + \cos\tau_2 - \sin\tau_2 \\ 1 + c^2(K-1)(K/2+\cos\tau_1) + [c^2(K-1)+n_3^2]\cos\tau_2 + \\ + n_3[\sin\tau_1 + \sin(\tau_2-\tau_1)] + c^2\cos\tau_1\cos\tau_2 \end{bmatrix}$$

$$\underset{\sim}{c}_2 = \begin{bmatrix} 2 \\ n_3\cos\tau_1 - \sin\tau_2 \end{bmatrix} \tag{37}$$

$$\underset{\sim}{c}_3 = \begin{bmatrix} n_3(K+1+2\cos\tau_1+\cos\tau_2) - \sin\tau_2 \\ 1 + c^2(K+1)(K/2+\cos\tau_1) + [c^2(K+1)-n_3^2]\cos\tau_2 + \\ + n_3[-\sin\tau_1 + \sin(\tau_2-\tau_1)] + c^2\cos\tau_1\cos\tau_2 \end{bmatrix}$$

erhält man anstelle von (34) und (35) die Matrizengleichung

$$\underset{\sim}{c}_1 x_4^2 + \underset{\sim}{c}_2 x_4 + \underset{\sim}{c}_1 = \underset{\sim}{0}\,. \tag{38}$$

Um den Zusammenhang der Lagewinkel τ_2 und τ_1 zu erhalten ist daraus x_4 zu eliminieren und das kann mit Hilfe der Resultanten-Methode bewerkstelligt werden.

Faßt man die Potenzen von x_4 (bis zur dritten Ordnung) zu der Spaltenmatrize $\underset{\sim}{x}_4$ zusammen und bildet mit Hilfe der $\underset{\sim}{c}_i$-Matrizen die 4×4 Resultantenmatrix M

$$\underset{\sim}{x}_4 = \begin{bmatrix} x_4^3 \\ x_4^2 \\ x_4 \\ x_1 \end{bmatrix} \qquad \underset{\sim}{M} = \begin{bmatrix} \underset{\sim}{0} & \underset{\sim}{c}_1 & \underset{\sim}{c}_2 & \underset{\sim}{c}_3 \\ \underset{\sim}{c}_1 & \underset{\sim}{c}_2 & \underset{\sim}{c}_3 & \underset{\sim}{0} \end{bmatrix} \tag{39}$$

so muß für diese Matrizen aufgrund von (38) die lineare homogene Gleichung

$$\underset{\sim}{M}\,\underset{\sim}{x}_4 = \underset{\sim}{0} \tag{40}$$

von $\underset{\sim}{M}$ verschwindet

(41) $$\det \underset{\sim}{M} = 0.$$

Diese Gleichung stellt den Zusammenhang zwischen dem Eingangswinkel τ_1 bzw. $\sigma_1 = \tau_1 - \Theta/2$ und dem Ausgangswinkel τ_2 bzw. $\sigma_2 = \tau_2 - \Theta/2$ her.

Ersetzt man τ_2 und τ_1 durch

(42) $$x_2 = \tan(\tau_2/2), \quad x_1 = \tan(\tau_1/2),$$

dann erhält man eine algebraische Gleichung 8-ter Ordnung in den neuen Variablen x_2 und x_1:

(43) $$\det \underset{\sim}{M}(\tau_2(x_2), \tau(x_1)) = F(x_2, x_1) = 0.$$

Diese Gleichung erlaubt die Abspaltung des in x_2 und in x_1 quadratischen Faktors

(44) $$F_s(x_2, x_1) = (K/2)(1 + x_2^2)(1 + x_1^2) + 2 - (x_1^2 + x_2^2)$$

der, gleich Null gesetzt, die bereits angegebene erste Übertragungsfunktion des liniensymmetrischen homogenen 4Π-Mechanismus liefert.

Die Übertragungsfunktion der *nichtsymmetrischen* homogenen 4Π-Mechanismen bestimmen demnach eine algebraische Gleichung 6-ter Ordnung.

Wegen der zyklischen Vertauschbarkeit der Argumente x_i gilt

(45) $$F(x_2, x_1) = F(x_1, x_4) \equiv H(x_4, x_1) = 0,$$

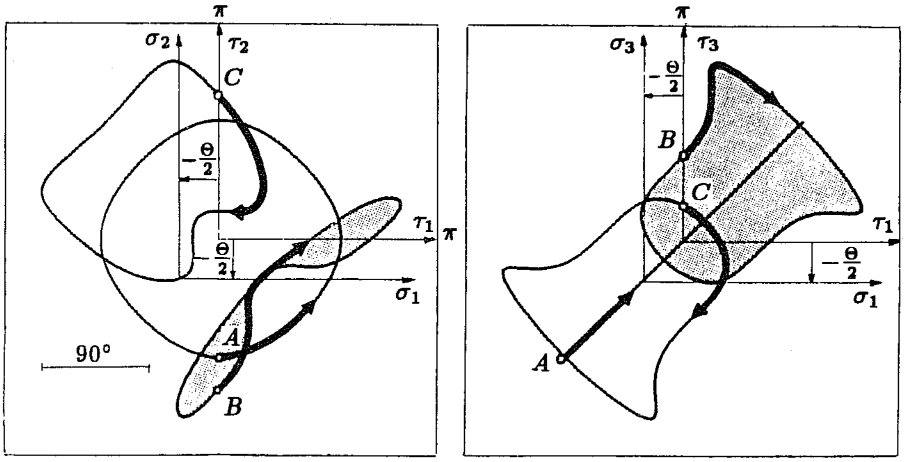
d.h., mit der ersten Übertragungsfunktion ist auch bereits die dritte bestimmt.

Bei der Berechnung der zweiten Übertragungsfunktion $G(x_3, x_1) = 0$ kann in ganz analoger Weise wie bei der Berechnung von $F(x_2, x_1) = 0$ vorgegangen werden. Allerdings ist mit der Kenntnis von zusammengehörigen Werten x_2, x_4, x_1 bzw. τ_2, τ_4, τ_1 der Lagewinkel τ_3 aus den beiden letzten Gleichungen von (19) bereits eindeutig zu bestimmen:

(46) $$\begin{aligned} \sin\tau_3 &= \sin\tau_1 + n_3(\cos\tau_2 - \cos\tau_4) \\ \cos\tau_3 &= \cos\tau_1 - (1/n_3)(\sin\tau_2 - \sin\tau_4) \end{aligned}$$

8. Rechenergebnisse

Die Abb. 7 gibt die Rechenergebnisse für einen bestimmten Systemparameter-Satz wieder. Die Annahme $\alpha = -\Theta$ führt über (10) auf $\alpha = \arccos\sqrt{\sqrt{2}-1} = 65.53019°$,



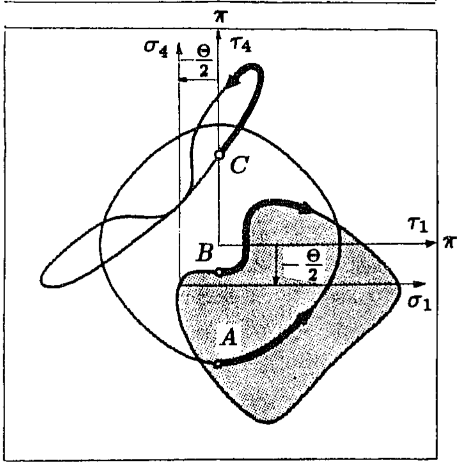


Abb. 7
Die drei Übertragungsfunktionen des homogenenen 4Π-Mechanismus mit den Systemparametern:

a	1
α	65.53019
s	5
Θ	-65.53019
h	8

$$n_3 = \tan(\Theta/2) = -0,6435942,\ n_1 = -n_3,\ K = 4[n_1 a/h + n_3 s/h]/ \\ /(n_1\sqrt{1+n_3^2}) = -1.681792.$$

Der Mechanismus besitzt zwei nichtsymmetrische Bauarten. Bei anderen Parametersätzen zeigt sich, daß entweder keine oder gleich vier nichtsymmetrische Bauarten möglich sind.

Abb. 8 zeigt die Übertragungsfunktionen nichtsymmetrischer Bauarten von verschiedenen 4Π-Mechanismen (mit gleichem n_3 und ungleichen K-Werten).

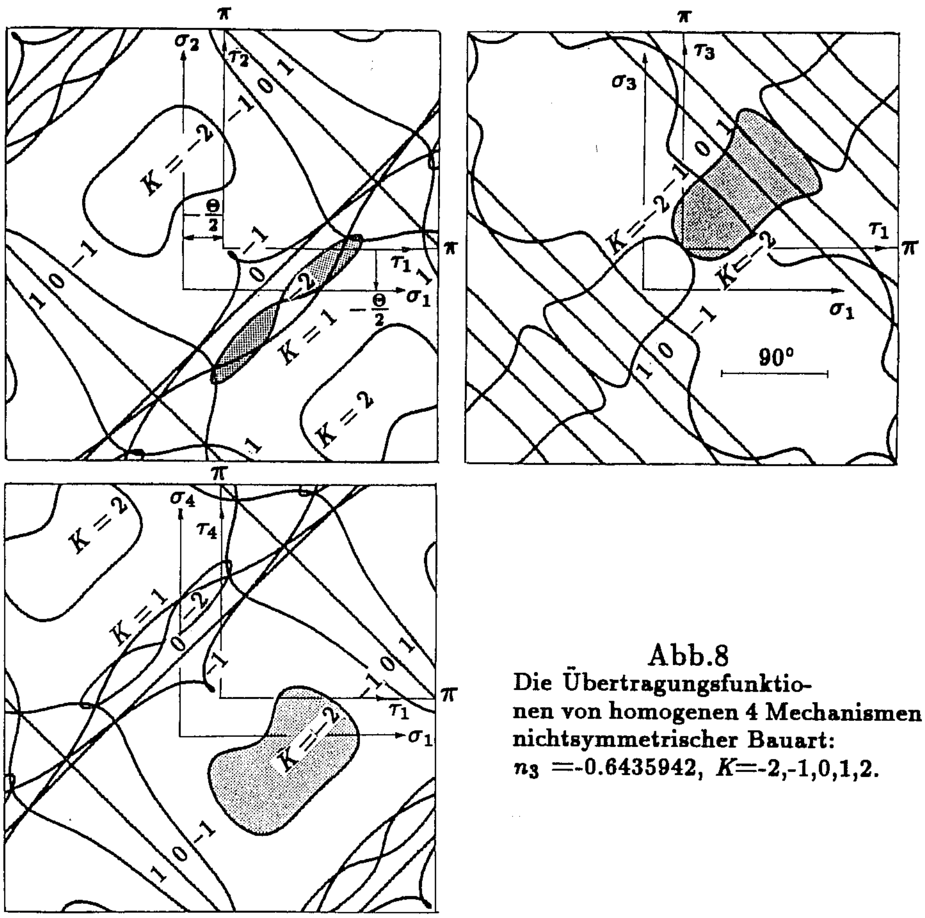


Abb.8
Die Übertragungsfunktionen von homogenen 4 Mechanismen nichtsymmetrischer Bauart:
n_3 **=-0.6435942,** K**=-2,-1,0,1,2.**

Die Graphen der Abbildungen 6,7 und 8 hat cand.ing. G. Moravi aufgrund eines von Dr. P. Dietmaier und ihm entwickelten Rechenprogramms erstellt, wofür ich ihm herzlich Dank sage.

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CONVERGENCE THEOREMS FOR A DANIELL-LOOMIS INTEGRAL

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Abstract: For a recent integral of Bobillo and Carrillo 1987, which subsumes the Daniell integral and the Dunford-Schwartz integration with respect to finitely additive measures, convergence theorems are obrtained, using local convergence in measure. Furthermore the relations between the Bobillo-Carrillo integral, the abstract Riemann integral and the Bourbaki integral are discussed.

For a semiring Ω of sets from an arbirtrary set X and $\mu : \Omega \to$ $\to [0, \infty)$ only finitely additive an analogue $R_1(\mu, \overline{\mathbb{R}})$ to the space of Lebesgue-μ-integrable functions $L^1(\mu, \overline{\mathbb{R}})$ and its Lebesgue integral has been introduced in Loomis [9], Dunford-Schwartz [5] and [8] and for which, following Loomis [9], we use the terms Riemann-μ-integrable and Riemann-μ-integral. The question, whether corresponding analogoues to the Daniell extension process, but without or weaker continuity assumptions on the elementary integral, exist, has been treated by Aumann [1], Loomis [9] and Gould [7]. Aumann's results are applicable only after the construction of a suitable integral seminorm; in Gould's paper [7] Stone's axiom is assumed, his results are therefore subsumed by the abstract Riemann integral (see for example [8], p.57, 268); Loomis [9] works without Stone's axiom, but still his three extension processes are only of Riemann power, for example if one starts with the Riemann integral on the continuous real-valued functions with compact support $C_0(\mathbb{R},\mathbb{R})$ or with the step functions $S(\Omega,\mathbb{R})$ corresponding to the ring Ω generated by the intervals $[a,b) \subset \mathbb{R}$ one gets $R_1(\mu_L|\Omega,\mathbb{R})$ and not $L^1(\mu_L,\mathbb{R})$.

The situation is different with the integral $\overline{I} : \overline{B} \to \mathbb{R}$ introduced recently by Bobillo and Carrillo [3]. It works for arbitrary function vector lattices B and non-negative linear $I : B \to \mathbb{R}$, and yields the usual L^1 in the two special cases above. In this note we prove convergence theorems for this integral, using an appropriate local "convergence in measure". In the case of Lebesgue's convergence theorem (§1 and 3), our results subsume the corresponding result for $R_1(\mu,\overline{\mathbb{R}})$ (§5); regarding the Monotone convergence theorem only a somewhat weaker version is true in $\overline{B}$ (§2). If $\mu : \Omega \to [0,\infty)$ is σ-additive, then the convergence used here is for sequences more general than pointwise convergence; if additionally Ω is a δ-ring, then $\overline{B} = R_1 = L^1$ modulo nulfunctions (§5 and 6) and one gets the usual Lebesgue convegence theorem. As an application we give in §5 a short proof of the recent result of Bobillo and Carrillo [4] that always $R_1 \subset \overline{B}$ modulo nulfunctions, in §6 we formulate a converse for the Lebesgue case, and discuss the possible relations between $\overline{B}$, R_1, L^1 and the Bourbaki extension.

Notations. $\overline{\mathbb{R}} := \{-\infty\} \cup$ reals $\mathbb{R} \cup \{\infty\} = [-\infty,\infty]$; we extend the usual $+$ in $\mathbb{R}$ to all of $\overline{\mathbb{R}} \times \overline{\mathbb{R}}$ by

$$(1) \quad \infty + (-\infty) = (-\infty) + \infty := 0, \infty \dot{+} (-\infty) = (-\infty) \dot{+} \infty := \infty, \\ \infty \underset{\cdot}{+} (-\infty) = (-\infty) \underset{\cdot}{+} \infty := -\infty,$$

$r - s := r + (-s)$, $r \dot{-} s := r \dot{+} (-s)$, $r \underset{\cdot}{-} s := r \underset{\cdot}{+} (-s)$ for $r, s \in \overline{\mathbb{R}}$. With $r \vee s := \max(r,s)$, $\wedge := \min$ and $r \cap t := (r \wedge t) \vee (-t)$ one has for $r, s \in \overline{\mathbb{R}}$, $0 \le t \in \overline{\mathbb{R}}$

$$(2) \qquad |r \cap t - s \cap t| \le 2(|r - s| \wedge t), |r \wedge t - s \wedge t| \le |r - s|, \\ |r \vee t - s \vee t| \le |r - s|;$$

for further properties used below see also Aumann's paper [1], p.442, ($^*a$ $-^*$ c). In all of the following we assume, with arbitrary set X

$$(3) \quad X \ne \emptyset, B \text{ function vector lattice} \subset \mathbb{R}^X, I : B \to \mathbb{R} \text{ linear}, I \ge 0,$$

i.e. under the on X pointwise defined $+$, $r\cdot$, $=, \le$, $\wedge$, $\vee$, $|\ |$ B is real linear space of functions $f : X \to \mathbb{R}$ containing with f, g also $f \wedge g$, $f \vee g$, $|f|$, and $0 \leq I(f)$ if $0 \leq f \in B$, where $|f|(x) := |f(x)|$ for $x \in X$.

From Bobillo's and Carillo's paper [3] we need the following definitions and results: $B^+ := \{g \in (-\infty, \infty]^X$: to each $x \in X$ exist $h_n \in B$ with $h_n \leq g$ and $h_n(x) \to g(x)\}$,

(4) $I^+(k) := \sup\{I(h) : h \in B \text{ and } h \leq k\}$, with $\sup \emptyset := -\infty$, $k \in \overline{\mathbb{R}}^X$,

with $I^+(k) \dot{+} I^+(l) \leq I^+(k \dot{+} l)$ for $k, l \in \overline{\mathbb{R}}^X$; $B^- : -B^+$, $I^-(k) := -I^+(-k)$, $B_+ := \{\varphi \in B^+ : I^+(\varphi + g) = I^+(\varphi) + I^+(g)$ for all $g \in B^+\}$, $B_- := -B_+$; B^+ and B_+ are $+$ and $\vee$ closed, B^+ is also $\wedge$ closed.

$$(5)\qquad \begin{aligned} \overline{I}(k) &:= \inf\{I^+(\varphi) : k \leq \varphi \in B_+\}, \text{ with } \inf \emptyset := \infty, k \in \overline{\mathbb{R}}^X; \\ \underline{I}(k) &:= -\overline{I}(-k), = \sup\{I^-(-\varphi) : \varphi \in B_+, -\varphi \leq k\}. \end{aligned}$$

$$(6)\qquad \overline{I}(k \dot{+} l) \leq \overline{I}(k) \dot{+} \overline{I}(l) \text{ for } k, l \in \overline{\mathbb{R}}^X,$$

$$(7)\qquad I^+(k) \leq \underline{I}(k) \leq \overline{I}(k) \leq I^-(k) \quad \text{for any } k \in \overline{\mathbb{R}}^X,$$

$I^+, \underline{I}, \overline{I}$ are monotone (increasing) on $\overline{\mathbb{R}}^X$.

The elements of $\overline{B} := \{k \in \overline{\mathbb{R}}^X : \underline{I}(k) = \overline{I}(k) \in \mathbb{R}\}$, $= B_0$ in [3], are called *I-summable*; $\overline{B}$ is a lattice, containing with f, g also $|f|$, rf with $r \in \mathbb{R}$, and any $h : X \to \overline{\mathbb{R}}$ with $h(x) = f(x) + g(x)$ only for those x, for which $f(x) \in \mathbb{R}$ and $g(x) \in \mathbb{R}$ (a slight extension of Theorem 5.2 of [3]), then $I(rf) = rI(f)$, $I(h) = I(f) + I(g)$, where $I := \underline{I} = \overline{I}$ on $\overline{B}$; $\overline{B}$ is closed under $\pm$, $\dot{\pm}$, $\underset{\cdot}{\pm}$. B is dense in $\overline{B}$ with respect to $||k||_I := \overline{I}(|k|)$. $B_{(+)} \cup B_{(-)} \subset \overline{B}$, where $B^{(+)} := \{g \in B^+ : I^+(g) < \infty\}$, $B_{(+)} := B_+ \cap B^{(+)}$, $B_{(-)} = -B_{(+)}$. $I|\overline{B}$ is the maximal extension of $I|B$ in the sense of Aumann [1] p.443 with respect to the integral(semi)norm $\overline{I}$.

1. Dominated convergence

To get convergence theorems also in the finitely additive case, a.e. or everywhere convergence is not sufficient; as in Dunford-Schwartz's work [5] (p. 101 – 104) one has to use a kind of convergence in measure, but localized (see §§4,5):

For any $T : [0,\infty]^X \to [0,\infty]$, arbitrary nets $(k_i)_{i\in J}$ with $k_i \in \overline{\mathbb{R}}^X$ for $i \in J =$ directed set, arbitrary $k \in \overline{\mathbb{R}}^X$ we need

Definition 1. $k_i \to k(T)$ means for each fixed $h \in B$ with $0 \le h$ one has $T(|k_i - k| \wedge h) \to 0$ (where e.g. $\infty - \infty = 0$ by (1)).

Lemma 1. *If $k_i,\ k \in \overline{\mathbb{R}}^X$, (k_i) net with $\overline{I}(|k_i - k|) \to 0$, then $\overline{I}(k_i) \to \to \overline{I}(k)$, $\underline{I}(k_i) \to \underline{I}(k)$, $k_i \to k(\overline{I})$.*

Proof. With (5) there are $z_i \in B_+$ with $k_i \dot{-} k \le |k_i - k| \le z_i$ and $I^+(z_i) \to 0$. Then $k_i \le k \dot{+} z_i$, $\overline{I}(k_i) \le \overline{I}(k \dot{+} z_i) \le \overline{I}(k) + \overline{I}(z_i)$ by (6), $\overline{\lim}\, \overline{I}(k_i) \le \overline{I}(k)$. Similarly $k - k_i \le z_i$, $k \le k_i \dot{+} z_i$, $\overline{I}(k) \le \le \overline{I}(k_i) + \overline{I}(z_i)$, $\overline{I}(k) \le \underline{\lim}\, \overline{I}(k_i)$, or $\overline{I}(k_i) \to \overline{I}(k)$, also if $\overline{I}(k) = \pm\infty$. Since $\overline{I}(|(-k_i) - (-k)|) = \overline{I}(|k_i - k|) \to 0$ and $\overline{I}(-l) = -\underline{I}(l)$, the $\underline{I}$-statement follows. ◇

Lemma 2. *If $k_i, k \in \overline{\mathbb{R}}^X$, $\varphi \in \overline{B}$, (k_i) net with $k_i \to k(\overline{I})$ and $\varphi \le k_i$ for $i \in J$, then $\overline{\lim}\, \underline{I}(k_i) \ge \underline{I}(k)$. If $k_i \le \varphi$, everything else unchanged, then $\underline{\lim}\, \overline{I}(k_i) \le \overline{I}(k)$.*

Proof. First with $\varphi = 0$: If $l \in B_+$, $0 \le -l \le k_+ := k \vee 0$, $l_i := k_i \wedge (-l) \ge 0$, by (2) one has $k_i = k_{i,+} \to k_+(\overline{I})$, $l_i \to k_+ \wedge (-l) = = -l(\overline{I})$, $0 \le (-l) - l_i \le -l \le$ some $h_0 \in B$ by definition of B^+, so $\overline{I}(|(-l) - l_i|) \to 0$. By Lemma 1 one has $\underline{I}(k_i) \ge \underline{I}(l_i) \to \underline{I}(-l)$ or $\underline{\lim}\, \underline{I}(k_i) \ge \underline{I}(-l) = -\overline{I}(l)$; since l was arbitrary $\ge -k_+$, $\underline{\lim}\, \underline{I}(k_i) \ge \ge \sup\{-I^+(l)\} = -\overline{I}(-k_+) = \underline{I}(k_+) \ge \underline{I}(k)$.

In the general case, one can assume $\underline{I}(k) > -\infty$ and $-\varphi = g \in \in B_{(+)}$. Then $0 \le l_i := k_i \dot{+} g \to k \dot{+} g(\overline{I})$, since $|(r \dot{+} t) - (s \dot{+} t)| \le |r - s|$ for $r, s, t \in \overline{\mathbb{R}}$. So by the above $\underline{\lim}\, \underline{I}(l_i) \ge \underline{I}(k + g)$. Now for $g \in B_+$, $k \in \overline{\mathbb{R}}^X$ with $\underline{I}(k) > -\infty$ one has

$$\underline{I}(k + g) = \underline{I}(k) + I^+(g): \tag{8}$$

$\ge$ follows from (6) since $\underline{I} = \overline{I} = I^+$ on B_+ and $\dot{+} = +$ on the right in (8). If $l \in B_+$ with $-l \le k + g$, then $-(l + g) \le k$ or $\underline{I}(k) = -\overline{I}(-k) \ge \ge -I^+(l + g) = -I^+(l) - I^+(g)$; this implies $I^-(-l) = -I^+(l) \le \underline{I}(k) + +I^+(g)$ or $\underline{I}(k + g) \le \underline{I}(k) + I^+(g)$. (8) applied to $\underline{\lim}\, \underline{I}(l_i) \ge \underline{I}(k + g)$ yields $\underline{\lim}\, \underline{I}(k_i) \ge \underline{I}(k)$ since $I^+(g) \in \mathbb{R}$.

The second statement of Lemma 2 follows from this since $\overline{B} \ni \ni -\varphi \le -k_i \to -k(\overline{I})$. ◇

Lemma 3. *If $k_i,\ k \in \overline{\mathbb{R}}^X$, $\varphi \in \overline{B}$, $k_i \to k(\overline{I})$ and $k_i - k \ge \varphi$ for $i \in J$, then $\underline{\lim}\, \underline{I}(k_i) \ge \underline{I}(k)$ and $\underline{\lim}\, \overline{I}(k_i) \ge \overline{I}(k)$; if instead $k_i - k \le \varphi$, then $\overline{\lim}\, \overline{I}(k_i) \le \overline{I}(k)$ and $\overline{\lim}\, \underline{I}(k_i) \le \underline{I}(k)$.*

Proof. If $\underline{I}(k) > -\infty$ there is $g \in B_{(+)}$ with $-g \leq k$; one can assume $-\varphi \in B_{(+)}$, then $k_i \geq k+\varphi \geq -(g+(-\varphi)) \in \overline{B}$. Lemma 2 yields $\underline{\lim}\,\underline{I}(k_i) \geq \underline{I}(k)$. Since $k_i - k \to 0(\overline{I})$, Lemma 2 gives $\underline{\lim}\,\underline{I}(k_i - k) \geq 0$; so to $\varepsilon > 0$ there are $i_\varepsilon \in J$ and $z_i \in B_{(+)}$ with $k_i - k \geq -z_i$ and $\overline{I}(z_i) = I^+(z_i) < \varepsilon$, $i \geq i_\varepsilon$; for such i then $\overline{I}(k) \leq \overline{I}(k_i + z_i) \leq \overline{I}(k_i) + \overline{I}(z_i) \leq \overline{I}(k_i) + \varepsilon$ by (6). The $\overline{\lim}$-statements follow as in Lemma 2. ◇

Lemma 3 applied to $l_i := |k_i - k|$, $l := 0$ (and Lemma 1) yield

Theorem 1. *If with k_i, $k \in \overline{\mathbb{R}}^X$, $\varphi \in \overline{B}$ one has $|k_i - k| \leq \varphi$ for $i \in J =$ directed set and $k_i \to k(\overline{I})$, then $\overline{I}(|k_i - k|) \to 0$, $\overline{I}(k_i) \to \overline{I}(k)$, $\underline{I}(k_i) \to \underline{I}(k)$.* ◇

Corollary 1. *If k_i, $k \in \overline{\mathbb{R}}^X$, $|k_i| \leq \varphi \in \overline{B}$, $k_i \to k(\overline{I})$, then $\overline{I}(k_i) \to \overline{I}(k \cap \varphi)$, $\underline{I}(k_i) \to \underline{I}(k \cap \varphi)$.*

Proof. $k_i \cap \varphi = k_i$, so $k_i \to k \cap \varphi(\overline{I})$ by (2), $|k_i - k \cap \varphi| \leq 2\varphi$. ◇

Corollary 2. *If with the assumptions of Theorem 1 additionally $k_i \in \overline{B}$, then $k \in \overline{B}$, $I(k_i) \to I(k)$. (Lebesgue's convergence theorem for $\overline{B}$).*

Corollary 3. *If $f_i \in \overline{B}$, $k \in \overline{\mathbb{R}}^X$, $f_i \to k(\overline{I})$, then $k \in \overline{B}$ if and only if $\overline{I}(|k|) < \infty$. Special case: $f_i \in \overline{B}$, $k \in \overline{\mathbb{R}}^X$, $\overline{I}(|f_i - k|) \to 0 \Rightarrow k \in \overline{B}$. ($\overline{B}$ is $\overline{I}$-closed).*

Proof. "If": $\overline{I}(|k|) < \infty$ is equivalent with $|k| \leq$ some $\varphi \in \overline{B}$, so $|f_i \cap \varphi - k| \leq 2\varphi$, $f_i \cap \varphi \to k \cap \varphi = k(\overline{I})$, $f_i \cap \varphi \in \overline{B}$ =lattice, Corollary 2. ◇

One has corresponding convergence theorems for B^+ (and B_+, = Corollary 7 in §3):

Corollary 4. *If in Lemma 2 additionally $k_i \leq k$ [resp. $k \leq k_i$], then $\underline{I}(k_i) \to \underline{I}(k)$ [resp. $\overline{I}(k_i) \to \overline{I}(k)$]. So if $k \in \overline{\mathbb{R}}^X$, $g_i \in B^+$, $\varphi \in \overline{B}$ with $\varphi \leq g_i \leq k$ and $g_i \to k(\overline{I})$, then $I^+(g_i) \to I^+(k) = \underline{I}(k)$.*

Proof. $\underline{I}(k_i) \leq \underline{I}(k)$, so $\overline{\lim}\,\underline{I}(k_i) \leq \underline{I}(k), \leq \underline{\lim}\,\underline{I}(k_i)$ by Lemma 2. Since $I^+ = \underline{I}$ on B^+ ($-p \leq g \in B^+$ with $p \in B_+$ implies $0 \leq g + p$, $0 \leq I^+(g+p) = I^+(g) + I^+(p)$, $\underline{I}(g) \leq I^+(g)$), if $k_i = g_i \in B^+$ one gets $I^+(g_i) \to \underline{I}(k)$, but $I^+(g_i) \leq I^+(k) \leq \underline{I}(k)$. ◇

In the above statements usually all assumptions are essential (see however §3): Domination by $\varphi \in B^{(+)}$ is in Theorem 1/Corollary 2 not enough: $k_i \equiv 0$, $k = 1T$ of Example 2 below; similarly for Lemma 2 ($k_i \equiv 1T$, $k = 0$), Lemma 3, "[ ]" of Corollary 4.

In Corollary 1 one cannot substitute k for $k \cap \varphi$ ($k_i \equiv 0$, $k = 1T$), so Corollary 2 is false if only $|k_i| \leq \varphi$; in Corollary 3 the existence of a $g \in B^{(+)}$ with $|k| \leq g$ cannot replace $\overline{I}(|k|) < \infty$, in Corollary 4 "$g_i \leq k$" is essential: $g_i \equiv 0$, $k = -1T$; the "if" in Corollary 3 also becomes false for "(f_i) is $\|\ \|_I$-Cauchy, i.e. $I(|f_i - f_j|) \to 0$" instead of "$\overline{I}(|k|) < \infty$", contrary to the R_1-spaces (§5) the $\overline{B}$ is in this sense not closed ($k = 1T$). In Corollary 4, $\varphi \in \overline{B}$ cannot be weakened as in Corollary 7 to $\varphi \in B^{(-)}$, there exist counterexamples B_Ω, I_μ (see §5) with μ even σ-additive.

2. Monotone convergence

For $k, l, p \in \overline{\mathbb{R}}^X$ we define $k \leq l(\overline{I})$ by $(k-l)_+ := (k-l) \vee 0 = 0(\overline{I})$ and $p = 0(\overline{I})$ by $p_n := p \to 0(\overline{I})$; by definition, $k \leq l(\overline{I}) \Leftrightarrow 0 \leq l - k(\overline{I})$.
Lemma 4. *If $k, l \in \overline{\mathbb{R}}^X$ with $k \leq l(\overline{I})$ and $\overline{I}(k) < \infty$ [resp. $\underline{I}(l) > -\infty$], then $\overline{I}(k) \leq \overline{I}(l)$ [resp. $\underline{I}(k) \leq \underline{I}(l)$].*
Proof. If $\overline{I}(l) < \infty$ to $\varepsilon > 0$ there is $g \in B_{(+)}$ with $l < g$ and $I^+(g) < < \overline{I}(l) + \varepsilon$ resp. $< -1/\varepsilon$, with (1) one has $0 \leq (k-g)_+ \leq (k-l)_+ \to 0(\overline{I})$; there is $p \in B_{(+)}$ with $k \leq p$ or $(k-g)_+ \leq (p-g)_+ \in \overline{B}$, so Theorem 1 yields $\overline{I}((k-g)_+) = 0$. Now $k \leq g \dot{+} (k-g)_+$, so $\overline{I}(k) \leq \overline{I}(g) + + \overline{I}((k-g)_+) = \overline{I}(g) = I^+(g) < \overline{I}(l) + \varepsilon$ resp. $-1/\varepsilon$. This applied to $(-l) \leq (-k)(\overline{I})$ yields $\underline{I}(k) \leq \underline{I}(l)$. ◊
Lemma 5. *If k_i, $k \in \overline{\mathbb{R}}^X$, (k_i) increasing net (i.e. $k_i \leq k_j$ if $i \leq j$) with $k_i \to k(\overline{I})$, then $k_i \leq k(\overline{I})$ for $i \in J$; if additionally $\underline{I}(k) > -\infty$, then $\underline{I}(k_i) \leq \underline{I}(k)$, $i \in J$; if furthermore $\underline{I}(k_{i_0}) > -\infty$ for some i_0 then $\underline{I}(k_i) \to \underline{I}(k)$.*
Proof. If $i \leq j$, $0 \leq (k_i - k)_+ \leq (k_j - k)_+ \leq |k_j - k|, \to 0(\overline{I})$, or $(k_i - k)_+ = 0(\overline{I})$; Lemma 4 yields the second statement. In the last there is g_0 with $\overline{B} \ni -g_0 \leq k_{i_0} \leq k_j$ if $i_0 \leq j$, $\underline{I}(k) \leq \underline{\lim}\, \underline{I}(k_j)$ by Lemma 2, so $\underline{I}(k_i) \to \underline{I}(k)$. ◊

Skipping a dualization of Lemma 5, we note, using Lemmas 3 – 5, the
Corollary 5. *If k_i, $k \in \overline{\mathbb{R}}^X$, (k_i) increasing net with $k_i \to k(\overline{I})$, $\overline{I}(|k|) < \infty$, $\underline{I}(k_{i_0}) > -\infty$ for some i_0 and all $\overline{I}(k_i) < \infty$, then $-\infty < < \underline{I}(k) = \lim \underline{I}(k_i) \leq \lim \overline{I}(k_i) = \overline{I}(k) < \infty$.*
Corollary 6. *If $f_i \in \overline{B}$, $k \in \overline{\mathbb{R}}^X$, (f_i) increasing net with $f_i \to k(\overline{I})$, $\overline{I}(|k|) < \infty$, then $k \in \overline{B}$, $\overline{I}(|f_i - k|) \to 0$, $I(f_i) \to I(k)$. (Monotone convergence theorem for $\overline{B}$).*

Proof. Corollary 3 gives $k \in \overline{B}$; $f_i \leq k(\overline{I})$ or $0 \leq k - f_i(\overline{I})$ by Lemma 4; so $(|t| - t)_+ = 2(-t)_+$ implies $|k - f_i| \leq k - f_i(\overline{I})$, Lemma 4/5 then $\overline{I}(|k - f_i|) \leq \overline{I}(k - f_i) = I(k) - I(f_i) \to 0$. ◊

Again $k_n, k \in \{0, \pm 1T\}$ with T of Example 2 show that e.g. $\overline{I} < \infty$ resp. $\underline{I} > -\infty$ are essential in the above, also $\overline{I}(|k|) < \infty$ cannot be weakened to $\overline{I}(k) < \infty$ in Corollary 6, even if additionally $\sup I(|f_n|) < < \infty$; so the usual Monotone convergence theorem is false for $\overline{B}$ with $\to (\overline{I})$ (it becomes true for a suitable extension of $\overline{B}$ which will be treated elsewhere; see also §3). The "increasing" also cannot be omitted ($k_n = 1[n, n+1) \to 0\ (\mu_L)$, §5).

3. Generalized dominated convergence

Lemma 6. *If $g \in B^+$ with $g \wedge |h| \in B_+$ for all $h \in B$, then $g \in B_+$.* This is due to Bobillo and Carrillo [4], p. 261, Remark 2b. Here $g \in B^+$ can be weakened to: $g \in \overline{\mathbb{R}}^X$ such that to each $x \in X$ with $g(x) \neq 0$ there is $h \in B$ with $h(x) \neq 0$. For $g \in B^+$ the assumptions $g \wedge |B| \subset B_+$, $g \wedge B \subset B_+$, $g \wedge |B| \subset \overline{B}$ are equivalent; without $g \in B^+$ however $g \wedge B \subset \overline{B}$ does not imply $g \in \overline{B}$ even if $I^X(g) < \infty$ (see (11); $g = 1T$ of Ex. 2).

Theorem 2. *If $f_i \in \overline{B}$, $g \in B^+$, $f_i \to g(\overline{I})$, then $g \in B_+$; if additonally $\underline{\lim}\, I(f_{i,+}) < \infty$, then $g \in \overline{B}$.*

Proof. With $h_0, h \in B$ with $h_0 \leq g$ and (2) one gets $p_i := (f_i \vee h_0) \wedge \wedge h \to (g \vee h_0) \wedge h = g \wedge h(\overline{I})$, $p_i \in \overline{B}$, $|p_i| \leq |h_0| \vee |h|$, Corollary 2 yields $g \wedge h \in \overline{B}$, $I(p_i) \to I(g \wedge h)$. Since $B^+ \wedge B \subset B^+$ and $B^+ \cap \overline{B} = = B_{(+)}$ by Guerrero-Carrillo ([4], p. 261, Rem. 2a), Lemma 6 shows $g \in B_+$. Furthermore $p_i \leq f_{i,+} + |h_0|$, so $I(g \wedge h) = \lim I(p_i) \leq \leq \underline{\lim}\, I(f_{i,+}) + I(|h_0|) =: c_0 < \infty$ independent of $h \in B$, so $I^+(g) < \infty$, $g \in B_{(+)} \subset \overline{B}$. ◊

Corollary 7. *If $g_i \in B_+$, g and $q \in B^+$ with $I^+(q) < \infty$, $-q \leq g_i \leq g$, $g_i \to g(\overline{I})$, then $g \in B_+$ and $I^+(g_i) \to I^+(g)$. Special case: $g_i \geq \varphi \in \overline{B}$.*

Proof. If $I^+(g) = \infty$, $g \in B_+$; else $g_i \in \overline{B}$, then $g \in B_+$ by Theorem 2. The assumptions imply $0 \leq g_i + q \leq g + q$ and $g_i + q \to g + q(\overline{I})$ (see before (8)), Corollary 4 yields $I^+(g_i) + I^+(q) = I^+(g_i + q) \to I^+(g + q) = = I^+(g_i) + I^+(q)$. ◊

Corollary 8. *If $f_i \in \overline{B}$, $g \in B^+$, $f_i \to g(\overline{I})$, $g_0 \in B^+$ with $|f_i| \leq g_0$ for $i \in J$ and $I^+(g_0) < \infty$, then $g \in \overline{B} \cap B_+$ and $I(|f_i - g|) \to 0$,*

$I(f_i) \to I(g)$.

Proof. (See Addendum and Lemma 7′ too). If $k \le g \in B^+$, $p \in B_+$, $-p \le k$, then $0 \le g + p$, $0 \le I^+(g+p) = I^+(g) + I^+(p)$, $I^-(-p) = -I^+(p) \le I^+(g)$, or

$$(9) \qquad k \in \overline{\mathbb{R}}^X,\ k \le g \in B^+ \text{ imply } \underline{I}(k) \le I^+(g) = \underline{I}(g).$$

Now $f_{i,+} \le |f_i| \le g_0$, so $I(f_{i,+}) \le I^+(g_0)$ and therefore $g \in \overline{B}$ by Theorem 2. Then $k_i := |f_i - g| \le g_0 + |g| \le g_0 + g + 2h_0 \in B^+$ with $h_0 \in B$, $h_0 \le g$, $I^+(g_0 + g + 2|h|) = I^+(g_0) + I^+(g) + 2I(|h|)$ since $g \in B_+ (= \overline{B} \cap B^+)$, $< \infty$; one can apply Lemma 7′, $I^X(k_i) \to 0$; but $k_i \in \overline{B}$, and $I^X = I$ on $\overline{B}$ by (12). ◇

Similar examples as above show that $\overline{\lim} I(f_i) < \infty$ does not suffice in Theorem 2, $f_{i,+} \le g_0$ or $I^+(g_0) = \infty$ do not imply $I(f_i) \to I^+(g)$ in Corollary 8; $g \in B^+$ is essential in Theorem 2 and Corollary 7/8, an analogue to Corollary 3 with $I^X(|k|) < \infty$ is false; replacing $g_i \to g(\overline{I})$ by $\to (I^+)$ or by $I^+(g_i) \to I^+(g)$ does not imply $g \in B_+$ in Corollary 7 ($g_i \equiv 0$, $g = g_0$ of Example 2), also $I^+(q) < \infty$ is essential.

Lemma 7. *If $k_i \in \overline{\mathbb{R}}^X$, $g_0 \in B^{(+)}$ with $k_i \le g_0$ for $i \in J$ and $k_i \to 0(I^+)$, then $\overline{\lim}\, I^+(k_i) \le 0$; if additionally $k_i \ge 0$ for $i \in J$, then $I^+(k_i) \to 0$.*

Proof. For $k \in \overline{\mathbb{R}}^X$, $h \in B$ one has

$$(10) \qquad I^+(k) = I^+(k \wedge h) \dot{+} I^+(k - k \wedge h):$$

by definition of I^+ one has "$\ge$" with "$=$" if $I^+(k) = -\infty$; if $p \in B$, $p \le k$, then $p \wedge h, p - p \wedge h \in B$ with $p \wedge h \le k \wedge h$, $p - p \wedge h \le k - k \wedge h$, $I(p) = I(p \wedge h) + I(p - p \wedge h) \le I^+(k \wedge h) + I^+(k - k \wedge h)$; p being arbitrary, "$\le$" follows. Choosing $h \in B$ with $h \le g_0$ one gets $k_i - k_i \wedge h \le g_0 - h$, (10) yields $I^+(k_i) \le I^+(|k_i| \wedge h) + I^+(g_0 - h)$ with $I^+(g_0 - h) = I^+(g_0) - I(h) < \varepsilon$ for suitable h, all i. Definition 1 yields Lemma 7. ◇

Addendum. For the proof of Corollary 8 we define, for any $k \in \overline{\mathbb{R}}^X$, with $\inf \emptyset = \infty$, $\sup \emptyset = -\infty$

$$(11) \qquad \begin{aligned} I^X(k) &:= \inf\{I^+(g) : k \le g \in B^+\}, \\ I_X(k) &: \sup\{I^-(p) : k \ge p \in B^-\}. \end{aligned}$$

With the definition of $\overline{I}$, $\underline{I}$, B_+, B_- and (7) one gets

$$(12) \qquad I^+ \le \underline{I} \le \min(I_X, I^X) \le \max(I_X, I^X) \le \overline{I} \le I^- \text{ on } \overline{\mathbb{R}}^X.$$

Lemma 7'. *If* $0 \le k_i \in \overline{\mathbb{R}}^X$, $k_i \le g_0 \in B^{(+)}$, $k_i \to 0(\overline{I})$, *then* $I^X(k_i) \to 0$.

Proof. To $\varepsilon > 0$ there exist $g_{i,\varepsilon} \in B^+$ with $k_i \le g_{i,\varepsilon} \le g_0$ and $I^+(g_{i,\varepsilon}) \le I^X(k_i) + \varepsilon$. So $k_i \le k_i \wedge h + (g_{i,\varepsilon} - g_{i,\varepsilon} \wedge h)$ if $0 \le h \in B$; there exist $z_i \in B_+$ with $k_i \wedge h \le z_i$ and $I^+(z_i) < \varepsilon$ if $i \ge i_\varepsilon$, then $I^+(g_{i,\varepsilon}) - \varepsilon \le I^+(k_i) \le I^+(z_i + (g_{i,\varepsilon} - g_{i,\varepsilon} \wedge h)) = I^+(z_i) + I^+(g_{i,\varepsilon} - g_{i,\varepsilon} \wedge h) = I^+(z_i) + I^+(g_{i,\varepsilon}) - I^+(g_{i,\varepsilon} \wedge h)$ by (10) and using

$$\text{(13)} \qquad \text{if } g \in B^+,\ h \in B, \text{ then } g - g \wedge h \in B^+.$$

So $g_{i,\varepsilon} \to 0(I^+)$ with directed set $J \times (0,\infty)$, Lemma 7 shows $I^+(g_{i,\varepsilon}) \to 0$, $I^+(g_{i,\varepsilon}) \to 0$, thus $I^X(k_i) \to 0$. ◇

There are analogues to Lemma 7/7' for certain certain other combinations from I^+, $\underline{I}$, I^X, $\overline{I}$, I^-, for example: Lemma 7' still holds if only $k_i \to 0\ (I^X)$, provided $\overline{B}$ satisfies Stone's axiom and $I(h \wedge \frac{1}{n}) \to 0$, $I(h - h \wedge n) \to 0$ as $n \to \infty$, $0 \le h \in B$.

4. Improper integrals

Under some additional assumptions, $\overline{B}$ is closed with respect to improper integrals, just as in the case of Riemann- or Lebesgue-integration ([8], p. 259/261):

$$\text{(14)} = S_+(B) \text{ means: (3), } 0 \le h \in B \Rightarrow h \wedge 1 \text{ and } h - h \wedge 1 \in B_+$$

$$\text{(15)} = C_\infty = C_\infty(B,I) : \text{(14), } 0 \le h \in B \Rightarrow I(h \wedge n) \to I(h) \text{ as } n \to \infty.$$

(14) implies $h \wedge t$ and $h - h \wedge t \in B_{(+)}$ if $0 \le h \in B$, $0 \le t \in \mathbb{R}$. Stone's condition "$0 \le h \in B \Rightarrow h \wedge 1 \in B$" implies (14).

Lemma 8. $S_+(B)$, $0 \le g \in B_{(+)}$, $0 \le t \in \mathbb{R}$ *imply* $g \wedge t$ *and* $g - g \wedge t \in B_{(+)}$; *conversely,* $C_\infty(B,I)$, $0 \le k \in \overline{\mathbb{R}}^X$ *with* $k \wedge n \in B_+$ *for* $n = 1, 2, \ldots$ *imply* $k \in B_+$ *with* $I^+(k \wedge n) \to I^+(k)$ *as* $n \to \infty$.

Proof. To g exist $0 \le h_n \in B$ with $I(h_n) \to I^+(g)$, so $\overline{I}(|g - h_n|) \to 0$; $\overline{I}(|g \wedge t - h_n \wedge t|) \le \overline{I}(|g - h_n|) \to 0$, implying $g \wedge t \in \overline{B}$, by definition of B^+ and with $B \wedge t \subset B^+$ the $g \wedge t$ are in B^+, so $g_1 t \in B^* \cap \overline{B} = B_{(+)}$; similarly $g - g \wedge t \in B_{(+)}$. If all $k \wedge n \in B_+$, $k \in B^+$ by the remark after Lemma 6; if $0 \le h \in B$, $k \wedge h$ and $k \wedge n \wedge h \in B^+$ with $0 \le k \wedge h - k \wedge h \wedge n \le h - h \wedge n$, Corollary 4 shows $I^+(k \wedge n) \ge I^+(k \wedge h \wedge n) \to I^+(k \wedge h)$,

$\approx I^+(k)$ for suitable h, so $I^+(k \wedge n) \to I^+(k)$. If $I^+(k) = \infty$, $k \in B_+$; else $(k \wedge n) \wedge h \in B_{(+)}$, which is min-closed by [3], p.248, Corollary 7 shows $k \wedge h \in B_+$. So $k \in B_+$ by Lemma 6. $\diamondsuit$

The first part of Lemma 8 becomes false if only $g \in B_+$.

Theorem 3. *If $C_\infty(B,I) = (15)$ holds and $k \in \overline{\mathbb{R}}^X$, then $k \in \overline{B}$ if and only if $k \cap n \in \overline{B}$ for $n = 1, 2, \dots$ and $\sup\{I(|k \cap n|) : n \in \mathbb{N}\} < \infty$; then $I(|k - k \cap n|) \to 0$, $I(k \cap n) \to I(k)$ (with $k \cap n = (k \wedge n) \vee (-n)$).*

Theorem 3 becomes false with $k \cap h$ instead of $k \cap n$, even if $0 \leq k \leq 1$, $0 \leq h \in B = B_\Omega$, $I = I_\mu$ as in (17) below, $I(k \wedge h) \equiv 0$: $k = 1T$ of Example 2, §5; so "improper" here is meant only with respect to unbounded functions, not with respect to "unbounded support". Theorem 3 also becomes false without C_∞ by

Example 1. $X = \mathbb{N}$, $B = \{(x_n)_{n \in \mathbb{N}} : \lim(x_n/n) \text{ exists} \in \mathbb{R}\}$, $I =$ this lim, $k = (n^2)$; $k \notin \overline{B}$, though X is a I-nulset, $I(1X) = 0$; even $h \wedge 1 \in B$ if $h \in B$.

Proof of Theorem 3. "If": Since $k_\pm \wedge n = (k \cap n)_\pm$, $k = k_+ - k_-$, $|k - k \cap n| = |k| - |k| \wedge n$, one can assume $k \geq 0$. To $k_n := k \wedge n - k \wedge (n-1) \in \overline{B}$ and $\varepsilon > 0$ there are $g_n \in B_{(+)}$ with $k_n \leq g_n$ and $I^+(g_n) \leq I(k_n) + \varepsilon 2^{-n}$, $n \in \mathbb{N}$. By recursive definition there is a unique sequence (z_n) with $z_n \in B_{(+)}$,

$$k_{n+1} \leq z_{n+1} = g_{n+1} \wedge (2(z_n - z_n \wedge \tfrac{1}{2})) \wedge 1 \text{ and}$$
$$I(z_n) \leq I(g_n),\ n \in \mathbb{N},\ z_1 = g_1:$$

$z_{n+1} \in B_{(+)}$ by Lemma 8 and the $\wedge$-closedness of $B_{(+)}$; if $k_{n+1}(x) > 0$, there $k > n$, $1 = k_n \leq z_n$, $z_{n+1} = 1 \geq k_{n+1}$. If $w_m := \Sigma_{j=m}^\infty z_j$, $w_m \in B^+$ by [3], p. 246. One has $w_m \wedge n = (\Sigma_{j=m}^{m+2n} z_j) \wedge n$: If $z_q(x) > 0$ for some $q > m + 2n$, $z_{q-1}(x) > \frac{1}{2}$ and thus $z_j(x) > 1/2$ if $1 \leq j \leq q$, implying " = ". Thus $w_m \wedge n \in B_{(+)}$ for $n \in \mathbb{N}$ by Lemma 8, again by Lemma 8 the $w_m \in B_+$ with $I^+(w_m) = \lim_{n \to \infty} I(w_m \wedge n) \leq$ $\leq \lim I(\Sigma_m^{m+2n} z_j) \leq \Sigma_m^\infty I(g_j) \leq \Sigma_m^\infty I(k_j) + \varepsilon$. Since $\Sigma_1^n k_j = k \wedge n$, $\Sigma_1^n I(k_j) \leq \sup_n I(k \wedge n) < \infty$, so $I^+(w_m) < 2\varepsilon$ if $m > m_\varepsilon$. Then $0 \leq k - k \wedge n = \Sigma_{n+1}^\infty k_j \leq w_{n+1}$ implies $\overline{I}(|k - k \wedge n|) \to 0$, $k \in \overline{B}$. $\diamondsuit$

Corollary 9. *If $C_\infty = (15)$ holds, $P \subset X$, $1P \in \overline{B}$, $I(1P) = 0$, then $\infty P \in \overline{B}$ with $I(\infty P) = 0$.*

Proof. $(\infty P) \cap n = n \cdot 1P \in \overline{B}$, $I(|\infty P \cap n|) = 0$, Theorem 3. $\diamondsuit$

Corollary 9 is false without C_∞ : $P = X$ in Example 1, even $1X \in B$.

5. Riemann-integrals

We consider now B, I arising from finitely additive set functions μ, with arbitrary set $X \neq \emptyset$:

(16) $= \mu|\Omega$ means: Ω is a semiring from X, $\mu : \Omega \to [0,\infty)$ is additive on Ω

(17) $B_\Omega :=$ step functions $S(\Omega,\mathbb{R})$, $I_\mu(h) := \int h d\mu$, $h \in B_\Omega$,

where $S(\Omega,\mathbb{R})$ contains all $h = \Sigma_1^n a_m A_m$ with $n \in \mathbb{N}$, $a_m \in \mathbb{R}$, $A_m \in \Omega$, $aA := a$ on A, $:= 0$ on $X - A$, $\int h d\mu = a_1\mu(A_1) + \cdots + a_n\mu(A_n)$ (see [8], p. 17); with $\mu|\Omega$ one has (3) for B_Ω and I_μ, B_Ω satisfies Stone's axiom and C_∞. In this situation one can define μ-local convergence, $k_i \to k(\mu)$, [8] p. 69, which localizes the convergence in μ-measure of Dunford-Schwartz ([5], p. 104). By the Lemma in [8], p. 70, A 2.72, for nets one gets with Definition 1 and $I_\mu^-(k) = \inf\{I_\mu(h) : k \leq h \in B_\Omega\}$

Lemma 9. *If $\mu|\Omega$ holds and k_i, $k \in \overline{\mathbb{R}}^X$, then $k_i \to k(\mu)$ if and only if $k_i \to k(I_\mu^-)$.*

By Lemma 9 and (7), $k_i \to k(\mu)$ always implies $k_i \to k(\overline{I}_\mu)$; the converse is in general false: $X = [0,1)$, $\Omega = \{[a,b) : 0 \leq a \leq b \leq 1\}$, $\mu =$ Lebesgue measure on Ω, $\mathbb{Q} =$ rationals $\subset X$; then $k_n := 0 \to \to 1\mathbb{Q}(\overline{\mathrm{I}}_\mu)$ by §6, (38), but not $\to (\mu)$. This is different for "Riemann-μ-integrable" functions: The space $L(\mu,\mathbb{R}) = \mathrm{L}(\mathrm{X},\Omega,\mu,\mathbb{R})$ of μ-integrable functions of Dunford-Schwartz ([5], III.2.17, p.112) has been generalized to $R_1(\mu,\mathbb{R})$ resp. $R_1(\mu,\overline{\mathbb{R}})$ in [8], p.70, 199; if $X \in \Omega$, then $L(\mu,\mathbb{R}) = = R_1(\mu,\mathbb{R})$, but even for $X = \mathbb{R}$, $\Omega = \{[a,b)\}$ and $\mu =$ Lebesgue-measure μ_L on Ω the $L(\mu_L,\mathbb{R})$ strictly $\subset R_1(\mu_L,\mathbb{R})$, there are $f \in \in R_1(\mu,\mathbb{R})$ which are not equivalent.

Lemma 10. *For $\mu|\Omega$ and f_i, $f \in R_1(\mu,\overline{\mathbb{R}})$ the convergences $f_i \to \to f(\mu)$ and $f_i \to f(\overline{I}_\mu)$ are equivalent.*

Proof. If $f_i \to 0(\overline{I}_\mu)$, $0 \leq h \in B := B_\Omega$, $\varepsilon > 0$, there are $i_\varepsilon = i_{\varepsilon,h}$, $z_i = z_{i,h} \in B_+$ with $g_i := |f_i| \wedge h \leq z_i$ and $I^+(z_i) < \varepsilon$, $i > i_\varepsilon$. Now $g_i \in \in R_1(\mu,\mathbb{R})$, the g_i are bounded with Ω-bounded support, so by [8], A 7.114, p.257 the g_i are "proper Riemann-μ-integrable", i.e. $\in R_e^1(\mu,\mathbb{R}) = \mathrm{I}_\mu^-$-closure of B in $\mathbb{R}^{\mathrm{X}}$ in the sense of Aumann [1]. For $g \in R_e^1(\mu,\mathbb{R})$ one has almost by definition Riemann-μ-integral $\int g d\mu = = I_\mu^+(g) = I_\mu^-(g)$, so $I_\mu^-(g_i) = I_\mu^+(g_i) \leq I_\mu^+(z_i) < \varepsilon$, $i > i_\varepsilon$; by Lemma 9, $f_i \to 0(\mu)$. $\diamondsuit$

By Bobillo and Carrillo [4], $R_1(\mu,\mathbb{R}) \subset \overline{\mathrm{B}}_\Omega$ mod μ-nulfunctions; a slight generalization of this follows easily with the results of §1:

Corollary 10. *If $\mu|\Omega$ holds, then $R_1(\mu,\overline{\mathbb{R}}) \subset \overline{B}_\Omega + \{k \in R_1(\mu,\overline{\mathbb{R}}) : \int |k| d\mu = 0\}$.*

Proof. If $0 \le f \in R_1 := R_1(\mu,\overline{\mathbb{R}})$, by [8]. A. 7.124 c, p. 259, there are $h_n \in B := B_\Omega$ with $0 \le h_n \le h_{n+1} \le f$, $h_n \to f(\mu)$, $I_\mu(h_n) \to \int f d\mu$. Then $g := \lim h_n \in B^+$, $\le f$, and $h_n \to g(\mu)$; Lemma 9, (7) and Corollary 8 give $g \in B_{(+)} \subset \overline{B}$, $I(h_n) \to I(g) = \int f d\mu$. Since (h_n) is Cauchy with respect to $\|\cdot\|_\mu := \int |\cdot| d\mu$, by definition $g \in R_1(\mu,\overline{\mathbb{R}})$, $\int g d\mu = \lim \int h_n d\mu = \int f d\mu$; if $0 \le k := f - g$, $0 = h_n - h_n \to k(\mu)$, $k \in R_1(\mu,\overline{\mathbb{R}})$, $\int |k| d\mu = 0$. With $f = f_+ - f_-$ and the linear and lattice properties of R_1 and $\overline{B}$, $\int \cdot \cdot d\mu$ and $I = I_\mu$ one gets (where $f_- \neq 0$, $f_+ = 0$, there the g, k for f_+ vanish too; one can arrange even $g(x) \neq \infty$ for $x \in X$): If $f \in R_1(\mu,\overline{\mathbb{R}})$, then

$$(18) \qquad \begin{aligned} &f = g + k, g \in \overline{B}_\Omega \cap R_1(\mu,\mathbb{R}),\ k \in R_1(\mu,\overline{\mathbb{R}}) \text{ with} \\ &\int |k| d\mu = 0,\ k = 0(\mu),\ \int f d\mu = I(g). \ \diamondsuit \end{aligned}$$

$R_1 \subset \overline{B}$ mod μ-nulfunctions of [4] is the only relation one has in general between R_1 and $\overline{B}_\Omega$: There exists X, a semiring Ω of sets from X and an even σ-additive $\mu : \Omega \to \{0,1\} \subset \mathbb{R}$ such that simultaneously $R_1(\mu,\mathbb{R}) - \overline{\mathrm{B}}_\Omega \neq \emptyset$, $\overline{B}_\Omega - (L_1(\mu,\overline{\mathbb{R}}) \cup R_1(\mu,\overline{\mathbb{R}})) \neq \emptyset$, $L_1(\mu,\mathbb{R}) - (\overline{\mathrm{B}}_\Omega \cup \cup R_1(\mu,\overline{\mathbb{R}})) \neq \emptyset$, $(\overline{B}_\Omega \cap L_1(\mu,\mathbb{R})) - \mathrm{R}_1(\mu,\overline{\mathbb{R}}) \neq \emptyset$ and $X = \cup\Omega$. We give only

Example 2. There is a semiring Ω, a σ-additive $\mu : \Omega \to \{0,1\}$, a set $T \subset X$ and a $g_0 \in B^{(+)}$ with $1T = 0(\mu)$, so $1T \in R_1(\mu,\mathbb{R})$, but $1T \notin \overline{B}_\Omega$, though even $1T \le g_0$ and $I_\mu^+(g_0) = 0$: $X := \mathbb{N}_0 \times \mathrm{J}$, $\mathbb{N}_0 = \{0,1,2,\ldots\}$, $J = [0,1) \subset \mathbb{R}$, Ω contains all M of the form $\{n\} \times E$, $\{n\} \times (J - E)$, $F \times \{y\}$ or $(N_0 - F) \times \{y\}$ with $0 \neq n \in \mathbb{N}_0$, E finite $\subset J$, $0 \notin F$ finite $\subset \mathbb{N}_0$, $y \in J$; $\mu : \Omega \to \{0,1\}$ is defined by $\mu(\{n\} \times (J - E)) = 1$, $\mu(M) = 0$ for all other $M \in \Omega$; $T := \{0\} \times J$, $g_0 := 1W$ with $W = \cup_{y \in J} A_y$ with $A_y := \mathbb{N}_0 \times \{y\}$ if $y \neq$ all $y_{n,k}$, $A_{y_{n,k}} := \{0, k, k+1, \ldots\} \times \{y_{n,k}\}$, where the $y_{n,k}$ are choosen as follows: for $n \in \mathbb{N}$, $I_n := (1 - \frac{1}{n}, 1 - \frac{1}{n+1})$, $y_{n,k} \in I_n$ for $k = 1, 2, \ldots$ with $y_{n,k} \neq y_{n,l}$ if $k \neq l$. Furthermore $\overline{B}_\Omega \subset R_1(\mu,\overline{\mathbb{R}})$. By definition $h_n := 0 \to 1T(\mu)$, but $1T \notin \overline{B}$, $B := B_\Omega$: Else $I(1T) = \int 1T d\mu = 0$ by Corollary 2, there is $g \in B_+$ with $1T \le g$, $I^+(g) < 1/2$, by Lemma 1 of [2] and since $B^+ \cap \overline{B} = B_{(+)}$ one can assume $g = 1M$. If $J_n := \{y \in J : \{0, n, n+1, \cdots\} \times \{y\} \subset M$, $J = \cup_1^\infty J_n$, so one J_{n_0}

is infinite; then $l := 1(\{n_0\} \times (J - J_{n_0})) \in B^+$ with $I^+(l) = 0$; since $1(\{n_0\}\times J) \leq 1M+l$, one has $1 \leq I^+(1M+l) = I^+(1M)+I^+(l) < \frac{1}{2}+0$, a contradiction. ($\overline{I}(1T) = \infty$ by Corollary 11.)

Finally $\overline{B} \subset R_1$ by the following criteria, since one can easily verify (c), for $\mu(A) > 0$ with (20), then $-g \in B^+$.

If $\mu|\Omega$ holds, the following conditions are equivalent for $B := B_\Omega$, I_μ:

(19) (a) $\overline{B} \subset R_1 := R_1(\mu,\overline{\mathbb{R}})$ (b) $B_{(+)} \subset R_1$
(c) if $0 \leq g \leq 1A$ with $A \in \Omega$, $g \in B_+$, then $g \in R_1$.

(20) $\Rightarrow \mu|\Omega,\ 0 \leq g \leq 1A,\ A \in \Omega,\ g \in B_{\Omega,+}\ \Rightarrow,$
$\Rightarrow [g \in R_1(\mu,\mathbb{R}) \Leftrightarrow \mathrm{I}^-(\mathrm{g}) = \mathrm{I}^+(\mathrm{g}) \Leftrightarrow \mathrm{I}^+(\mathrm{g}) + \mathrm{I}^+(-\mathrm{g}) = 0]$.

(19) follows from the closure properties of R_1 ([8], A 7.124 (f) $\Rightarrow$ (a), A 7.121, A 3.56) with Lemma 10 for $p_n + g_n$, $B_- \ni -p_n \leq f \leq g_n \in B_+$; [8], A 7.114 gives (20).

So generally $R_1 \not\subset \overline{B}_\Omega$; we can however characterize the sets on which $R_1 \subset \overline{B}_\Omega$. For this we need

Corollary 11. *If $\mu|\Omega$ holds and $f \in R_1(\mu,\overline{\mathbb{R}})$, then $f \in \overline{B}_\Omega$ if and only if there is $g \in \overline{B}_\Omega$ with $|f| \leq g$; then $\int f d\mu = I_\mu(f)$.*

Proof. Only "if": $g := |f| \in \overline{B}_\Omega$. For the "if", one can assume $f \geq 0$ as in the proof of Corollary 10; with h_n as there one has $\int h_n d\mu \to \int f d\mu$, $0 \leq f - h_n \leq g$, $h_n \to f(\overline{I}_\mu)$, so $f \in \overline{B}_\Omega$ and $I_\mu(f) = \lim I_\mu(h_n) = \int f d\mu$ by Corollary 2. ◇

Definition 2 with $\mu|\Omega$: $\mathcal{R}(\mu) := \{M \subset X : f \in R_1(\mu,\overline{\mathbb{R}}),\ f = 0$ on $X - M \Rightarrow f \in \overline{B}_\Omega\}$.

$\mathcal{R}(\mu)$ is complete, i.e. if $P \subset M \in \mathcal{R}(\mu)$, then $P \in \mathcal{R}(\mu)$.

Theorem 4. *If $\mu|\Omega$ holds and $M \subset X$, then $M \in \mathcal{R}(\mu)$ if and only if $1P \in \overline{B}_\Omega$ for each strong μ-nulset $P \subset M$.*

Here P is called a *strong μ-nulset* iff $1P \in R_1(\mu,\mathbb{R})$ and $\int 1P d\mu = 0$, or equivalently iff $1P \to 0(I_\mu^-)$ ([8], p. 69).

Proof of "if": By corollary X and as there we can assume $0 \leq f \in R_1(\mu,\overline{\mathbb{R}})$ with $\int f d\mu = 0$, $f = 0$ outside M; with Theorem 3 it is enough to show $f \wedge 1 \in \overline{B}$, $B = B_\Omega$, $C_\infty(B, I_\mu)$ holds since step functions are bounded, so we assume $f \leq 1$. The $P_n := \{x \in X : f(x) > 1/n\}$ are strong μ-nulsets $\subset M$, since $\frac{1}{n}P_n \leq f = 0(\mu)$, so $1P_n \in \overline{B}$, $I(1P_n) = 0$ by Corollary 11, $n \in \mathbb{N}$. $0 \leq f_n := f \wedge 2^{-n} - f \wedge 2^{-n-1} \leq 2^{-n}P_{n+1}$ with $f = \Sigma_1^\infty f_n$; to $\varepsilon > 0$ there are $g_n \in B_+$ with $2^{-n}P_{n+1} \leq g_n \leq 2^{-n}$, $I(g_n) < \varepsilon \cdot 2^{-n}$, using Lemma 8. So $g := \Sigma_1^\infty g_n \in B^+$ and $l_n := \Sigma_1^n g_j \to$

$\to g$ uniformly on X; this implies $l_n \to g(\mu)$, with $I(l_n) < \varepsilon$ for $n \in \mathbb{N}$. Then $g \in B_+$ with $I^+(g) \le \varepsilon$, $g \in \overline{B}$, by Corollary 7. Obviously $f \le g$, Corollary 11 yields $f \in \overline{B}$. ◊

Corollary 12. *If $\mu|\Omega$ holds, $\mathcal{R}(\mu)$ of Definition 2 is a ring containing all $M \subset X$ to which there is $g \in \overline{B_\Omega}$ with $1M \le g$, especially*

$$\Omega \subset \{P \subset M : 1M \in R_1(\mu,\mathbb{R}) \cap B_\Omega^+\} \subset \{P \subset M : 1M \in \overline{B_\Omega}\} \subset \mathcal{R}(\mu).$$

Proof. Theorem 4, Corollary 11 and the linearity of $\to (I_\mu^-)$ give the "ring" and $M \in \mathcal{R}(\mu)$ if $1M \le g \in \overline{B}$, $B = B_\Omega$. One even has

$$R_1(\mu,\overline{\mathbb{R}}) \cap B^+ \subset B_{(+)} \subset \overline{B}: \tag{21}$$

If $g \in R_1 \cap B^+$, then $g \ge$ some $h \in B$, so one can assume $g \ge 0$; the proof of Corollary 10 yields $g \in B_{(+)}$. ◊

One can also show that $\mathcal{R}(\mu)$ is closed with respect to certain countable unions: If $M = \cup_1^\infty M_m$ with $1M_m \in \overline{B_\Omega}$, $1M_n \to 1M(\overline{I}_\mu)$, $1M \in B_\Omega^+$, then $M \in \mathcal{R}(\mu)$. Special case: $M = \cup_1^\infty A_m \in \mathcal{R}(\mu)$ if $A_m \in \Omega$ and $1 \cup_1^n A_m \to 1M(\mu)$ (directly: Theorem 4, Corollary 7, 11). $M = X$ gives $R_1(\mu,\mathbb{R}) \subset \overline{B_\Omega}$ if one of the following conditions is true:

(22) $1P \in \overline{B_\Omega}$ if P strong μ-nulset

(23) there are $A_m \in \Omega$ with $X = \cup_1^\infty A_m$ and $1(\cup_1^n A_m) \to 1X(\overline{I}_\mu)$

(24) there is a locally finite countable Ω-partition of X

(25) there are $A_m \in \Omega$ with $X = \cup_1^\infty A_m$ and μ is σ-additive on Ω

(26) $I^+(1X) < \infty$ (equivalently: μ is bounded on the ring generated by Ω, or $1X \in \overline{B_\Omega}$, or $1X \in R_1(\mu,\mathbb{R})$)

(27) all $\{x\} \in \Omega$, $x \in X$ (equivalently: $[0,\infty]^X \subset B_\Omega^+$), then even $R_1(\mu,\overline{\mathbb{R}}) = \overline{B}_\Omega$ by (19), (20), (21).

Example for (24) or (25): $X = \mathbb{R}^n$, $\mu =$ Lebesgue measure μ_L^n,

$$\Omega = \Omega_n := \{\textstyle\prod_1^n [a_j, b_j) : a_j \le b_j,\ a_j,\ b_j \in \mathbb{R}\}. \tag{28}$$

By (25) an "example 2" with $X =$ countable union of $A_m \in \Omega$ does not exist.

Finally, one can always force $R_1(\mu,\overline{\mathbb{R}}) \subset$ some $\overline{B}$ with $\int \cdot\cdot\, d\mu = = I$; $\Sigma := \{D = (A - M) \cup P : A \in \Omega$, M and P strong μ-nulsets$\}$, $\nu(D) := \mu(A)$, $B = B_\Sigma$, $I = I_\nu$; then $R_1(\nu,\overline{\mathbb{R}}) = R_1(\mu,\overline{\mathbb{R}})$, integrals and strong nulsets coincide ([8], p.199, A 6.148).

Also always $0 \leq f \in R_1(\mu,\overline{\mathbb{R}}) \Rightarrow f \in (B_\Omega)^*_+$ of [3], p.235, though $\int f d\mu < \overline{I}(f)$ if $\overline{I}(f) = \infty$ ([4], p.263, Proposition 1).

6. Lebesgue-, Daniell- and Bourbaki-integrals

In this section we additonally assume Daniell's continuity condition, = σ-stetig in Floret's work [6], p.43:

(29) (3) and $I(h_n) \to 0$ if $0 \leq h_{n+1} \leq h_n \in B$, $n \in \mathbb{N}$, with $h_n(x) \to 0$ for each $x \in X$;

then the space $L^1 := L^1(B,I) := L^1(B,I,\overline{\mathbb{R}}) \subset \overline{\mathbb{R}}^X$ of Daniell-I-integrable functions with integral extension $J : L^1 \to \mathbb{R}$ is well defined (e.g. [6], p. 77; Daniell- "summable" in Pfeffer's work [10] (p. 60); = closure of B in $\overline{\mathbb{R}}^X$ with respect to a suitable integral-seminorm in Aumann's paper [1] (p. 448 – 450)). Here one has an analogue to the statement of Corollary 10:

Theorem 5. *In the Daniell situation* (29), $\overline{B} \subset L^1(B,I) \cap \overline{B} \cap \mathbb{R}^X + \overline{B}_n$, *with* $\overline{B}_n := \{f \in \overline{B} : \overline{I}(|f| = 0\}$, *and* $I = J$ *on* $L^1 \cap \overline{B}$.

Our proof is fundamentally similar to that of Corollary 10, but more involved and somewhat lengthy, so we omit it here.

Corresponding to the remarks before Example 2, $\overline{B} \subset L^1 + \overline{B}_n$ is the only generally true relation of this type:

Example 3. There is a σ-algebra Ω and a σ-additive $\mu : \Omega \to \{0,1\}$ such that

$$(30) \qquad R_1(\mu,\overline{\mathbb{R}}) = L^1(\mu,\overline{\mathbb{R}}) = L^1(B_\Omega, I_\mu) = L_1(\mu,\overline{\mathbb{R}}) \subsetneqq \overline{B}_\Omega:$$

X uncountable, $\Omega = \{M \subset X : M \text{ or } X - M \text{ countable and } \not\ni x_0\}$, $\mu =$ Dirac measure δ_{x_0} in $x_0 \in X$, even τ-continuous (see after (35)). There are also algebras Ω for which with $\mu = \delta_{x_0}$,

$$(31) \qquad R_1 \subsetneqq L^1 = L_1 := L^1 + \{k \in \overline{\mathbb{R}}^X : k \cdot 1A = 0 \ \mu - a.e. \text{ for each } A \in \Omega\} \subsetneqq \overline{B}_\Omega.$$

Example 4. There is an algebra K and a σ-additive $\mu : K \to [0,1]$ such that

$$(32) \ R_1(\mu,\overline{\mathbb{R}}) = \overline{B}_K \subsetneqq \overline{B}_K + \{f \in \overline{\mathbb{R}}^X : f = 0 \mu - a.e.\} \subsetneqq L^1(\mu|K,\overline{\mathbb{R}}) = \overline{B}_{\Omega_1},$$

i.e. L^1 differs from $\overline{B}$ by more than just L^1-nulfunctions (see (38)):

$X = [0,1]$, Ω =ring K generated by all intervals $\subset X$ (thus $\{t\} \in K$, $t \in X$), μ = Lebesgue measure $\mu_L^1|K$, $\Omega_1 = \{[a,b)\}$ of (28); here $L^1 = L^1(B_K, I_\mu)$ = usual {Lebesgue integrable $f : X \to \overline{\mathbb{R}}$}, if $G :=$ $:= \cup_1^\infty (r_m, r_m + 3^{-m})$, $P := \{r_m : m \in \mathbb{N}\}$:= rationals $\subset X$, $B := B_K$, then $1P \notin \overline{B}$, $1G \notin \overline{B} + \{f = 0 \ \mu - a.e.\}$: In $f = 1G + p \in \overline{B}$ with $p = 0$ a.e. one can assume $0 \le f \le 1$ since $\overline{B}$ is a lattice ([3], p. 252), thus f and $1 - f \in [0,\infty)^X \subset B^+$, $f \in B_{(+)}$, $1 = I^+(f) + I^+(1-f)$, $I^+(f) \le \int f dx = \int 1G dx \le \frac{1}{2}$; by definition of G to each $x \in X$ and $\varepsilon > 0$ there is $y \in (G - \{p \ne 0\}) \cap (x - \varepsilon, x + \varepsilon)$, $f(y) = 1$, thus $I^+(1-f) = 0$, a contradiction. $\overline{B} = R_1(\mu|K,\overline{\mathbb{R}}) = R_1(\mu|\Omega_1,\overline{\mathbb{R}}) \subset L^1 =$ $= \overline{B}_{\Omega_1}$ by (27) and (38).

By simple disjoint union one can combine Examples 2 – 4 into one X, Ω, μ. Example 4 shows that a converse of Theorem 5 is false; it also shows that the extension process $B \to \overline{B}$ is in general not monotone in B – contrary to the Lebesgue, Daniell and Bourbaki extensions.

Furthermore the convergence $\to$ $(\overline{I})$ used here is in general not comparable with that of L^1, i.e. pointwise (almost everywhere) convergence, not even in the situation $\mu|\Omega$ with σ-additive μ.

Only under additional assumptions can one say more:

If $I|B$ is monotone-net-continuous, = Bourbaki-integral (Pfeffer [10], p. 44), = τ–stetig (Floret [6], p.336), then the space $L^\tau :=$ $:= L^\tau(B,I) = L^\tau(B,I,\overline{\mathbb{R}})$ of Bourbaki-I-integrable functions ($\mathcal{L}^\#$ in [10], ${}^\tau\mathcal{L}_1$ in [6], p.338) and the corresponding integral extension $I^\tau :$ $: L^\tau \to$ R are well defined with $L^1(B,I,\overline{\mathbb{R}}) \subset L^\tau(B,I,\overline{\mathbb{R}})$, $J = I^\tau|L^1$; then I^+ is additive on B^+, i.e. $B_+ = B^+$ and $\overline{I}$ = upper Bourbaki integral (Bobillo-Carrillo [3], p. 247), thus

(33) $\qquad$ (3), I τ-continuous on $B \Rightarrow \overline{B}$ = Bourbaki extension
$$L^\tau(B,I,\overline{\mathbb{R}}),\ I = I^\tau,$$

Daniell-$L^1(B,I) =: L^1 \subset \overline{B} = L^\tau = L^1 + L_n = L^1 + \overline{B}_n$ by [6] (p. 340) or our Theorem 5, n := nulfunctions, with $\subset$ generally strict by Example 3.

If $B = C_0(X,\mathbb{R})$ with arbitrary Hausdorff space X, then any nonnegative linear $I : B \to \mathbb{R}$ is τ-continuous ([6], p.337), L^τ is defined and (33) holds. With Pfeffer ([10], p.37) one gets for $B = C_0 :=$ $:= C_0(X,\mathbb{R})$ and any nonnegative linear $I : C_0 \to \mathbb{R}$, automatically τ-continuous

(34) $\qquad$ X locally compact, all open $G \subset X$ are σ-compact $\Rightarrow$

$$\Rightarrow L^1(C_0,I) = L^\tau(C_0,I) = \overline{C}_0;$$

example: $X = \mathbb{R}^n$, I =Riemann/Lebesgue integral (see (38)).

In the situation $\mu : \Omega \to [0,\infty)$ with σ-additive μ on the semiring Ω, $B = B_\Omega$,

(35) $\quad I_\mu = \int \cdot \cdot \, d\mu$ is τ-continuous on B_Ω iff μ is τ-additive on Ω (see (17))

(there are such μ, Ω = algebra, but μ not τ-continuous – the converse holds for rings). As a sufficient criterium on has:

If $\mu|\Omega$ = (16) holds, μ is σ-additive on Ω and if for any index set S

(36) to $A, A_s \in \Omega$, $s \in S$, with $A_s \supset A$, exist countable $S_0 \subset S$, μ-nulset P with $\bigcup_{s\in S} A_s = P \cup \bigcup_{s\in S_0} A_s$

is true, then for $B = B_\Omega, I = I_\mu$ of (17) one has with $\overline{B}_n$, L_1 of Theorem 4, (33)

(37) $\quad I|B$ is τ-continuous,

$$L^1(B,I) = L^1(\mu,\overline{\mathbb{R}}) \subset \overline{B} = L^\tau(B,I) = L^1 + \overline{B}_n \subset L_1(\mu,\overline{\mathbb{R}}) :$$

By (36) μ is τ-additive on Ω, by (35) I τ-continuous on B; with (33) and Theorem 5 it is enough to show $\overline{B}_n \subset L_{1,n}$. This being a local property, it suffices to show that $g \in L^1$ if $0 \le g \le 1A$, $g \in B^+$, $A \in \Omega$, since then $I^+(g) = \int g d\mu$ by (33); $g \in L^1$ follows if g is μ-measurable, and the latter is an immediate consequence of (36) and the definition of B^+.

Example 5. There is a ring Ω and a τ-additive $\mu : \Omega \to [0,\infty)$ with (36) and

$$L^1 \subsetneqq \overline{B} = L^\tau \subsetneqq L_1:$$

X := disjoint $\cup_{i\in S} S_i$ with $S_i := S := [0,1)$, Ω_0, μ_0 as in Example 3, $\Omega_i := \Omega_1$ of (28) in S_i and μ_i := Lebesgue measure μ_L, $0 < i < 1$, Ω := ring generated by $\cup_{i\in S}\Omega_i$, $\mu = \mu_i$ on Ω_i; (36) for $\mu_L|\Omega_1$ holds by the remarks following: if $f := 1$ on all $0_i \in S_i$ with $i > 0$, else $f := 0$, then $f \in L_{1,n}$ but $\overline{I}(f) = \infty$.

The condition (36) follows from an abstract Vitali covering condition for $\mu|\Omega$; the latter is true for example for $X = \mathbb{R}^n$, μ = Lebesgue measure μ_L^n, Ω = intervals Ω_n of (28); since here additionally $X = \cup_1^\infty A_m$ with $A_m \in \Omega$, one has $L^1 = L_1$.

We collect the above results with (27) and $R_1 = L_1$ for δ-rings

([8], p. 265), $L^1(C_0, I_\mu, \overline{\mathbb{R}}) = L^1(\mu|M_n, \overline{\mathbb{R}})$:
If $\Omega_n \subset \Sigma$ semiring $\subset M_n :=$ {Lebesgue-measurable sets $\subset \mathbb{R}^n$ with finite L-measure}, $\Omega \in \{\Omega_n, \Sigma, M_n\}$, $\mu = \mu_L^n$, $I = I_\mu$, $B = B_\Omega$ or $C_0 := C_0(\mathbb{R}^n, \mathbb{R})$ (see (17), (28)), $X = \mathbb{R}^n$ or more generally open $\subset \mathbb{R}^n$ with corresponding $\Omega, \cdots$, then the following L-spaces, and their integrals, all coincide

$$(38) \qquad \begin{aligned} L^1(\mu|\Omega,\overline{\mathbb{R}}) &= L_1(\mu|\Omega,\overline{\mathbb{R}}) = L^1(B,I) = L^\tau(B_{\Omega_n}, I) = \\ &= L^\tau(C_0, I) = \overline{C}_0 = \overline{B}_{\Omega_n} = \overline{B}_{M_n} = R_1(\mu|M_n,\overline{\mathbb{R}}). \end{aligned}$$

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EXTERIOR PARALLELISM FOR POLYHEDRA

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Abstract: The aim of this note is to extend the theory of parallel differentiable immersions to the piecewise linear case. Parallelism for differentiable immersions has been established by H.R. Farran and S.A. Robertson [4] and was studied in several subsequent papers ([3], [8] and [9]). It has strong relations to the geometry of the normal bundle ([9]) and to the theory of focal points ([7]).

We mainly shall concentrate on the 1-dimensional case, because there good motivations can be obtained for the study of higher-dimensional polyhedra. The main results obtained in [3] and [8] for the parallelism of differentiable curves can be transferred to the piecewise linear situation. The arguments are rather elementary and therefore proofs are only sketched in these cases.

The behaviour of polyhedral 2-manifolds in E^3 and E^4 with respect to exterior parallelism is representative for that of higher-dimensional polyhedra. In addition to the 1-dimensional case two local obstructions to the existence of parallel polyhedra to a given one occur. In this context some kind of normal curvature will be developed for polyhedra.

1. Parallel polygons

The introduction of the notion of parallelity for polygons can be motivated by the construction of parallel differentiable curves in the plane (or hypersurfaces in E^n). There the evolute plays an important role for the regularity of this construction. To see the analogy in the piecewise linear case look at the different situations in Figure 1, where parallel polygons to P are obtained according to our subsequent definition:

1) The polygon Q corresponds to the regular case where the parallel curve is located between the original one and its focal set.

2) The polygon S corresponds to the singular case, where the parallel curve meets the nearest focal point of the original one.

3) The polygon T corresponds to the singular case, where the evolute is met.

4) The polygon R corresponds to the regular case, where the focal set remains between the parallel curve and the original one.

All these cases are exhibited for a rectangle in Figure 2. The regular cases Q and R will be considered as parallel polygons to P while the other cases S and T will not have this property. Also we shall exclude the degenerate situation where the original polygon has angles 0 or π.

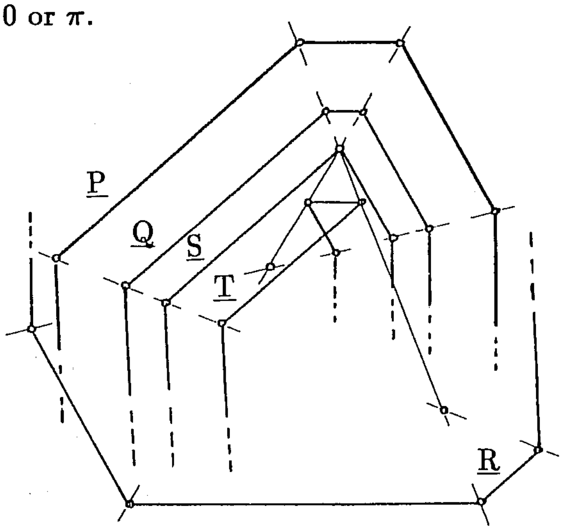

Figure 1

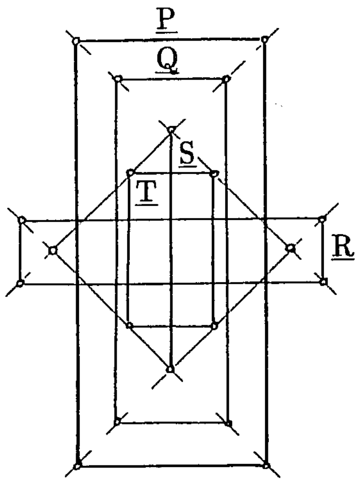

Figure 2
Parallel rectangles

For the development of the general theory let P be a *polygon* in Euclidean n-space E^n, given by its vertices. $\{p_i | i \in I\}$ and its connecting oriented line segments $s_i = p_i p_{i+1}$ from p_i to p_{i+1}, where $I = \mathbb{Z}$, $\mathbb{Z}_k$ or $\{i \in \mathbb{Z} | m \leq i \leq n\}$ for some pair $m, n \in \mathbb{Z}$. We shall restrict our considerations to the *generic* situation where the angle between s_{i-1}

and s_i at p_i lies between 0 and π for all $i \in I$. At every vertex p_i of $\underline{P}$ there is a hyperplane of local symmetry through p_i which divides the angle between the oriented line segments from p_i to p_{i-1} and p_{i+1} into equal parts. This hyperplane is called the ***symmetric normal*** of $\underline{P}$ at p_i, denoted by N_i. The intersection between N_i and N_{i+1} is called the *focal* $(n-2)$*-plane* F_i of $\underline{P}$ at s_i (if it exists) (see Figures 3a,b).

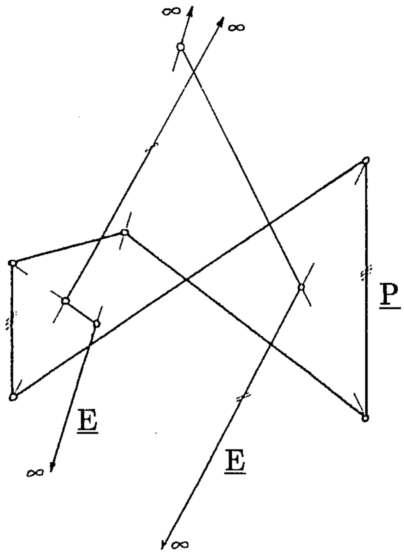

Figure 3a

Focal set and evolute in the planar case

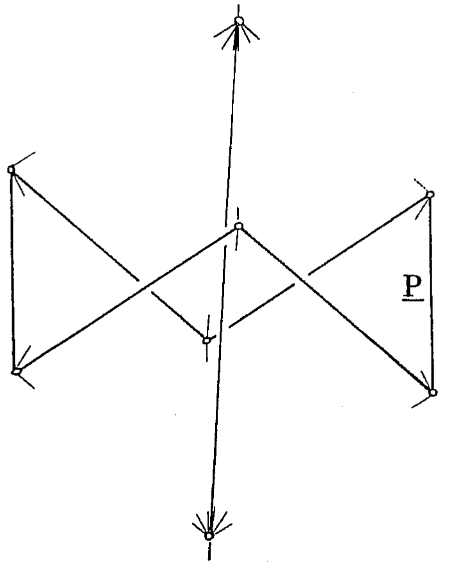

Figure 3b

Degenerated focal set in the spatial case

The focal planes of $\underline{P}$ can be used to construct a kind of *evolute* for $\underline{P}$. The following will show this in the planar case (see Figure 3a) and can be generalized easily to higher dimensions: If F_{i-1} and F_i exist, then take the connecting line segment between F_{i-1} and F_i on N_i if they are located on the same side of $\underline{P}$, and take its closed complement in the other case. If F_i exists and F_{i-1} does not exists then take the closed halfline on N_i which begins at F_i and does not meet $\underline{P}$. The similar procedure is applied, if F_{i-1} exists and F_i does not. The resulting composition of line segments and half lines gives the evolute of $\underline{P}$.

Definition 1. Two polygons $\underline{P}= \{p_i|i \in I\}$ and $\underline{Q}= \{q_i|i \in I\}$ of the same combinatorial type are called *parallel*, if for every $i \in I$ $p_i p_{i+1}$ is parallel to $q_i q_{i+1}$ and the symmetric normal of $\underline{P}$ at p_i coincides with that of $\underline{Q}$ at q_i.

Remark 1. Parallelism of polygons is an equivalence relation. Furthermore parallel polygons have coinciding focal planes and evolutes.

Definition 2. A *self-parallelism* of a polygon $\underline{P} = \{p_i | i \in I\}$ is a permutation σ of the index set such that $\underline{P \circ \sigma} := \{p_{\sigma(i)} | i \in I\}$ has the same line segments as $\underline{P}$ and is parallel to $\underline{P}$.

Remark 2. The self-parallelisms of a polygon form a group under composition of maps, the *self-parallel group* $G(\underline{P})$ *of* $\underline{P}$. Furthermore it can be seen like in the differentiable case that this group must be cyclic, because $\underline{P}$ is 1-dimensional.

Remark 3. If $\underline{P}$ and $\underline{Q}$ are parallel, then the lines which correspond under this parallelism have constant distance from each other, not depending on $i \in I$. This implies that $G(\underline{P})$ acts transitively and isometrically on the point set which is obtained by the intersection of the lines of $\underline{P}$, corresponding to a given one under the operation of $G(\mathrm{P})$, with their common normal hyperplane. This set may be called the *parallel frame* of $\underline{P}$ in our case (see [4] for the differentiable version).

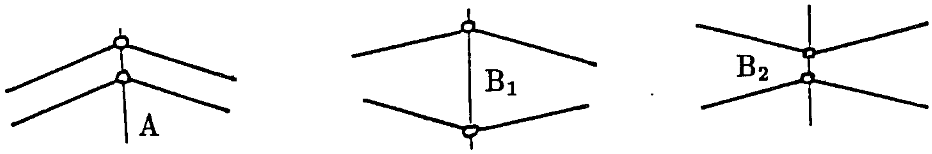

Figure 4

2. Polygons in the plane

The study of parallelism for polygons in the plane is rather simple because the choice of the unit normals to the line segments of the polygon is unique up to sign. First it should be observed that in this case there are only three possibilities for the location of the corresponding vertices and neighboring line segments (see Figure 4). Also, looking at the orientations, we see that the cases A and B cannot occur simultaneously for the same polygon. In the case A the focal points on the common symmetric normal lie outside of the segment from p_i to q_i while in the cases B they must be in the interior of that segment.

Now we shall mainly concentrate on self-parallelisms of a closed polygon $\underline{P} = \{p_i | i \in \mathbb{Z}_k\}$. According to Remarks 2 and 3 a parallel frame of $\underline{P}$ admits a transitive isometric operation of the self-parallel group $G(\underline{P})$ which can be assumed to be fixed point free, if multiple coverings are excluded. This implies $G(\underline{P}) = \mathbb{Z}_2$ in the non-trivial case, and thus k must be even and the only non-trivial self-parallelism is given by $\sigma(i) = i + k/2$.

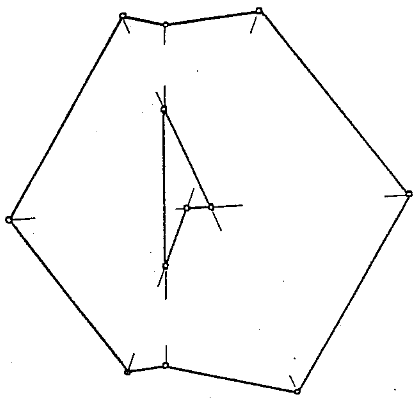

Figure 5
Self-parallel octogon in the plane

An explicit example for this situation is given by Figure 5. The relation to plane curves of constant width [1] is given by the fact that every closed tangent polygon to such a curve serves as an example for a self-parallel polygon, if the set of osculating points always contains both of the intersections of the corresponding binormal with the curve. Similarly examples of self-parallel polygons with self-intersections can be obtained from rosettes of constant width [2], for which explicit constructions have been given in [10]. Since case A of Figure 4 easily can be excluded for closed self-parallel curves, we have

Theorem 1. *Let $\underline{P}$ be a closed polygon in the plane admitting a nontrivial self-parallelism. Then $G(\underline{P}) = \mathbb{Z}_2$ and for every $i \in \mathbb{Z}_k$ the sides s_i and $s_{i+k/2}$ have a common focal point given by the intersection of the line segments from p_i to $p_{i+k/2}$ and from p_{i+1} to $p_{i+k/2+1}$.* ◊

Corollary 1. *Let $\underline{P}$ be a closed convex polygon admitting a nontrivial self-parallelism. Then the evolute of $\underline{P}$ is contained in the convex domain bounded by $\underline{P}$.* ◊

Remark 4. This corresponds to main results in [3] or [8]. Also by a lengthy argument a little more general version of Corollary 1 can be proved avoiding the assumption of convexity. Furthermore it can be shown that in the non-closed case non-trivial self-parallelisms are not possible.

Remark 5. Using methods established by P.C. Hammer and A. Sobczyk ([5], [6]) it can be seen that every closed convex polygon with a nontrivial self-parallelism admits an inscribed curve of constant width.

3. Normal holonomy of a polygon in space

The aim of this section is to establish some kind of parallel transfer in the normal bundle of a polygon in 3-space and to exhibit its relation to parallelism for polygons. Most constructions and results extend to higher dimensions.

Let $\underline{P} = \{p_i | i \in I\}$ be a generic polygon in Euclidean 3-space with sides s_i. Let A_i be the reflection at the symmetric normal plane of $\underline{P}$ at p_i. A ***normal vector field*** along the subarc $\underline{P}' = \{p_i | i \in J\}$ of $\underline{P}$ is a choice of normal vectors ξ_i to s_i for every $i \in J$ such that the segment s_i belongs to $\underline{P}$.

Definition 3. A normal vector field $\{\xi_i\}_{i \in J}$ along the subarc $\underline{P}'$ of $\underline{P}$ is called *parallel*, if $\xi_{i+1} = A_{i+1}(\xi_i)$ for all $i, i+1 \in J$. The *parallel transfer* of the normal vector ξ_{i_0} at s_{i_0} to s_{i_1} is given by the value of the parallel vector field along $\underline{P}' = \{p_i | i_0 \le i \le i_1 + 1\}$ at s_{i_1} which is uniquely determined by its initial value ξ_{i_0} at s_{i_0}.

Remark 6. In the closed case $\underline{P} = \{p_i | i \in \mathbb{Z}_k\}$ the parallel transfer of normal vectors along one period of $\underline{P}$ is a proper linear isometry of the normal space of $\underline{P}$ at s_i onto itself, given by $\prod_{v=1}^{k} A_{i+v}$. Hence it is a rotation around an angle $\alpha(\underline{P})$ which does not depend on $i \in \mathbb{Z}_k$. This angle is called the ***normal rotation angle*** of $\underline{P}$.

Remark 7. Smoothing the vertices of $\underline{P}$ by small circles tangent to the corresponding adjacent sides of $\underline{P}$, we get a C^1-curve having the same differential geometric normal holonomy as $\underline{P}$.

Example 1. a) Every closed polygon contained in a plane in 3-space has normal rotation angle 0. The same is true for closed tangent polygons to a sphere and in particular for closed edge polygons on Platonic solids.

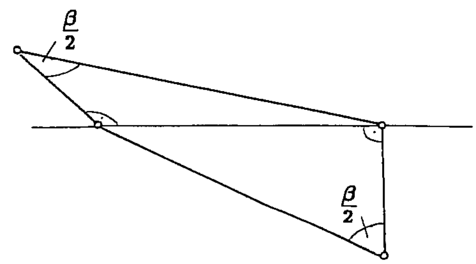

Figure 6

b) For a given $\beta \in (0, \pi)$ a closed quadrangle with normal rotation angle β is demonstrated in Figure 6. There the planes spanned by p_1, p_2, p_3 and by p_2, p_3, p_4 are assumed to be perpendicular to each other.
c) The center polygon in Figure 7 is a closed hexagon with normal rotation angle π. We conjecture that there is no quadrangle or pentagon with this property.

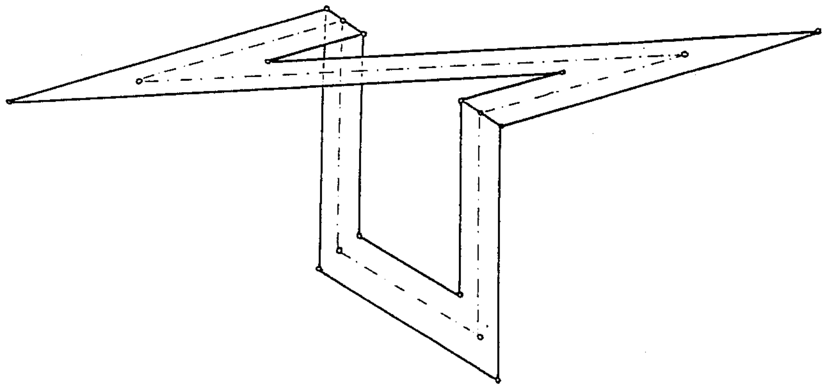

Figure 7
Möbius strip bounded by a self-parallel 12-gon with a center hexagon.

Proposition 1. *Two generic polygons* $\underline{P} = \{p_i | i \in I\}$ *and* $\underline{Q} = \{p_i | i \in \in I\}$ *are parallel if and only if the segments* $p_i q_i$ *and* $p_{i+1} q_{i+1}$ *have equal normal parts* ξ_i *with respect to* s_i *for every* $i \in I$ *and if* $\{\xi_i | i \in I\}$ *is a parallel normal vector field to* $\underline{P}$.
Proof. If $\underline{P}$ and $\underline{Q}$ are parallel then their sides with equal subscript are parallel and the local symmetry with respect to the common symmetric normal implies parallelism of the normal vector field given above. Conversely, the assumed equality of the normal parts implies that the corresponding sides are parallel. Since they constitute a parallel normal vector field to $\underline{P}$ it is easily seen that the symmetric normal planes of $\underline{P}$ and $\underline{Q}$ coincide at corresponding vertices. ◊

As in [8] this shows that non-vanishing normal rotation angle constitutes an obstruction to the existence of parallel polygons.
Corollary 2. *A closed generic polygon admits a (non-identical) parallel polygon if and only if it has vanishing normal rotation angle.*

The sufficiency of the second condition is obtained as a special case from the following
construction: Let α be the normal rotation angle of $\underline{P} = \{p_i | i \in \mathbb{Z}_k\}$.

Assume α is a rational multiple of 2π; $\alpha = 2\pi l/m$ with (l, m) relatively prime, $m = 1$ for $\alpha = 0$. Let $\varepsilon_i > 0$ be such the ε_i-tube around $p_i \vee p_{i+1}$ does not meet the focal line of $\underline{\mathrm{P}}$ at s_i, and take $\varepsilon > 0$ as the minimum of these ε_i, $i \in \mathbb{Z}_k$. Choose some unit normal ξ_1 to s_1 and extend it by parallel transfer of normal vectors to the m-fold covering of $\underline{\mathrm{P}}$. By the assumption on the normal holonomy of $\underline{\mathrm{P}}$ this gives a parallel normal vector field $\{\xi_i | i \in \mathbb{Z}_{km}\}$ along the m-fold covering of $\underline{\mathrm{P}}$. The line $l_{i+\nu k}$, $i = 1, \ldots, k$, $\nu = 0, \ldots, m-1$, is obtained from $p_i \vee p_{i+1}$ by parallel displacement about $\varepsilon\xi_{i+\nu k}$. Let $q_j := l_{j-1} \wedge l_j$, $j \in \mathbb{Z}_{km}$, which is non-empty by our construction. Then $\underline{\mathrm{Q}} = \{q_j | j \in \mathbb{Z}_{km}\}$ defines a closed polygon which is parallel to the m-fold covering of $\underline{\mathrm{P}}$ and has self-parallel group $\mathbb{Z}_m$ (see Figures 8,9). ◊

Corollary 3. *For every natural number m there exists a generic polygon with self-parallel group $\mathbb{Z}_m$.*

This follows directly from the given examples together with the construction described above. ◊

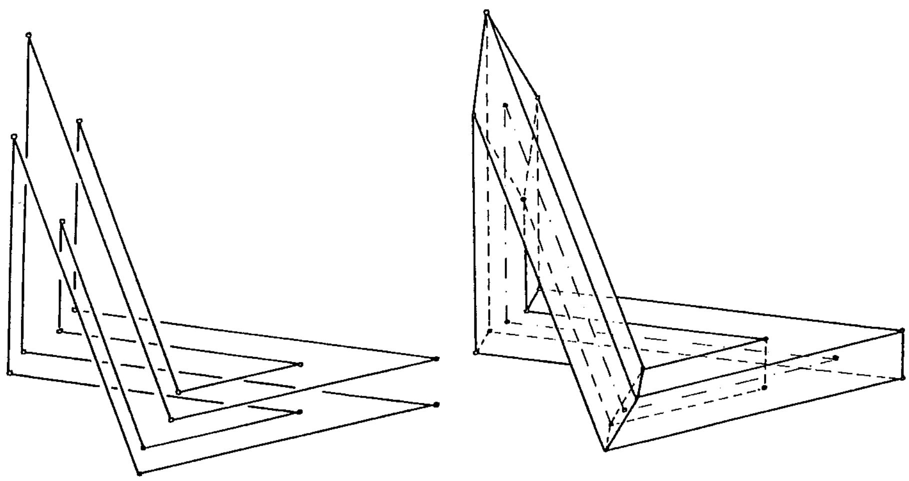

Figure 8
Self-parallel 16-gon with self-parallel group $\mathbb{Z}_4$ and its interpretation as an edge polygon on a PL-torus.

Remark 8. a) If the normal rotation angle of $\underline{\mathrm{P}}$ is an irrational multiple of 2π, then in a similar way an infinite polygon can be constructed having self-parallel group $\mathbb{Z}$ and being everywhere dense on a tubular surface around $\underline{P}$.

b) Considering higher dimensions than three, it can be observed that only in the odd case obstructions to the existence of parallel polygons may occur. In even dimensions there always exists a parallel polygon to a given closed one because the corresponding normal holonomy map has at least one fixed direction.

Theorem 2. *Let $\underline{P}$ be a self-parallel polygon with k vertices and self-parallel group $\mathbb{Z}_m$ satisfying $m \geq 3$. Then k is an integer multiple of m and there is a polygon $\underline{C}$ with k/m vertices, the center of $\underline{P}$, from which $\underline{P}$ can be reconstructed by the construction given above.*

Proof. According to Remark 3 the set of line segments of $\underline{P}$ which correspond to a given one are located in a regular way on a circular cylinder. Intersecting the axes of these cylinders appropriately we shall get a polygon $\underline{C}$ with k/m vertices. As in [8] it can be seen that the focal lines of $\underline{P}$ remain outside of the convex hull of every m related parallel segments of $\underline{P}$. This leads to the conclusion that $\underline{P}$ is parallel to the m-fold covering of $\underline{C}$. ◇

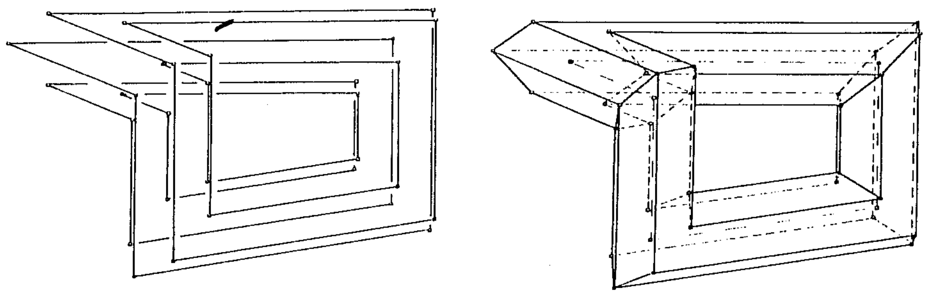

Figure 9
Self-parallel 30-gon with self-parallel group Z_6 and its interpretation as an edge polygon on a PL-torus.

Remark 9. The same result can be shown in the case $m = 2$, if no focal line meets the strip between the associated parallel lines of $\underline{P}$. Then $\underline{P}$ bounds a piecewise linear immersion of the Möbius strip (see Figure 7). The case where this condition is not valid is also possible (see Figure 3).

Using arguments from the preceding proof and elementary geometry we also get

Theorem 3. *Let* P *be a self-parallel polygon with self-parallel group* $\mathbb{Z}_k$ *and center polygon* $\underline{Q}$. *Then length* $(\underline{P}) = k \cdot$ *length*$(\underline{Q})$

4. Obstructions to exterior parallelism in higher dimensions

A theory of exterior parallelism for piecewise linear submanifolds of dimension greater than one in E^n can be developed only for special types. There are two local obstructions which will be sketched by the following considerations:

Let $\underline{P} = \{\mathcal{V}, \mathcal{E}, \mathcal{S}\}$ be a polyhedral 2-manifold in E^3 (possibly with self-intersections) where $\mathcal{V}$, $\mathcal{E}$ and $\mathcal{S}$ denote the sets of vertices, edges and sides respectively. The existence of a polyhedral 2-manifold $\underline{Q}$ of the same combinatorial type such that the corresponding sides of $\underline{P}$ and $\underline{Q}$ are parallel and have constant distance from each other implies that for all vertices of $\underline{P}$ (and $\underline{Q}$) the following is satisfied:

Definition 4. A vertex $p \in \mathcal{V}$ of the polyhedral 2-manifold $\underline{P}$ in E^3 is called *pa-admissible*, if for all edges $l \in \mathcal{E}$ ending at p the planes, which intersect the angle between the corresponding adjacent sides of $\underline{P}$ into equal parts, have a common line. This line is uniquely determined and called the *symmetric normal* of $\underline{P}$ at p.

For polyhedral 2-manifolds, having *pa*-admissible vertices only, parallelism can be defined in the same way as in Definition 1. Examples for polyhedra possessing non-*pa*-admissible edges can be obtained easily. Clearly, if only three edges end at some vertex, then this vertex is *pa*-admissible. Sufficient and necessary for the *pa*-admissibility of a vertex is that every four consecutive unit normals (suitably oriented and labelled) of the sides of $\underline{P}$ meeting at this vertex lie in a common plane. Examples for polyhedra having *pa*-admissible vertices only are given by the boundaries of the Platonic solids or the polyhedral tori obtained by suitable connections of the vertices of a self-parallel polygon (see Figures 8 and 9).

Using symmetric normals, focal points can be introduced as previously. These can be used to develop criteria for the construction of parallel polyhedra along fields of unit normals as above. Thus the focal points of a suitably constructed polyhedral torus show some similarity to the corresponding situation for the standard torus.

If we consider polyhedral 2-manifolds $\underline{P}$ in E^4, we get an additional obstruction to the existence of parallel polyhedra. This corresponds to the fact that in the case of differentiable 2-manifolds in E^4 sometimes parallel sections of the normal bundle do not exist, i.e., the normal connection is not flat. The visualization in the piecewise linear case is prepared by the following

Definition 5. Let $\underline{P} = \{\mathcal{V}, \mathcal{E}, \mathcal{S}\}$ be a polyhedral 2-manifold in E^4. For a given $p \in \mathcal{V}$ let k be the number of sides of $\underline{P}$ meeting at p, and choose a cyclic labelling of these sides $\{s_i | i \in \mathbb{Z}_k\}$, such that two consecutive sides have a common edge at p. Let A_i denote the linear map given by the reflection at the 3-plane which divides the angle between s_i and s_{i+1} into equal parts. Then the *normal curvature* of $\underline{P}$ at p is given by $\Gamma(p) = \prod_{i=1}^{k} A_i$.

Remark 10. The normal curvature is a linear orientation preserving isometry of the normal plane of s_1, i.e. it is given by a rotation about an angle α, the normal curvature angle of $\underline{P}$ at p. This angle is uniquely determined up to sign. The definition of a parallel normal vector field along some part of $\underline{P}$ can be given in the obvious way, but for the existence of such a field on the simplex star around p, the vanishing of the normal curvature angle at p is necessary and sufficient. Clearly this represents another local obstruction to the existence of parallel polyhedra.

That the normal curvature angle can attain many values can be seen from

Example 2. Take a quadrangle in 3-space with normal rotation angle $\beta \in (0, \pi)$ (see Figure 6 and Example 1). Consider the line l in 4-space which is orthogonal this 3-space and passes through the center of gravity of the quadrangle. Look at a point p on l as a vertex of a polyhedral 2-manifold $\underline{P}$ having the simplex star around p bounded by the given rectangle. If p lies in the 3-space of the quadrangle, then the normal curvature angle of $\underline{P}$ at p vanishes. But if p tends to infinity, then the normal curvature angle tends to $\pm\beta$.

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ON MATRIX NEAR-RINGS

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Abstract: This paper extends the work on matrix near-rings $\mathcal{M}_n(R)$, the near-rings of $n \times n$ matrices over right near-rings with identity [3]. Our main aim is to investigate matrix near-rings constructed over right near-rings, not necessarily with identity. We show many similarities to the ring case. It is of interest that one can find some striking contrasts as well. For example, unlike the ring case, not all ideals of $\mathcal{M}_n(R)$ are full (Theorem 2.13), which solves a problem posed in [3].

Introduction

Since the construction of matrix near-rings over arbitrary near-rings by using a functional view of matrices [3], a number of very satisfying structural results have been obtained ([1], [4], [5], [6], [7]). This encourages one to believe that matrix near-rings will play a very important role in the theory of near-rings similar to the role played by matrix rings in ring theory.

This work is divided into two sections. In Section 1 we deal with matrix near-rings constructed over near-rings which need not have an identity element.

It turns out that the existence of an identity element in the base near-ring R is an important condition (Theorem 1.19). Thus, like other researchers in matrix near-ring theory, in Section 2 we will be studying matrix near-rings $\mathcal{M}_n(R)$, where R is a near-ring with identity.

1. In Meldrum and Van der Walt [3], matrix near-rings $\mathcal{M}_n(R)$ are defined over near-rings with identity, and without identity, separately. We use their first definition for arbitrary near-rings, not necessarily with identity.

Let R be a right near-ring and $n \in \mathbb{N}$, the set of all natural numbers. The direct sum of n copies of the group $(R, +)$ is denoted by R^n. The elements of R^n are thought of as column vectors, but for typographical reasons we write then in transposed form with pointed brackets. We define elementary matrices

$$f_{ij}^r : R^n \to R^n$$

by

$$f_{ij}^r = \iota_i f^r \pi_j, \text{ for } r \in R, 1 \leq i, j \leq n$$

where ι_j and π_j are jth coordinate injection and projection functions and

$$f^r(s) = rs \text{ for all } s \in R.$$

For typographical reasons, we use the symbol $[r; i, j]$ for f_{ij}^r.

Definition 1.1. The *near-ring of* $n \times n$ *matrices over* R, denoted by $\mathcal{M}_n(R)$ is the subnear-ring of $M(R^n)$, the near-ring of all maps from R^n to itself, generated by the set $\{[r; i, j] : r \in R, 1 \leq i, j \leq n\}$.

We emphasise that R need not have an identity in this definition. We wish to carry over the additive laws of $\mathcal{M}_n(R)$ to R.

Definition 1.2. [6] An R-module G is called a *connected R-module* if for any g_1, g_2 in G, there are g in G and x, y in R such that $g_1 = xg$ and $g_2 = yg$.

Lemma 1.3. *Let G be a connected R-module. If $(R, +) \in V$, a variety of additive groups, then $G \in V$.*

Proof. Let $w(x_1, \ldots, x_p)$ be a law of V. If $g_1, \ldots, g_p \in G$ then there exists g in G and $x_1, \ldots, x_p$ in R such that $g_1 = r_1 g, \ldots, g_p = r_p g$, by 3.2 of [6]. Now $w(g_1, \ldots, g_p) = 0_G$ by 12.9 of [2], the hypothesis and

2.12 of [2]. So the law $w(x_1, \dots, x_p)$ holds in G. This is true for all the laws of V. Hence $G \in V$. ◇

Theorem 1.4. *Let $(R, +)$ be a connected R-module. Then $(R, +) \in V$ if and only if $(\mathcal{M}_n(R), +) \in V$.*

Proof. The necessary condition follows by Lemma 8 of [1] and the converse follows from 3.3 of [6] and Lemma 1.3. ◇

The following are some immediate consequences of this result.

Corollary 1.5. *Let $(R, +)$ be a connected R-module. Then $(R, +)$ is in V if and only if $(\mathcal{M}_n(R), +)$ is in V, where V is one of abelian, nilpotent or soluble.* ◇

We have analogous results to Lemma 1.3 and to Corollary 1.5 for a monogenic R-module as every monogenic R-module is connected.

To extend Theorem 9 of [1], we first state a rewording of 12.9, [2].

Lemma 1.6. *Let $w(v_1, \dots, v_p)$ be a word in p variables $v_1, \dots, v_p$. Then $w(x_1, \dots, x_p)\alpha = w(x_1\alpha, \dots, x_p\alpha)$ where $x_1, \dots, x_p \in \mathcal{M}_n(R)$ and $\alpha \in R^n$.* ◇

We remind the reader here that if I is an ideal of R, then I^+ is the ideal of $\mathcal{M}_n(R)$ generated by $\{[a; i, j];\ a \in I,\ 1 \le i,\ j \le n\}$ and $I^* := \{X \in \mathcal{M}_n(R);\ X\alpha \in I \text{ for all } x \in R^n\}$ is also an ideal of $\mathcal{M}_n(R)$. Also, if J is an ideal of $\mathcal{M}_n(R)$, then $J_* := \{a \in R;\ a = \pi_j X\alpha$, for some $j, 1 \le j \le n,\ X \in J,\ \alpha \in R^n\}$ is an ideal of R. These results and definitions come from [3].

Theorem 1.7. *Let R be a near-ring and I an ideal of R. If $(I, +) \in V$, then $(I^*, +) \in V$.*

Proof. Exactly the same method of proof as for Theorem 1.3, and Lemma 1.6 enable us to get this result. ◇

We shall show in Theorem 2.4. that in the case of near-rings with identity the converse of the above statement also holds.

Lemma 1.8. $w([a_1; i, j]), \dots, [a_p; i, j]) = [w(a_1, \dots, a_p);\ i,\ j]$ *where* $a_1, \dots, a_p \in R$ *and* $1 \le i,\ j \le n$.

Proof. By using induction on the length q of the word $w(v_1, \dots, v_p)$, and 3.1 (1) of [3], we get what we want. ◇

Theorem 1.9. *Let R be a near-ring and I be an ideal of R. If $(I, +) \in$ $\in V$, then $(I^+, +) \in V$.*

Proof. Immediate from Proposition 1 of [7] and Theorem 1.7. ◇

Some immediate consequences of Theorems 1.7 and 1.9 are as follows.

Corollary 1.10. *Let R be a near-ring and I be an ideal of R. If $(I, +)$*

is in V, then $(I^+,+)$ and $(I^*,+)$ are in V where V is one of abelian, nilpotent or soluble. ◇

The next result we aim to prove is

Theorem 1.11. *If R is distributive over I, then $\mathcal{M}_n(R)$ is distributive over I^*.*

The proof will be based on the following lemmas.

Lemma 1.12. *If R is distributive over I, then*

$$xa + yb = yb + xa$$

where $x, y \in R$ and $a, b \in I$.

Proof. Expand $(x+y)(a+b)$, first using the hypothesis, then the right distributivity of R and vice versa. ◇

Lemma 1.13. *If R is distributive over I, then*

$$[x; i, j]\alpha + [y; k, l]\beta = [y; k, l]\beta + [x; i, j]\alpha$$

where $x, y \in R$, $\alpha, \beta \in R^n$ and $1 \leq i, j, k, l \leq n$.

Proof. Follows from 3.1 of [2], [3], Lemma 1.12 and simple calculation. ◇

Lemma 1.14. *Let R be a zero-symmetric near-ring. If R is distributive over I, then*

$$X\alpha + Y\beta = Y\beta + X\alpha$$

where $X, Y \in \mathcal{M}_n(R)$ and $\alpha, \beta \in I^n$.

Proof. Follows by induction on $w(X) + w(Y)$, Lemma 1.13 and 2.16 of [2], 3.2 of [3] and 4.1 of [3]. ◇

Lemma 1.15. *If R is distributive over I, then*

$$[x; i, j](\alpha + \beta) = [x; i, j]\alpha + [x; i, j]\beta$$

where $x \in R$, $\alpha, \beta \in I^n$ and $1 \leq i, j \leq n$.

Proof. Simple calculation. ◇

Lemma 1.16. *If R is a zero-symmetric near-ring and R is distributive over I, then $\mathcal{M}_n(R)$ is distributive over I^n.*

Proof. Let $X \in \mathcal{M}_n(R)$ and $\alpha, \beta \in I^n$. By using induction on the weight $w(X)$ of X, Lemmas 1.15, 1.14, 2.16 of [2] and 4.1 of [3], we can show that

$$X(\alpha + \beta) = X\alpha + X\beta. \ \diamond$$

We are now able to prove Theorem 1.11.

Proof of Theorem 1.11. Let $X \in \mathcal{M}_n(R)$, $A, B \in I^*$, $\alpha \in R^n$. Then $X(A+B)\alpha = X(A\alpha + B\alpha)$. Now the rest of the proof follows immediately from Lemma 1.16 and the definition of I^*. $\diamondsuit$

Theorem 1.17. *Let R be a zero-symmetric near-ring. If R is distributive over I, then $\mathcal{M}_n(R)$ is distributive over I.*

Proof. Immediate from Proposition 1 of [7] and Theorem 1.11. $\diamondsuit$

Corollary 1.18. *If R is distributive then $\mathcal{M}_n(R)$ is distributive.* $\diamondsuit$

We conclude this section with a result which shows the convenience of considering the case in which R has an identity element. Recall that if R is a near-ring with identity then the map $I \to I^*$ is an injection.

Theorem 1.19. *The map $I \to I^*$ need not be an injection, in general.*

Proof. Let R be a zero-symmetric near-ring without identity and I be a non-trivial proper ideal of R such that $xy \in I$ for all x, y in R; we aim to show that $R^* = I^*$. Let $X \in \mathcal{M}_n(R)$. By using induction on the weight $w(X)$ of X, and 2.16 of [2], 3.2 and 4.1 of [3], we can show that $X \in I^*$. $\diamondsuit$

2. We start this section with a couple of results which show the similarities to the ring case. R is, henceforth, a near-ring with identity.

Theorem 2.1. *Let R be a zero-symmetric near-ring. If $n > 1$, then $\mathcal{M}_n(R)$ cannot be integral.*

Proof. Assume the result to be false and choose two non-zero elements, say x and y, of R. The hypothesis and 3.1 (3) of [3] imply that $[x;i,j][y;k,l] = 0$ if $j \neq k$. Therefore $[x;i,j] = 0$ or $[y;k,l] = 0$. Hence $x = 0$ or $y = 0$. This is a contradiction. $\diamondsuit$

Theorem 2.2. *A sum of distinct matrix units E_{kk}, $1 \leq k \leq n$, is an idempotent in $\mathcal{M}_n(R)$.*

Proof. E_{ii} for $i = 1, 2, \ldots, n$ is an idempotent, by 3.1 (3) of [3]. By right distributivity in $\mathcal{M}_n(R)$ and 3.1 (5) of [3], it can be seen easily that

$$(E_{11} + \ldots + E_{ll})(E_{11} + \ldots + E_{ll}) = E_{11} + \ldots + E_{ll}.$$

This completes the proof. $\diamondsuit$

Corollary 2.3. *If $n > 1$, then $\mathcal{M}_n(R)$ cannot be a local near-ring.* $\diamondsuit$

Our next result takes Theorem 1.7 further.

Theorem 2.4. $(I, +) \in V$ *if and only if* $(I^+, +) \in V$.

Proof. We use a technique similar to that of the proof of Lemma 1.3.

Proposition 1 of [7], the definition of I^+, the hypothesis, and Lemma 1.8 give us the desired result. ◇

Theorem 2.5. *$(I,+) \in V$ if and only if $(I^*,+) \in V$.*

Proof. Only the converse needs a proof which is exactly the same as that of the above result. ◇

Corollary 2.6. *$(I,+)$ is in V if and only if $(I^*,+)$ is in V, if and only if $(I^+,+)$ is in V, where V is one of abelian, nilpotent or soluble.* ◇

R is, henceforth, assumed to be a zero-symmetric near-ring.

Theorem 2.7. *If J is an ideal of $\mathcal{M}_n(R)$, then R is distributive over J_* if and only if $\mathcal{M}_n(R)$ is distributive over J.*

Proof. Let $X \in \mathcal{M}_n(R)$ and $A, B \in J$. By Proposition 3 of [7], 4.6 of [3] and Theorem 1.11, we get $X(A+B) = XA + XB$. To prove the sufficiency, let $x \in R$, $a, b \in J_*$. By 4.5 of [3], 3.1 of [3] and the hypothesis, we get $[x(a+b);1,1] = [xa + xb;1,1]$. Hence $x(a+b) = xa + xb$. ◇

Exactly the same method of proof as for the sufficiency of the condition of Theorem 2.7 enables us to show the converse of Theorems 1.11 and 1.17.

Theorem 2.8. *R is distributive over I if and only if $\mathcal{M}_n(R)$ is distributive over I^*.* ◇

Theorem 2.9. *R is distributive over I if and only if $\mathcal{M}_n(R)$ is distributive over I^+.* ◇

To end, we answer the question posed in [3]: Does, in general, $\mathcal{M}_n(R)$ possess ideals which are not full?

First we establish the following lemmas.

Lemma 2.10. *If $I^+ = I^*$ for any ideal I of R then all ideals of $\mathcal{M}_n(R)$ are full.*

Proof. If J is an ideal of $\mathcal{M}_n(R)$, it can be seen easily that $J = (J_*)^*$ (by 4.6 of [3], the hypothesis and Proposition 3 of [7]). ◇

Furthermore

Lemma 2.11. *$I^+ = I^*$ for each ideal I of R if and only if all ideals of $\mathcal{M}_n(R)$ are full.*

Proof. For sufficiency, take an ideal I^+ of $\mathcal{M}_n(R)$. $I^+ = L^*$ for some ideal L of R. We aim to show that $I = L$. Let $a \in L$, then $[a;1,1] \in I^+ = L^*$, therefore $a \in L$. Now Proposition 1 of [7], the hypothesis and Proposition 2 of [7] imply $L \subseteq I$. This completes the proof. ◇

Lemma 2.12. *If there exists an ideal I of R such that $I^+ \neq I^*$ then not all ideals of $\mathcal{M}_n(R)$ are full.*
Proof. Assume that all ideals of $\mathcal{M}_n(R)$ are full. Then $I^+ = I^*$ for each ideal I of R by Lemma 2.11. This contradicts the hypothesis. ◊
Theorem 2.13. *In general $\mathcal{M}_n(R)$ possesses ideals which are not full.*
Proof. Immediate from Lemma 2.12 and Example 4 of [7]. ◊

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TWO FIXED POINT THEOREMS FOR ABSTRACT MARKOV OPERATORS

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Abstract: Two fixed point theorems for Markov operators on (L)-spaces are proven: first under the assumption of existence of positive lower element h for the Markov operator P, i.e. $||(P^n d-h)^-|| \to 0$ for all probability distributions d, and second under the assumption of existence of prositive upper element h for P, i.e. $||h|| <2$ and $||(P^n d-h)^+|| \to 0$ for all probability distributions d. Both of them are abstract version of Lasota-Yorke theorems for Markov operators on L^1 but their proofs are something different.

Let L be a (L)-space, i.e. Banach lattice in which the norm has the following properties

$$|a| \leq |b| \Rightarrow ||a|| \leq ||b||,$$
$$a \geq 0,\ b \geq 0 \Rightarrow ||a+b|| = ||a|| + ||b||,$$

where

$$|a| = a^+ + a^-,\ a^+ = a \vee 0,\ a^- = (-a) \vee 0.$$

Lemma 1. *If $a, b \in L$, $a \geq 0$, $b \geq 0$, then*

$$||a-b|| = ||a|| - ||b|| + 2||(a-b)^-|| = ||b|| - ||a|| + 2||(a-b)^+||.$$

Proof. It is true because

$$||(a-b)^+|| + ||b|| = ||(a-b) + (a-b)^- + b|| = ||a|| + ||(a-b)^-||. \ \diamondsuit$$

Denote by

$$L_+ = \{a \in L : a > 0\}$$

the set of all *positive elements* of L and by

$$L_p = \{d \in L_+ : ||d|| = 1\}$$

the set of all *probability distributions* of L.

A linear mapping $P : L \to L$ is called a *Markov operator* on L iff

$$P(L_p) \subset L_p.$$

Every Markov operator P on a (L)-space L has the following properties

$$Pa > 0,\ ||Pa|| = ||a|| \text{ for } a \in L_+$$
$$Pa \leq Pb \text{ for } a \leq b$$
$$(Pa)^+ \leq Pa^+,\ (Pa)^- \leq Pa^-$$
$$|Pa| \leq P|a|,\ ||Pa|| \leq ||a||.$$

An element $h \in L_+$ will be called a *lower element* for the Markov operator P iff

$$\lim_{n\to\infty} ||(P^n d - h)^-|| = 0 \text{ for every } d \in L_p.$$

Denote by H_0 the set of all lower elements for a Markov operator P.

Theorem 1. *If the set H_0 is nonempty, then the Markov operator P has a unique fixed probability distribution d_0. Moreover*

$$P^n d \to d_0 \ \textit{for all}\ d \in L_p.$$

We begin the proof of this theorem with a set of lemmas.

Lemma 2. *If $h \in H_0$, then $||h|| \leq 1$.*

Proof. For $d \in L_p$ and $n \in \mathrm{N}$ we have

$$h \leq P^n d + (P^n d - h)^-$$

and consequently

$$||h|| \leq 1 + ||(P^n d - h)^-||. \ \diamondsuit$$

Lemma 3. *If $h \in H_0 \cap L_p$, then h is a unique fixed probability distribution for P and $P^n d \to h$ for all $d \in L_p$.*

Proof. By Lemma 1 we have for all $d \in L_p$
$$||P^n d - h|| = 2||(P^n d - h)^-|| \to 0.$$
So
$$P^n h \to h,\ P^{n+1} h \to h$$
and by continuity of P
$$P^{n+1} h \to Ph.$$
Hence $Ph = h$. Now if $\overline{h}$ is a fixed probability distribution too, then
$$\overline{h} = P^n \overline{h} \to h.$$
Finally $\overline{h} = h$. ◇

Lemma 4. *If $h_1, h_2 \in H_0$, then $h_1 \vee h_2 \in H_0$.*

Proof. It is obvious that $h_1 \vee h_2 \in L_+$. For the proof that
$$||(P^n d - h_1 \vee h_2)^-|| \to 0$$
it is enough to verify that
$$||(d - h_1 \vee h_2)^-|| \le ||(d - h_1)^-|| + ||(d - h_2)^-||$$
for $d \in L_p$. It is true because
$$(d - h_1 \vee h_2)^- = (h_1 \vee h_2 - d) \vee 0 \le (h_1 - d) \vee 0 + (h_2 - d) \vee 0 = \\ = (d - h_1)^- + (d - h_2)^-.\ ◇$$

Lemma 5. *If $h \in H_0$, then $Ph \in H_0$.*

Proof. It is obvious that $Ph \in L_+$. Now observe that
$$||(P^n d - Ph)^-|| \le ||P(P^{n-1} d - h)^-|| = ||(P^{n-1} d - h)^-|| \to 0.\ ◇$$

Lemma 6. *If $h \in H_0$ and $Ph = h$, then $(2 - ||h||)h \in H_0$.*

Proof. Let $x = ||h||$ and assume that $x < 1$. For a given $d \in L_p$ consider the sequence
$$r_n = (1 - x)^{-1}(P^n d - h).$$
Since $h \in H_0$ we have $||r_n^-|| \to 0$ and (see Lemma 1)
$$||r_n|| = 1 + 2(1 - x)^{-1}||(P^n d - h)^-|| \to 1.$$
Therefore, for any given $\varepsilon > 0$ there exists $m \in \mathrm{N}$ such that
$$||r_m^-|| < \varepsilon/8,\ ||r_m|| < 1 + \varepsilon/4.$$
For

$$s = (|r_m|/||r_m||) - r_m$$

we have

$$||s|| = ||(|r_m|/||r_m||) - |r_m| + 2 \cdot r_m^-|| \leq (||r_m|| - 1) + 2||r_m^-|| < \varepsilon/2$$

and $r_m + s \in L_p$. Since $h \in H_0$ we have

$$||(P^n(r_m + s) - h)^-|| < \varepsilon/2 \text{ for } n \geq n_0(m).$$

Multiplication by $1 - x \in (0,1)$ gives

$$||(P^{n+m}d - h + (1-x)P^n s - (1-x)h)^-|| < \varepsilon/2 \text{ for } n \geq n_0(m).$$

Consequently for $n \geq n_0(m)$

$$||(P^{n+m}d - (2-x)h)^-|| \leq \varepsilon/2 + (1-x)||P^n s|| \leq \varepsilon/2 + ||s|| < \varepsilon.$$

Finally $(2-x)h \in H_0$. ◇

Proof of Theorem 1. Let (see Lemma 2)

$$x_0 = \sup\{||h|| : h \in H_0\} > 0$$

and $\{\overline{h}_n\}$ be a sequence of lower elements such that $||\overline{h}_n|| \to x_0$. Replacing, if necessary, $\{\overline{h}_n\}$ by the sequence $\{h_n\}$ defined by

$$h_1 = \overline{h}_1,\ h_{n+1} = h_n \vee \overline{h}_{n+1},\ n \in \mathrm{N}$$

we get an increasing sequence of lower elements (see Lemma 4) such that $||h_n|| \to x_0$. Since

$$||h_m - h_n|| = ||h_m|| - ||h_n|| < \varepsilon$$

for $m \geq n \geq n_0(\varepsilon)$, there exists $h_0 \in L_+$ such that $h_n \to h_0$ and $||h_0|| = x_0$. Moreover $h_0 \in H_0$, because

$$||(P^n d - h_0)^-|| \leq ||(P^n d - h_k)^-|| + ||h_0 - h_k||$$

for all $d \in L_p$ and $n, k \in \mathrm{N}$. Finally h_0 is the largest element in H_0. Suppose it is not. Then there exists $h \in H_0$ such that the inequality $h \leq h_0$ is not true and for the lower element $\overline{h} = h \vee h_0$ we have $||\overline{h}|| >$ $> x_0$ which is impossible. Now by Lemma 5, $Ph_0 \in H_0$ and consequently $Ph_0 \leq h_0$. Moreover $Ph_0 = h_0$, because the operator P preserves the norm on L_+. Therefore, according to Lemma 6, $(2 - x_0)h_0 \in$ $\in H_0$. Hence $(2 - x_0)h_0 \leq h_0$ and consequently $(2 - x_0)h_0 = h_0$, because $x_0 \leq 1$ (see Lemma 2). Finally $||h_0|| = 1$ and applying Lemma 3 finishes the proof. ◇

An element $h \in L_+$ will be called an *upper element* for the Markov operator P iff $||h|| < 2$ and

$$\lim_{n\to\infty} ||(P^n d - h)^+|| = 0 \text{ for every } d \in L_p.$$

Denote by H^0 the set of all upper elements for the Markov operator P.

Theorem 2. *If the set H^0 is non-empty, then the Markov operator P has a unique fixed probability distribution d_0. Moreover $P^n d \to d_0$ for all $d \in L_p$.*

Lemma 7. *If $h \in H^0$, then $||h|| \geq 1$ and $Ph \in H^0$. If $h_1, h_2 \in H^0$, then $h_1 \wedge h_2 \in H^0$.*

The proofs of these facts are analogous to the proofs of Lemmas 2, 4 and 5. $\diamondsuit$

Lemma 8. *If the set H^0 is non-empty, then for all $d_1, d_2 \in L_p$*

$$\lim_{n\to\infty} ||P^n(d_1 - d_2)|| = 0.$$

Proof. Fix two arbitrary probability distributions d_1 and d_2. For $a = d_1 - d_2$ we have

$$||a^+|| = ||a^-|| = ||a||/2 = \alpha$$

because

$$||a^+|| + 1 = ||a^+ + d_2|| = ||a + a^+ + d_2|| = ||a^- + d_1|| = ||a^-|| + 1.$$

Assume for a moment that $\alpha > 0$ and $h \in H^0$. Then

$$||P^n a|| = \alpha||(P^n(a^+/\alpha) - h) - (P^n(a^-/\alpha) - h)|| \leq$$
$$\leq \alpha(||P^n(a^+/\alpha) - h|| + ||P^n(a^-/\alpha) - h||).$$

Since a^+/α, $a^-/\alpha \in L_p$ then there exists $n_1 \in \mathrm{N}$ such that

$$||(P^{n_1}(a^+/\alpha) - h)^+|| \leq (2 - ||h||)/4,$$
$$||(P^{n_1}(a^-/\alpha) - h)^+|| \leq (2 - ||h||)/4.$$

Therefore, by Lemma 1

$$||P^n a|| \leq ||a||\,||h||/2.$$

For $\alpha = 0$ this inequality is obvious. Finally, for d_1, $d_2 \in L_p$ we have

$$||P^{n_1}(d_1 - d_2)|| \leq ||d_1 - d_2||\,||h||/2.$$

In the same way we can find a $n_2 \in \mathrm{N}$ such that

$$||P^{n_1+n_2}(d_1 - d_2)|| \leq ||P^{n_1} d_1 - P^{n_1} d_2||\,||h||/2 \leq ||d_1 - d_2||(||h||/2)^2$$

because P preserves the norm on L_+. After k steps we obtain

$$||P^{n_1+\dots+n_k}(d_1 - d_2)|| \leq ||d_1 - d_2||(||h||/2)^k,$$

where $n_1, \dots, n_k$ are suitable choosen natural members. Hence

$$\lim_{n\to\infty} ||P^{n_1+\dots+n_k}(d_1 - d_2)|| = 0$$

and since the sequence $\{||P^n a||\}$ is decreasing, for $a \in L$, we get

$$\lim_{n\to\infty} ||P^n(d_1 - d_2)|| = 0. \diamondsuit$$

Proof of Theorem 2. For $h \in H^0$ we define the decreasing sequence $\{h_n\}$ of upper elements (see Lemma 7) by

$$h_1 = h,\ h_{n+1} = h_n \wedge Ph_n,\ n \in \mathbb{N}.$$

Since the sequence $\{||h_n||\}$ is decreasing and bounded, and

$$||h_m - h_n|| = ||h_m|| - ||h_n|| < \varepsilon$$

for $m \geq n \geq n_0(\varepsilon)$, there exists $h^0 \in L^+$ such that $h_n \to h^0$ and $||h^0|| < 2$. Moreover $h^0 \in H^0$, because

$$||(P^n d - h)^+|| \leq ||(P^n d - h_k)^+|| + ||h_k - h^0||$$

for all $d \in L_p$ and $n, k \in \mathrm{N}$. An element h^0 is a fixed point of P, because from inequality $h_{n+1} \leq Ph_n$ we have $h^0 \leq Ph^0$ and because $||Ph^0|| = ||h^0||$ the inequality $h^0 < Ph^0$ is impossible. Finally $d^0 = h^0/||h^0||$ is a fixed probability distribution of P. Moreover by Lemma 8 we have

$$||P^n d - d^0|| \to 0 \text{ as } n \to \infty$$

for all $d \in L_p$, and consequently if d^1 is a fixed probability distribution of P too then

$$||d^1 - d^0|| = ||P^n(d^1 - d^0)|| \to 0,$$

which finishes the proof. $\diamondsuit$

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