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NOTE ON PROJECTIVE MODULES

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Abstract: We give an elementary proof of the fact that over a commutative ring a projective module of constant finite rank is finitely generated.

Let P be a projective module over a commutative ring; a well-known theorem of Kaplansky asserts that P is locally free (see [1]). This allows one to define the rank of P at any prime ideal q – it's just the cardinality of any basis for the free module P localized at q. A fact which apparently is not as well known is that P must be finitely generated if the rank of P is finite and *constant* as q varies. Vasconcelos gives a proof of this in [2] where he uses the wedge product to reduce to the rank one case and employs an idempotent argument from there. The purpose of this note is to give an elementary proof of this fact which avoids the wedge product. We use induction and a familiar localization argument to reduce to the rank zero case where the result is obvious.

Theorem. ([2], Proposition 1.3): *Let R be a commutative ring and P a projective R-module of constant finite rank. Then P is finitely generated.*

Proof. Induct on n, the constant rank of P. If $n = 0$, then P is locally zero, hence zero and therefore finitely generated (by the emtpy set).

Suppose $n > 0$ and the result is true for all projective modules (over all commutative rings) of constant rank less than n. Since P is projective, there exist R-modules F and K, with F free, sucht that $F = P \oplus K$.

Let I be the trace ideal of P i.e., the ideal of R generated by the set

$$\{y \in R | y = f(x) \text{ for some } x \in P \text{ and } f \in \text{ Hom } (P, R)\}.$$

Then $I = R$. Indeed, suppose not. Then $I \subseteq M$ for some maximal ideal $M \subseteq R$. It readily follows that $P \subseteq MF$. Hence $P \subseteq MP \oplus MK$ so $P \subseteq MP$. Thus $P = MP$. Since P_M, i.e. P localized at M, is finitely generated, Nakayama's Lemma implies that $P_M = 0$. Thus the rank of P at M is zero, a contradiction. Therefore $I = R$.

It follows that there exist $f_i \in \text{Hom } (P, R)$ and $x_i \in P$ satisfying

$$(*) \qquad 1 = f_1(x_1) + \ldots + f_r(x_r).$$

We may assume that no $f_i(x_i)$ is nilpotent by shortening $(*)$ and replacing 1 by a unit in R.

Let S_i be the multiplicatively closed subset of R generated by $f_i(x_i)$. Then for all i we have induced homomorphisms on the localizations

$$\widehat{f_i} : P_{S_i} \rightarrow R_{S_i}$$

(defined by $\widehat{f_i}({}^P/_{s_i}) = {}^{f_i(P)}/_{s_i}$) which, by construction, are surjective. Thus for each i, there exists a projective R_{S_i}-module K_i satisfying

$$P_{S_i} = R_{S_i} \oplus K_i.$$

Since P has constant rank n, each K_i has constant rank $n - 1$. By induction, each K_i is finitely generated, thus P_{S_i} is finitely generated (as an R_{S_i}-module). By $(*)$, P is finitely generated.

Remark. (i) The idea of using the trace ideal comes from Vasconcelos'paper [2].

(ii) Let R be a countable direct product of fields and P the corresponding countable direct sum. Then P is a projective R-module, its rank at each prime is either zero or one and P is not finitely generated.

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A METHOD FOR CONSTRUCTION OF SPLINE CURVES

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Abstract: This paper deals with the construction of so-called $L_{Q,p}$-splines based on Lienhard's interpolation method [see Lienhard].

1. Generalization of Lienhard's interpolation method

Let $Q \geq 1$ be an integer. In the space R^m ($m > 1$ integer) let n distinct points $P_i = x_j^{(i)}$ ($i = 1, \dots, n; j = 1, \dots, m$) be given. The symbol $x_j^{(i)}$ denotes also the corresponding ordered m-tuple of coordinates, or rather the vector which has these coordinates. Thus, the elements of the set $\mathbf{R}^m$ are either pionts or vectors, according to which of the notions corresponds more to our conception in the given context. As a rule, we use the notion of a point in situations when location in the space $\mathbf{R}^m$ is discussed while the notion of a vector indicates that we are interested in the direction.

We shall look for polynomials in the real variable t of degree at most K (not determined as yet)

$$P_{x_j}^{(i)}(t) = \sum_{k=0}^{K} a_{jk}^{(i)} t^k \quad (i = 1, ..., n-1) \tag{1.1}$$

such that

$$P_{x_j}^{(i)}(-1) = x_j^{(i)}, P_{x_j}^{(i)}(1) = x_j^{(i+1)} \tag{1.2}$$

$$\frac{d^q}{dt^q} P_{x_j}^{(i)}(1) = \frac{d^q}{dt^q} P_{x_j}^{(i+1)}(-1) \quad (q = 1, ..., Q). \tag{1.3}$$

Conditions (1.2) guarantee that the interpolation arc parametrized with the aid of the functions $P_{x_j}^{(i)}(t)$ $(j = 1, \ldots, m)$ passes through the supporting points (nodes) P_i, P_{i+1}. Conditions (1.3) guarantee the fluent transition from arc to arc, in the first till the Q derivatives. To satisfy conditions (1.3) we have to know the values of the first till Q-th derivatives of the functions $P_{x_j}^{(i)}(t)$ at the points P_i, P_{i+1} :

$$\frac{d}{dt} P_{x_j}^{(i)}(-1) = Dx_j^{(i)}, \frac{d}{dt} P_{x_j}^{(i)}(1) = Dx_j^{(i+1)},$$

(1.4) ....

$$\frac{d^Q}{dt^Q} P_{x_j}^{(i)}(-1) = D^Q x_j^{(i)}, \frac{d^Q}{dt^Q} P_{x_j}^{(i)}(1) = D^Q x_j^{(i+1)};$$

by convention, $Dx_j^{(i)} = D^1 x_j^{(i)}, D^2 x_j^{(i)}, ..., D^Q x_j^{(i)}$, $Dx_j^{(i+1)} = D^1 x_j^{(i+1)}$, $D^2 x_j^{(i+1)}$,..., $D^Q x_j^{(i+1)}$ is the notation of these values. The manner of their determination will be discussed in Section 2. By (1.2), (1.3) we have $2Q + 2$ definite conditions for every polynominal (1.1). With their aid each of these polynomials is thus uniquely determined as a polynomial of degree at most $K = 2Q + 1$:

$$P_{x_j}^{(i)}(t) = \sum_{k=0}^{2Q+1} a_{jk}^{(i)} t^k. \tag{1.5}$$

For the q-th derivative of the function $P_{x_j}^{(i)}(t)$ we have

$$\frac{d^q}{dt^q} P_{x_j}^{(i)}(t) = \sum_{k=q}^{2Q+1} k(k-1) \ \dots \ (k-q+1) a_{jk}^{(i)} t^{k-q}. \tag{1.6}$$

If we substitute the values $t = -1, 1$ into (1.5), (1.6), we obtain (taking into account (1.2), (1.4)) the following system of $2Q+2$ linear equations for the $2Q + 2$ unknown coefficients $a_{jk}^{(i)}$ of the polynomial (1.5):

$$\begin{aligned}
&\sum_{k=0}^{2Q+1} (-1)^k a_{jk}^{(i)} = x_j^{(i)}, \\
&\sum_{k=q}^{2Q+1} (-1)^{k-q} k(k-1) \ \dots \ (k-q+1) a_{jk}^{(i)} = D^q x_j^{(i)}, \\
&\qquad\qquad (q = 1, 2, \dots, Q) \\
&\sum_{k=0}^{2Q+1} a_{jk}^{(i)} = x_j^{(i+1)}, \\
&\sum_{k=q}^{2Q+1} k(k-1) \ \dots \ (k-q+1) a_{jk}^{(i)} = D^q x_j^{(i+1)}, \\
&\qquad\qquad (q = 1, 2, \dots, Q).
\end{aligned} \tag{1.7}$$

We introduce the matrices

$$A_{ij} = (a_{j0}^{(i)}, \dots, a_{j,2Q+1}^{(i)}), \tag{1.8}$$

$$X_{ij} = (x_j^{(i)}, D x_j^{(i)}, \dots, D^Q x_j^{(i)}, x_j^{(i+1)}, D x_j^{(i+1)}, \dots, D^Q x_j^{(i+1)}) = \tag{1.9}$$

$$= (x_j^{(i)}, X_{ij}^*, x_j^{(i+1)}, X_{i+1,j}^*),$$

$$X_{ij}^* = (Dx_j^{(i)}, \dots, D^Q x_j^{(i)}), \tag{1.10}$$
$$X_{i+1,j}^* = (Dx_j^{(i+1)}, \dots, D^Q x_j^{(i+1)}).$$

The matrix of the coeffients of system (1.7), which is necessarily regular in consequence of the uniqueness of the determination of the desired polynomials, is denoted by the symbol A_Q; it is a matrix of type $(2Q+ +2, 2Q+2)$. Then the solution of system (1.7) is expressed in matrix notation by the relation

$$A_{ij}^T = A_Q^{-1} \circ X_{ij}^T. \tag{1.11}$$

Here the superscript T denotes the transposed matrices to matrices (1.8), (1.9), and A_Q^{-1} is the inverse matrix of A_Q. By (1.11) we then have for polynomials (1.5) the expression

$$P_{x_j}^{(i)}(t) = (P_{x_j}^{(i)}(t)) = (1, t, \dots, t^{2Q+1}) \circ A_{ij}^T. \tag{1.12}$$

Here we have identified the type (1,1) matrix $(P_{x_j}^{(i)}(t))$ with the element $P_{x_j}^{(i)}(t)$.

2. Determination of the values $\mathbf{D}^q\mathbf{x}_j^{(i)}$, $\mathbf{D}^q\mathbf{x}_j^{(i+1)}$

Values of the first till the Q-th derivatives of the functions $P_{x_j}^{(i)}(t)$ at the points P_i, P_{i+1} (cf. (1.4)) are determinated as follows: In the plane with the rectangular coordinate system t, s_j we construct the points $(2h, x_j^{(i+h)})$, $-Q+p \le h \le Q-p$, h integer.
Here the fixed chosen integer p satisfies the inequality $0 \le p \le \le Q-1$. According to Fig. 1 the points determine uniquely the following polynomial of degree at most $2Q-2p$:

$$_{Q,p}R_{x_j}^{(i)}(t) = \sum_{k=0}^{2Q-2p} {}_{Q,p}b_{jk}^{(i)} t^k. \tag{2.1}$$

With the aid of this polynomial we put

$$(2.2)\qquad \begin{aligned} Dx_j^{(i)} &= \frac{d}{dt}{}_{Q,p}R_{x_j}^{(i)}(0) = {}_{Q,p}b_{j1}^{(i)}, \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ D^Q x_j^{(i)} &= \frac{d^Q}{dt^Q}{}_{Q,p}R_{x_j}^{(i)}(0) = Q(Q-1)\ \dots\ 3.2\,{}_{Q,p}b_{jQ}^{(i)}. \end{aligned}$$

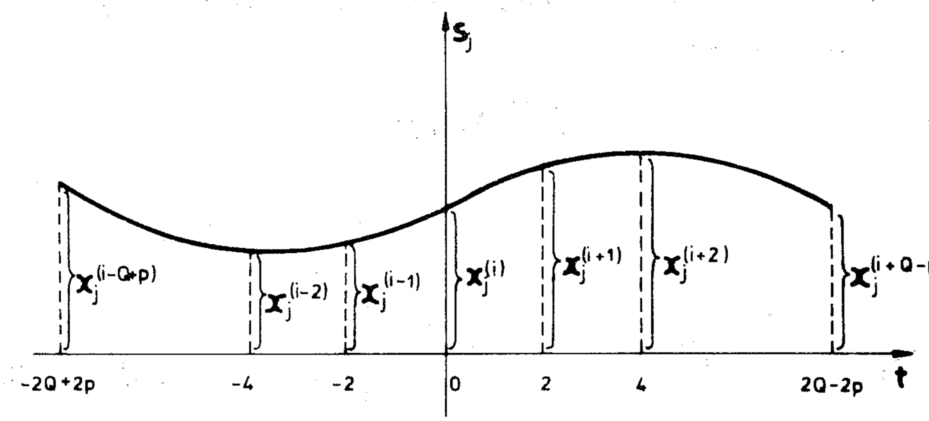

Fig. 1

The originality of Lienhard's interpolation method consists precisely in this approach to the determination of the values (2.2). The method yields the "missing" values of the derivatives in the mentioned manner from the auxiliary polynomials (2.1). In brief, we will speak of the $L_{Q,p}$ interpolation method.

Since every coefficient of the polynomial (2.1) is a certain linear combination of the values $x_j^{(i-Q+p)}, \dots, x_j^{(i+Q-p)}$, every derivative $Dx_j^{(i)}, \dots, D^Q x_j^{(i)}$ is also a certain linear combination of the same values. Therefore, there exists a matrix $B_{Q,p}$ of type $(Q, 2Q-2p+1)$ such that we have (see (1.10))

$$(2.3)\qquad X_{ij}^* = (x_j^{(i-Q+p)}, \dots, x_j^{(i+Q-p)}) \circ B_{Q,p}^T.$$

Then

$$(2.4)\qquad (x_j^{(i)}, X_{ij}^*) =$$

$$= (x_j^{(i-Q+p)}, \dots, x_j^{(i+Q-p)}, x_j^{(i+Q-p+1)}) \circ \begin{pmatrix} 0 & \\ \vdots & \\ \vdots & B_{Q,p}^T \\ 1 & \\ \vdots & \\ 0 & 0\dots0 \end{pmatrix} \; Q-p+1,$$

where the unit in the first column stands in the $(Q-p+1)$-st row. Analogously we have

$$(2.5) \qquad (x_j^{(i+1)}, X_{i+1,j}^*) =$$

$$= (x_j^{(i-Q+p)}, \dots, x_j^{(i+Q-p)}, x_j^{(i+Q-p+1)}) \circ \begin{pmatrix} 0 & 0\dots0 \\ \vdots & \\ \vdots & B_{Q,p}^T \\ 1 & \\ \vdots & \\ 0 & \end{pmatrix} \; Q-p+2,$$

where the unit in the first column stands now in the $(Q-p+2)$-nd row. Employing (2.4), (2.5) it is then possible to express (1.9) in the form

$$X_{ij} = (x_j^{(i-Q+p)}, \dots, x_j^{(i+Q-p)}, x_j^{(i+Q-p+1)}) \circ$$

$$Q+2$$

$$(2.6) \qquad \circ \begin{pmatrix} 0 & & 0 & 0\dots0 \\ \vdots & & \vdots & \\ \vdots & B_{Q,p}^T & \vdots & \\ 1 & & 0 & B_{Q,p}^T \\ 0 & & 1 & \\ \vdots & & \vdots & \\ 0 & 0\dots0 & 0 & \end{pmatrix} \begin{matrix} \\ \\ \\ Q-p+1 \\ Q-p+2 \\ \\ \\ \end{matrix}$$

Substitution of (2.6) into (1.11) yields

$$A_{ij}^T = C_{Q,p} \circ \begin{pmatrix} x_j^{(i-Q+p)} \\ \vdots \\ x_j^{(i+Q-p)} \\ x_j^{(i+Q-p+1)} \end{pmatrix}, \tag{2.7}$$

where

$$C_{Q,p} = A_Q^{-1} \circ \begin{pmatrix} 0 & \dots & \overset{Q-p+1}{1} & & \overset{Q-p+2}{0} & \dots & 0 \\ & & & & & & 0 \\ & & & B_{Q,p} & & & \vdots \\ & & & & & & \vdots \\ 0 & \dots & 0 & & 1 & \dots & 0 \\ 0 & & & & & & \\ \vdots & & & B_{Q,p} & & & \\ 0 & & & & & & \end{pmatrix} \; Q+2. \tag{2.8}$$

3. Grouping of nodes

In the case of a unclosed interpolation curve we construct groups consisting of $2Q-2p+2$ points each from the given nodes $P_1, \dots, P_n$:

$$\begin{array}{ll} \text{Group 1:} & P_{Q-p+1}, \dots, P_3, P_2, P_1, P_2, P_3, \dots, P_{Q-p+2}, \\ \text{Group 2:} & P_{Q-p}, \dots, P_2, P_1, P_2, P_3, P_4, \dots, P_{Q-p+3}, \\ \dots\dots\dots & \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ \text{Group (n-1):} & P_{n-Q+p-1}, \dots, P_{n-3}, P_{n-2}, P_{n-1}, P_n, \\ & P_{n-1}, \dots, P_{n-Q+p}. \end{array} \tag{3.1}$$

In this grouping we replace the eventually "missing" points by those points which are obtained by the "mirror image" of the given points with respect to the point P_1, or to the point P_n.
For instance, for $n = 3$, $Q = 2$, $p = 0$ we form the following groups of six points each:

$$\text{Group 1: } P_3, P_2, P_1, P_2, P_3, P_2,$$
$$\text{Group 2: } P_2, P_1, P_2, P_3, P_2, P_1.$$

If formula (2.7) is applied to group 1 now, we obtain by (1.12) the polynomials $P_{x_j}^{(1)}(t)$ $(j = 1, \dots, m)$ which parametrize the arc P_1P_2. Similarly we obtain the other arcs.
Then the unclosed interpolation curve $P_1P_2 \dots P_{n-1}P_n$ is composed of these arcs.
In the case of a closed interpolation curve $P_1P_2 \dots P_nP_1$ we construct the nodes as follows:

$$\begin{array}{ll} \text{Group 1: } & P_{n-Q+p+1}, \dots, P_{n-1}, P_n, P_1, P_2, P_3, \dots, P_{Q-p+2}, \\ \text{Group 2: } & P_{n-Q+p+2}, \dots, P_n, P_1, P_2, P_3, P_4, \dots, P_{Q-p+3}, \\ \text{Group 3: } & P_{n-Q+p+3}, \dots, P_1, P_2, P_3, P_4, P_5, \dots, P_{Q-p+4}, \\ \dots\dots & \dots\dots\dots\dots\dots\dots\dots\dots \\ \text{Group n: } & P_{n-Q+p}, \dots, P_{n-2}, P_{n-1}, P_n, P_1, P_2, \dots, P_{Q-p+1}. \end{array} \tag{3.2}$$

In this grouping we replace the eventually "missing" points by those points which follow in one or other direction the point P_1, or P_n.

4. The interpolation method $L_{1,0}$

In formula (1.11) we have in this case

$$A_1^{-1} = \frac{1}{4} \begin{pmatrix} 2 & 1 & 2 & -1 \\ -3 & -1 & 3 & -1 \\ 0 & -1 & 0 & 1 \\ 1 & 1 & -1 & 1 \end{pmatrix}. \tag{4.1}$$

To determine matrix $B_{1,0}$ of type (1.3) (cf. (2.3)) we fit the polynomial ${}_{1,0}R_{x_j}^{(i)}(t)$ (cf. (2.1)) to the points $(2h, x_j^{(i+h)})$, $h = -1, 0, 1$, and put $Dx_j^{(i)} =_{1,0} b_{j1}^{(i)}$ in accordance with (2.2). Then we obtain (cf. (2.3))

$$X_{ij} = (x_j^{(i-1)}, x_j^{(i)}, x_j^{(i+1)}) \circ B_{1,0}^T, \tag{4.2}$$

where

$$B_{1,0} = \frac{1}{4}(-1, 0, 1). \tag{4.3}$$

By (2.8) and with the aid of (4.3) we than have

$$C_{10} = \frac{1}{4}\begin{pmatrix} 2 & 1 & 2 & -1 \\ -3 & -1 & 3 & -1 \\ 0 & -1 & 0 & 1 \\ 1 & 1 & -1 & 1 \end{pmatrix} \circ \frac{1}{4}\begin{pmatrix} 0 & 4 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}. \tag{4.4}$$

Multiplication of these matrices and substitution into (2.7) yield

$$16A_{ij}^T = \begin{pmatrix} -1 & 9 & 9 & -1 \\ 1 & -11 & 11 & -1 \\ 1 & -1 & -1 & 1 \\ -1 & 3 & -3 & 1 \end{pmatrix} \circ \begin{pmatrix} x_j^{(i-1)} \\ x_j^{(i)} \\ x_j^{(i+1)} \\ x_j^{(i+2)} \end{pmatrix}. \tag{4.5}$$

Example 1. In the plane $\mathbf{R}^2$ let us consider the points $P_1 = (0,0)$, $P_2 = (2,3)$, $P_3 = (15,-6)$, $P_4 = (2,-10)$, $P_5 = (10,5)$. In Fig. 2 the unclosed interpolation curve $P_1\ P_2\ P_3\ P_4\ P_5$ is shown. The parametric equations of the individual arcs of this interpolation curve are obtained using formulas (4.5) and (1.12) (for $Q = 1$).

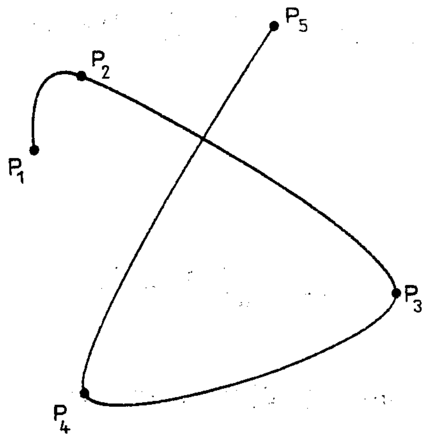

Fig. 2

It can be shown that in the cases of the interpolation methods $L_{2,0}$ and $L_{2,1}$ we have

$$(4.6)\qquad C_{2,0} = \frac{1}{768}\begin{pmatrix} 9 & -75 & 450 & 450 & -75 & 9 \\ -13 & 81 & -562 & 562 & -81 & 13 \\ -10 & 78 & -68 & -68 & 78 & -10 \\ 18 & -106 & 228 & -228 & 106 & -18 \\ 1 & -3 & 2 & 2 & -3 & 1 \\ -5 & 25 & -50 & 50 & -25 & 5 \end{pmatrix},$$

and

$$(4.7)\qquad C_{2,1} = \frac{1}{32}\begin{pmatrix} -2 & 18 & 18 & -2 \\ 3 & -25 & 25 & -3 \\ 2 & -2 & -2 & 2 \\ -4 & 12 & -12 & 4 \\ 0 & 0 & 0 & 0 \\ 1 & -3 & 3 & -1 \end{pmatrix}$$

respectively.

5. An alternative determination of the values $\mathbf{D}^q\mathbf{x}_j^{(i)}$, $\mathbf{D}^q\mathbf{x}_j^{(i+1)}$

Instead of (2.3) we now put

$$X_{ij}^* = (x_j^{(i-Q+p)}, \ldots, x_j^{(i+Q-p)}) \circ B_{Q,p}^T + M_Q b_j^{(i)}, \tag{5.1}$$

where $M_Q = (m_1, \ldots, m_Q)$ is a non-zero constant matrix while $b_j^{(1)}, \ldots, b_j^{(n+1)}$ are vectors which are undetermined as yet. It can be shown that

$$A_{ij}^T = \left(\boxed{C_{Q,p}} \; \boxed{D_Q} \right) \circ \begin{pmatrix} x_j^{(i-Q+p)} \\ \vdots \\ x_j^{(i+Q-p)} \\ x_j^{(i+Q-p+1)} \\ b_j^{(i)} \\ b_j^{(i+1)} \end{pmatrix}, \tag{5.2}$$

where $C_{Q,p}$ is the matrix (2.8), and further

$$D_Q = A_Q^{-1} \circ \begin{pmatrix} 0 & 0 \\ \boxed{M_Q^T} & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ 0 & 0 \\ \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} & \boxed{M_Q^T} \end{pmatrix} Q+2 \tag{5.3}$$

Here A_Q^{-1} is the inverse to the matrix A_Q of the coefficients of the system (1.7).

6. The unclosed interpolation $\mathbf{L}_{Q,p}$ – spline

According to (5.2) the relation

(6.1) $$P_{x_j}^{(i)}(t) = (1, t, t^2, \ldots, t^{2Q+1}) \circ A_{ij}^T$$

(see (1.12)) yields, upon differentiation and substitution of the values $t = 1, -1$, the relation

(6.2) $$\frac{d^q}{dt^q} P_{x_j}^{(i)}(1) = \frac{d^q}{dt^q} P_{x_j}^{(i+1)}(-1)$$

for $q = 1, \ldots, Q$ (cf. (1.3)). Further differentiation and substitution of the values $t = 1, -1$ yield

(6.3) $$\frac{d^{Q+1}}{dt^{Q+1}} P_{x_j}^{(i)}(1) =$$

$$= -G(x_j^{(i-Q+p)}, \ldots, x_j^{(i+Q-p+1)}) + f(M) b_j^{(i)} + g_1(M) b_j^{(i+1)},$$

(6.4) $$\frac{d^{Q+1}}{dt^{Q+1}} P_{x_j}^{(i+1)}(-1) =$$

$$= F(x_j^{(i-Q+p+1)}, \ldots, x_j^{(i+Q-p+2)}) - g_2(M) b_j^{(i+1)} - h(M) b_j^{(i+2)};$$

here F and G are certain linear combinations of the values in the parantheses while f, g_1, g_2, h are certain linear forms of the point $M = (m_1, \ldots, m_Q) \in \mathbb{R}^Q$. Since we wish to construct an interpolation spline of degree $2Q+1$, we compare expressions (6.3), (6.4). Then we obtain the following equality between vectors of the space $\mathbb{R}^m$:

(6.5) $$f(M) b_j^{(i)} + g(M) b_j^{(i+1)} + h(M) b_j^{(i+2)} = p_j^{(i)},$$

where $g = g_1 + g_2$, $F + G = p_j^{(i)}$. Let the form g differ from zero at least at one point, let us denote

(6.6) $$Y_1 = \{M \in \mathbb{R}^Q | g(M) \neq 0\}.$$

For $i = 1, \dots, n-1$, (6.5) is a system of n-1 linear equations in the unknown $b_j^{(1)}, \dots, b_j^{(n+1)}$. Therefore we add to the system (6.5) the following boundary conditions (one as the first equation, the other as the last equation):

$$g(M)b_j^{(1)} + c_j b_j^{(2)} = z_j, d_j b_j^{(n)} + g(M)b_j^{(n+1)} = u_j, \tag{6.7}$$

where c_j, d_j, z_j and u_j are real numbers. Then the resulting system will be of the form

$$\begin{array}{llllll} g(M)b_j^{(1)} + & c_j b_j^{(2)} & & & & = z_j, \\ f(M)b_j^{(1)} + & g(M)b_j^{(2)} & + \; h(M)b_j^{(3)} & & & = p_j^{(1)}, \\ & f(M)b_j^{(2)} & + \; g(M)b_j^{(3)} & + \; h(M)b_j^{(4)} & & = p_j^{(2)}, \\ \dots\dots & \dots\dots & \dots\dots & \dots\dots & \dots\dots & \dots\dots \\ & & f(M)b_j^{(n-1)} & + \; g(M)b_j^{(n)} & + \; h(M)b_j^{(n+1)} & = p_j^{(n-1)}, \\ & & & d_j b_j^{(n)} & + \; g(M)b_j^{(n+1)} & = u_j. \end{array} \tag{6.8}$$

The matrix of the coefficients of system (6.8) is

$$E_{Q,p;j} = \begin{pmatrix} g(M) & c_j & 0 & \dots & 0 & 0 & 0 \\ f(M) & g(M) & h(M) & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & f(M) & g(M) & h(M) \\ 0 & 0 & 0 & \dots & 0 & d_j & g(M) \end{pmatrix} \tag{6.9}$$

Assume that the set

$$Y = \{M \in \mathbf{R}^Q \mid |g(M)| > |f(M)| + |h(M)|\} \tag{6.10}$$

is nonempty. Then this set is a part of the set (6.6). For an arbitrary point $M \in Y$ and for

$$|c_j| < |g(M)|, \; |d_j| < |g(M)| \tag{6.11}$$

the matrix (6.9) has a dominant main diagonal, i.e. it is regular. Then the system of equations (6.8) is uniquely solvable.

Then the individual arcs of an unclosed interpolation spline

$P_1P_2\dots P_{n-1}P_n$ are constructed as follows: We form groups of $2Q-$ $-2p+2$ points each of the given nodes $P_1, P_2, \dots, P_n$ (see (3.1)). If formula (6.5) is applied to the first group of points and to the vectors $b_j^{(1)}, b_j^{(2)}$, we obtain, by (6.1), the polynomials $P_{x_j}^{(1)}(t)$ ($j = 1, \dots, m$) which parametrize the arc P_1P_2. Similarly we obtain the other arcs. The desired interpolation spline $P_1P_2\dots P_{n-1}P_n$ is then composed of the mentioned arcs. To be more exact, we shall speak of the so-called $L_{Q,p}$-spline of degree $2Q+1$ since we have started from the $L_{Q,p}$ Lienhard interpolation method in the derivation (see Section 2). The behaviour of this spline may be modified by the choice of the feasible point $M \in Y$ (see (6.10)), the feasible numbers c_j, d_j (see (6.11)) and the numbers z_j, u_j.

In contradistinction to the $L_{Q+1,p}$ interpolation method which uses polynomials of degree at most $2Q+3$ and which guarantees the continuity of the first till $(Q+1)$-st derivatives (see (1.3) for $q = 1, \dots Q+1$) the $L_{Q,p}$-spline has identical properties (with reference to the derivatives) while polynomials of degree at most $2Q+1$ are sufficient.

Example 2. In the plane $\mathbf{R}^2$ let us consider the same nodes as in Example 1. The unclosed interpolation $L_{1,0}$-spline $P_1P_2P_3P_4P_5$ for $m_1 = 1/12$, $c_j = d_j = z_j = u_j = 0$ is shown in Fig. 3.

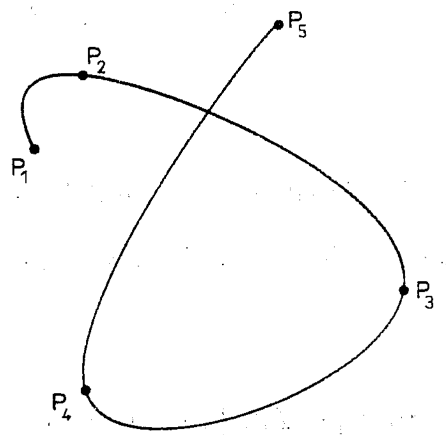

Fig. 3

7. The closed interpolation $L_{Q,p}$-spline

When constructing the closed interpolation $L_{Q,p}$-spline $P_1P_2\dots P_nP_1$ we group the nodes $P_1, P_2, \dots, P_n$ according to (3.2). To the equations (6.5) for $i = 1, \dots, n-1$ we add an additional equation for $i = n$ and two boundary conditions

$$b_j^{(1)} = b_j^{(n+1)},\ b_j^{(2)} = b_j^{(n+2)}. \tag{7.1}$$

According to (6.5), for $i = n$ and (7.1) we have

$$p_j^{(n)} =$$

$$= F(x_j^{(n-Q+p+1)}, \dots, x_j^{(n+Q-p+2)}) + G(x_j^{(n-Q+p)}, \dots, x_j^{(n+Q-p+1)}) =$$

$$= f(M)b_j^{(n)} + g_1(M)b_j^{(n+1)} + g_2(M)b_j^{(1)} + h(M)b_j^{(2)},$$

whence we obtain

$-G(x_j^{(n-Q+p)}, \dots, x_j^{(n+Q-p+1)}) + f(M)b_j^{(n)} + g_1(M)b_j^{(n+1)} =$
$= F(x_j^{(n-Q+p+1)}, \dots, x_j^{(n+Q-p+1)}) - g_2(M)b_j^{(1)} - h(M)b_j^{(2)},$
i.e. $d^{Q+1}P_{x_j}^{(n)}(1)/dt^{Q+1} = d^{Q+1}P_{x_j}^{(1)}(-1)/dt^{Q+1}$ (see (6.3), (6.4) for $i = n$).

Further, by (6.2) we have, for $i = n$, $d^qP_{x_j}^{(n)}(1)/dt^q = d^qP_{x_j}^{(1)}(-1)/dt^q$ for $q = 1, \dots, Q$. Instead of system (6.8) for the case of the unclosed interpolation $L_{Q,p}$-spline we now have the following system of equations:

$$\begin{array}{lllllll}
g(M)b_j^{(2)} & + & h(M)b_j^{(3)} & & & & = p_j^{(1)} \\
f(M)b_j^{(2)} & + & g(M)b_j^{(3)} & + & h(M)b_j^{(4)} & & = p_j^{(2)}, \\
 & & f(M)b_j^{(3)} & + & g(M)b_j^{(4)} & + \ h(M)b_j^{(5)} & = p_j^{(3)}, \\
\multicolumn{7}{c}{\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots} \\
 & & f(M)b_j^{(n-1)} & + & g(M)b_j^{(n)} & + \ h(M)b_j^{(n+1)} & = p_j^{(n-1)}, \\
h(M)b_j^{(2)} & & & + & f(M)b_j^{(n)} & + \ g(M)b_j^{(n+1)} & = p_j^{(n)}.
\end{array} \tag{7.2}$$

The matrix of coefficients of system (7.2) is

$$F_{Q,p;j} = \begin{pmatrix} g(M) & h(M) & 0 & \dots & 0 & f(M) \\ f(M) & g(M) & h(M) & \dots & 0 & 0 \\ 0 & f(M) & g(M) & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ h(M) & 0 & 0 & \dots & f(M) & g(M) \end{pmatrix}. \tag{7.3}$$

For an arbitrary point $M \in Y$ (see (6.10)) the matrix (7.3) has a dominant main diagonal, i.e. it is regular. Then the system of equations (7.2) is uniquely solvable.

Example 3. In the space $\mathbb{R}^3$ let us consider the points $P_1 = (5,0,0)$, $P_2 = (10,5,5)$, $P_3 = (0,10,15)$, $P_4 = (-5,3,8)$. The closed interpolation $L_{1,0}$-spline $P_1P_2P_3P_4P_1$ for $m_1 = 1/12$ is shown in axonometric projection in Fig. 4. For the sake of simplicity, the symbol P_i is also used here to denote the axonometric projection of a node while the symbol P_i' denotes its axonometric first projection.

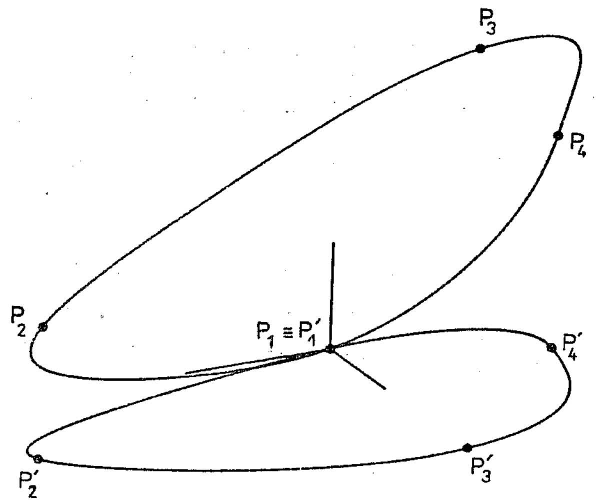

Fig. 4

Example 4. Let us construct a closed $L_{2,0}$-spline $P_1P_2P_3P_4P_1$ with the same nodes as in Example 3. For chosen $m_1 = 1/12$, $m_2 = 1/4$ the constructed curve is shown in axonometric projection in Fig. 5.

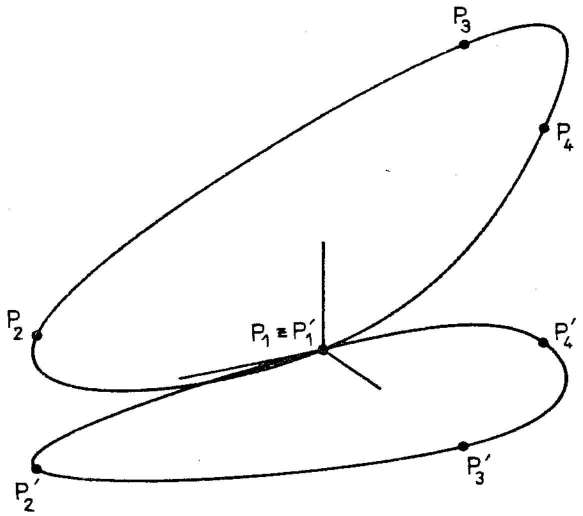

Fig. 5

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ON SOME ARITHMETICAL PROPERTIES OF WEIGHTED SUMS OF S-UNITS

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Abstract: We prove some new arithmetical properties of sums of the form $\alpha_0 x_0 + \alpha_1 x_1 + \cdots + \alpha_n x_n$ where $\alpha_0, \alpha_1, \cdots, \alpha_n$ are non-zero S-integers and $x_0, x_1, \cdots, x_n$ are S-units in a given algebraic number field K. By using a result of Evertse and Győry [6] on weighted S-unit equations, we derive in §1 a general but ineffective result. In §2, we obtain some effective results for $n = 1$ by means of Baker's method and its p-adic analogue. As

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a consequence, we get some information about the arithmetical properties of the solutions of certain decomposable form equations as well as of the terms of recursive sequences.

1. Ineffective results

Let K be an algebraic number field of degree d with ring of integers $\mathcal{O}_K$ and let M_K be the set of places (i.e. equivalence classes of multiplicative valuations) on K. A place v is called finite if v contains only non-archimedian valuations, and infinite otherwise. Let S be a finite subset of M_K containing all infinite places. A number $\alpha \in K$ is called an S-integer (resp. an S-unit) if $|\alpha|_v \leq 1$ (resp. $|\alpha|_v = 1$) for every valuation $|\ |_v$ from a place $v \in M_K \setminus S$. The S-integers form a ring which is called the ring of S-integers and is denoted by $\mathcal{O}_S$. The S-units form a multiplicative group which is denoted by $\mathcal{O}_S^*$. For each $\beta \in \mathcal{O}_S \setminus \{0\}$, we write

$$N_S(\beta) = \prod_{v \in S} |\beta|_v$$

which is a positive rational integer called the S-norm of β. If in particular S consists exactly of the infinite places then $N_S(\beta) = |N_{K/Q}(\beta)|$.

Let $n \geq 1$ be an integer. Denote by $\mathbb{P}^n(K)$ the n-dimensional projective space over K, that is the set of all $(n+1)$-tuples $(x_0, x_1, \cdots, x_n)$ with $x_i \in K$, where two tuples are identical if they differ by a non-zero scalar multiple. Further, we denote by $\mathbb{P}^n(\mathcal{O}_S^*)$ the set of $(x_0, x_1, \cdots, x_n)$ with $x_i \in \mathcal{O}_S{}^*$. For given $\underline{\alpha} = (\alpha_0, \alpha_1, \cdots, \alpha_n) \in (\mathcal{O}_S \setminus \setminus \{0\})^{n+1}$, we consider those $\beta \in \mathcal{O}_S$ which can be represented in the form

$$\text{(1)} \qquad \beta = \alpha_0 x_0 + \alpha_1 x_1 + \cdots + \alpha_n x_n \quad \text{with} \quad x_0, x_1, \cdots, x_n \in \mathcal{O}_S^*.$$

Van der Poorten and Schlickewei [8] and Evertse [5], independently, proved that for given non-zero $\beta \in \mathcal{O}_S$, the equation (1) has at most finitely many solutions such that

$$(2) \qquad \sum_{j=1}^{r} \alpha_{i_j} x_{i_j} \neq 0 \text{ for each subset } \{i_1, \cdots, i_r\} \text{ of } \{0, 1, \cdots, n\}.$$

Later, Evertse and Győry [6] proved that there is a constant C depending only on K, S and n but not on $\underline{\alpha}$ such that the number of solutions of (1) having property (2) is at most C. The proofs of these results of [8], [5] and [6] involve the p-adic analogue of the Thue-Siegel-Roth-Schmidt method. Very recently, Everest [2], [3] gave an asymptotic formula for the number of $\underline{x} = (x_0, x_1, \cdots, x_n) \in \mathbb{P}^n(\mathcal{O}_S^*)$ with $N_S(\alpha_0 x_0 + \cdots + \alpha_n x_n) \leq q$ and (2) as $q \to \infty$. Tijdeman and Wang [15] applied the above result of Evertse and Győry [6] to simultaneous weighted sums of elements of finitely generated mulitplicative groups. As another application, we shall deduce the following theorem.

For a rational integer ν with $|\nu| > 1$, we denote by $P(\nu)$ the greatest prime factor of ν and we write $P(0) = P(\pm 1) = 1$. In what follows in **1**, $C_1(\ )$, $C_2(\ ), \cdots$ will denote positive numbers depending only on parameters occuring between parantheses.

Theorem 1. *Let $P > 1$ be an integer. The number of values $N_S(\beta)$ with $\beta \in \mathcal{O}_S$ and $P(N_S(\beta)) \leq P$ for which* (1) *holds is at most $C_1(K, S, P, n)$.*

It is a remarkable fact that C_1 does not depend on the coefficients $\alpha_0, \alpha_1, \cdots, \alpha_n$ in (1). We remark that in general we are not able to make C_1 explicit. This is due to the non-explicit character of the number $C = C(K, S, n)$ mentioned above. Further, we note that in Theorem 1 all β are taken into account which are represented in the form (1) (independently of the fact that (2) holds or not).

It follows from the above mentioned results of [8] or [5] that the set of values $N_S(\beta)$ with $\beta \in \mathcal{O}_S$ and (1) is not bounded. Theorem 1 implies immediately the following result.

Corollary 1. *$P(N_S(\beta)) \to \infty$ as $N_S(\beta) \to \infty$ with $\beta \in \mathcal{O}_S$ and* (1).

For $n = 1$, we shall give in **2** effective and quantitive versions of this assertion. We note that Corollary 1 can also be deduced from the results of [8] or [5]. We shall now give a consequence of Corollary 1 to decomposable form equations. Let

$$F(\underline{X}) = F(X_1, \cdots, X_m) \in \mathcal{O}_S[X_1, \cdots, X_m]$$

be a decomposable form in $m \geq 2$ variables which factorises into linear forms, say $l_1(\underline{X}), \cdots, l_n(\underline{X})$ over K. For a non-zero element b of $\mathcal{O}_S$, we consider the decomposable form equation

$$F(\underline{x}) = F(x_1, \cdots, x_m) = b \text{ in } x_1, \cdots, x_m \in \mathcal{O}_S. \tag{3}$$

Corollary 2. *Suppose that for some i with $1 \leq i \leq m, X_i$ can be expressed as a linear combination of $l_1(\underline{X}), \cdots, l_n(\underline{X})$. If (3) has infinitely many solutions and if $N_S(x_i)$ is unbounded for the solutions $\underline{x} = (x_1, \cdots, x_m)$ of (3) then, for these solutions, $P(N_S(x_i))$ is also unbounded.*

Important examples to which Corollary 2 can be applied are the full norm form equations, i.e. equations of the form

$$F(\underline{x}) = N(x_1 + \omega_2 x_2 + \cdots + \omega_n x_n) = b \text{ in } x_1, \cdots, x_n \in \mathbb{Z}$$

where $\{1, \omega_2, \cdots, \omega_n\}$ is a basis of $\mathbb{Q}(\omega_2, \cdots, \omega_n)$ over $\mathbb{Q}$. In this case, every X_i can be expressed as a linear combination of the linear factors of F, and if the equation is solvable and $n \geq 3$ or $n = 2$ and $\mathbb{Q}(\omega_2)$ is real, then it has infinitely many solutions $\underline{x} = (x_1, \cdots, x_n)$. Then $\max_i |x_i|$ is obviously unbounded. Moreover, it follows from a recent result of Everest [4] that, for these solutions, $|x_i|$ is unbounded for each i, and hence Corollary 2 implies that $P(x_i)$ is not bounded. For effective and quantitive versions of this assertion with $m = 2$, $n = 2$, see Corollary 4.

We shall now prove Theorem 1. As was mentioned above, the proof will be based on the following result on weighted unit equations. Let $\alpha'_0, \cdots, \alpha'_n \in K \setminus \{0\}$. A solution of the S-unit equation

$$\alpha'_0 x_0 + \cdots + \alpha'_n x_n = 1 \text{ in } x_0, x_1, \cdots, x_n \in \mathcal{O}_S^* \tag{4}$$

is called degenerate if $\alpha'_0 x_0 + \cdots + \alpha'_n x_n$ has a vanishing subsum, and non-degenerate otherwise. Now, we state the following theorem of Evertse and Győry [6].

Lemma 1. *The number of non-degenerate solutions of* (4) *is at most* $C_2(K, S, n)$.

As was mentioned above, the number C_2 cannot be made explicit by means of the method of proof used in [6]. At the last conference on Diophantine approximations in Oberwolfach (March 14-18, 1988), H.P. Schlickewei announced that in the special case when $K = \mathbb{Q}$ and S is generated by s distinct prime numbers, he is able to make explicit

$$C_2(\mathbb{Q}, S, n) = (8(s+1))^{2^{26n+4}(s+1)^6}.$$

Using this explicit value of C_2, in this special case we can make C_1 explicit in Theorem 1.

Proof of Theorem 1. It is enough to deal with the case $\beta \neq 0$. If $\beta \in \mathcal{O}_S \setminus \{0\}$ is represented in the form (1), then it is also represented by a non-empty subsum of $\alpha_0 x_0 + \cdots + \alpha_n x_n$ which has no non-empty vanishing subsum. Since $\alpha_0 x_0 + \cdots + \alpha_n x_n$ has at most 2^{n+1} subsums, it will be sufficient to prove the assertion for those β for which (1) holds and $\alpha_0 x_0 + \cdots + \alpha_n x_n$ has no vanishing subsum.

Let S' be the smallest subset of M_K with $S' \supseteq S$ such that all elements $\beta \in \mathcal{O}_S \setminus \{0\}$ with $P(N_S(\beta)) \leq P$ belong to $\mathcal{O}^*_{S'}$. It is easy to see that S' is finite and depends only on K, S and P. If $\beta \in \mathcal{O}_S \setminus \{0\}$ with $P(N_S(\beta)) \leq P$ is represented in the form (1), then we have

$$1 = \alpha_0(x_0/\beta) + \cdots + \alpha_n(x_n/\beta) \quad \text{where} \quad x_i/\beta \in \mathcal{O}^*_{S'}.$$

Hence, it follows from Lemma 1 that there exists a subset U_0 of $(\mathcal{O}^*_{S'})^{n+1}$ of cardinality at most $C_3(K, S', n) \leq C_4(K, S, P, n)$ with the following property: If $\beta \in \mathcal{O}_S \setminus \{0\}$ with $P(N_S(\beta)) \leq P$ such that

(5)

$$\beta = \alpha_0 x_0 + \cdots + \alpha_n x_n \text{ and } \alpha_0 x_0 + \cdots + \alpha_n x_n \text{ has no vanishing subsum,}$$

then $(x_0, \cdots, x_n) = \eta(x_0^0, \cdots, x_n^0)$ for some $\eta \in \mathcal{O}^*_{S'}$, and $(x_0^0, \cdots, x_n^0) \in$ $\in U_0$. Fix such a tuple $(x_0^0, \cdots, x_n^0) \in U_0$ and suppose that $\beta' \in \mathcal{O}_S \setminus \{0\}$ with $P(N_S(\beta')) \leq P$ is another element such that

$$(6) \quad \begin{cases} \beta' = \alpha_0 x'_0 + \cdots + \alpha_n x'_n \text{ holds, } \alpha_0 x'_0 + \cdots + \alpha_n x'_n \\ \quad \text{has no vanishing subsum and } (x'_0, \cdots, x'_n) = \\ \quad = \eta'(x_0^0, \cdots, x_n^0) \text{ with some } \eta' \in \mathcal{O}^*_{S'}. \end{cases}$$

Then, it follows that $(x'_0, \cdots, x'_n) = \eta'/\eta(x_0, \cdots, x_n)$ and hence we have $\eta'/\eta \in \mathcal{O}_S^*$. But this, together with (5) and (6), implies that $\beta' = (\eta'/\eta)\beta$ and so $N_S(\beta') = N_S(\beta)$. Consequently, the number of values $N_S(\beta)$ with $\beta \in \mathcal{O}_S \setminus \{0\}$ for which (1), (2) and $P(N_S(\beta)) \leq P$ hold does not exceed the cardinality of U_0 which is bounded above by $C_4(K, S, P, n)$. $\diamondsuit$

Proof of Corollary 2. Suppose that

$$X_i = c_{i_1} l_{i_1}(\underline{X}) + \cdots + c_{i_k} l_{i_k}(\underline{X}) \tag{7}$$

for some distinct $i_i, \cdots, i_k$ and $c_{i_1}, \cdots, c_{i_k} \in K\setminus\{0\}$. By assumption, (3) has infinitely many solutions $\underline{x} = (x_1, \cdots, x_m)$ and $N_S(x_i)$ is unbounded for these solutions. Then it follows from (3) that, for these solutions, $l_{i_j}(\underline{x})$ can assume only finitely many values apart from a factor from $\mathcal{O}_S^*$, $j = 1, \cdots, k$. Consequently, there is a subset χ of solutions $\underline{x} = (x_1, \cdots, x_m)$ of (3) with unbounded $N_S(x_i)$ such that, for each of these solutions, $l_{i_j}(\underline{x}) = \delta_{i_j} u_{i_j}$ with some fixed $\delta_{i_j} \in K \setminus \{0\}$ and with $u_{ij} \in \mathcal{O}_S^*, j = 1, \cdots, k$. There is a $t \in \mathrm{N}$ for which $\alpha_{i_j} := tc_{i_j}\delta_{i_j} \in \mathcal{O}_S \setminus \{0\}$ for $j = 1, \cdots, k$. Now (7) implies that

$$tx_i = \alpha_{i_1} u_{i_1} + \cdots + \alpha_{i_k} u_{i_k}. \tag{8}$$

For $k = 1$, this gives

$$N_S(x_i) N_S(t) = N_S(tx_i) = N_S(\alpha_{i_1})$$

which implies that $N_S(x_i)$ is bounded. For $k \geq 2$, Corollary 1 can be applied to (8). Then Corollary 1 together with the unboundedness of $N_S(x_i)$ implies that $P(N_S(tx_i))$ is unbounded, whence $P(N_S(x_i))$ is also unbounded. $\diamondsuit$

2. Effective results

In this section, we consider the effective versions of Corollary 1 for $n = 1$ and some of their consequences. Let K, $\mathcal{O}_K$, d, S, $\mathcal{O}_S$ and $\mathcal{O}_S^*$

have the same meaning as in **1**. For given $\underline{\alpha} = (\alpha_0, \alpha_1) \in (\mathcal{O}_S \setminus \{0\})^2$, consider now those $\beta \in \mathcal{O}_S \setminus \{0\}$ which can be represented in the form

$$\beta = \alpha_0 x_0 + \alpha_1 x_1 \quad \text{with} \quad x_0, x_1 \in \mathcal{O}_S^*. \tag{9}$$

Then it follows from an effective result of Győry ([7], Lemma 6) on S-unit equations that

$$P(N_S(\beta)) > C_5 \log\log N_S(\beta) \tag{10}$$

provided that $N_S(\beta) > C_6$, where C_5, C_6 are effectively computable positive numbers depending only on K, S and $\underline{\alpha}$. The proof of the above mentioned result of [7] involves Baker's theory on linear forms in logarithms and its p-adic analogue. By using the same theory as well as its p-adic analogue we shall prove the following improvement of (10).

For a rational integer ν with $|\nu| > 1$, we denote by $Q(\nu)$ the greatest square free factor of ν and we set $Q(0) = Q(\pm 1) = 1$.

Theorem 2. *There are effectively computable positive numbers C_7, C_8, depending only on K, S and $\underline{\alpha}$, such that if* (9) *and $N_S(\beta) > C_7$ hold then*

$$Q(N_S(\beta)) > \exp\{C_8 \frac{(\log\log N_S(\beta))^2}{\log\log\log N_S(\beta)}\}. \tag{11}$$

It follows from a well-known result (cf. [9]) that, for large $N_S(\beta)$,

$$\log Q(N_S(\beta)) \le 1.02 P(N_S(\beta)).$$

This, together with (11), implies

$$P(N_S(\beta)) > C_9 \frac{(\log\log N_S(\beta))^2}{\log\log\log N_S(\beta)}$$

with some effectively computable positive number $C_9 = C_9(K, S, \underline{\alpha})$.

For some applications, it will be more convenient to consider (9) and state Theorem 2 in a slightly different form. In what follows, $C_{10}(\ )$, $C_{11}(\ ), \cdots$ will denote effectively computable positive numbers depending only on parameters occuring between parantheses. For brevity, we write $N(\beta)$ for $N_{K/\mathbb{Q}}(\beta), \beta \in K$. We denote by $\mathcal{L}$ the multiplicative semigroup $\mathcal{O}_S^* \cap \mathcal{O}_K$, by $\overline{|\alpha|}$ the maximum of the absolute values of the conjugates of an algebraic number α, and by $H(\alpha)$ the (usual) height of α (i.e. maximum of the absolute values of the coefficients of the minimal defining polynomial of α over $\mathbf{Z}$). There is a positive integer a with $a \le C_{10}(\underline{\alpha})$ such that $a\alpha_i \in \mathcal{O}_K$ and $\overline{|a\alpha_i|} \le C_{11}(\underline{\alpha})$ for $i = 0, 1$. Further, for each pair x_0, x_1 satisfying (9), there is an $x \in \mathcal{L}$ such that $x\ x_i \in \mathcal{L}$ for $i = 0, 1$. Hence, we may assume without loss of generality that, in (9), $\underline{\alpha} = (\alpha_0, \alpha_1) \in (\mathcal{O}_K \setminus \{0\})^2$, $\beta \in \mathcal{O}_K \setminus \{0\}$ and

$$(9') \qquad \beta = \alpha_0 x_0 + \alpha_1 x_1 \quad \text{with} \quad x_0, x_1 \in \mathcal{L}.$$

Further, it is easy to see that we may also assume that

$$(12) \qquad \min(\mathrm{ord}_\wp(x_0), \mathrm{ord}_\wp(x_1)) \le C_{12}(K, S)$$

for every prime ideal $\wp$ in $\mathcal{O}_K$. Since $|N(\beta)| \ge N_S(\beta)$ and, for large $N_S(\beta), Q(N_S(\beta)) \ge C_{13} Q(N(\beta))$ with some $C_{13} = C_{13}(K, S, \underline{\alpha})$, Theorem 2 immediatly follows from the following.

Theorem 3. *Suppose that* $\beta \in \mathcal{O}_K \setminus \{0\}$ *is represented in the form* (9') *with* (12) *and* $|N(\beta)| > e^{e^e}$. *Then*

$$(13) \qquad \log|Q(N(\beta))| \ge C_{14} \frac{(\log\log|N(\beta)|)^2}{\log\log\log|N(\beta)|}$$

where C_{14} *is an effectively computable positive number depending only on* K, S *and* $\underline{\alpha}$.

Theorem 3 with $K = \mathbf{Q}$ is due to Shorey [11]. Theorem 3 and Theorem 4 below will be proved in **3**. To formulate Theorem 4, we write in (9')

$$(14) \qquad X = \max(\overline{|x_0|}, \overline{|x_1|}, e)$$

and

$$P_1 = P(N_{K/\mathbf{Q}}(\beta)).$$

Further, we set

$$(15) \qquad D = \begin{cases} 2 & \text{if } d = 1, \\ d & \text{if } d > 1. \end{cases}$$

The following result is an analogue of Corollary 1.2 of [13] which was established in the case $K = \mathbb{Q}$.

Theorem 4. *There are effectively computable positive numbers C_{15}, C_{16}, depending only on K, S and $\underline{\alpha}$, such that if (9') and (12) hold then*

$$(16) \quad \log(\prod_{\sigma} \max(|x_0^{(\sigma)}|, |x_1^{(\sigma)}|)) \leq C_{15} P_1^{D+1}(\log\log X)/\log(P_1+1)$$

where the product is taken over all the embeddings of K in C *and*

$$(17) \qquad \log H(\frac{x_1}{x_0}) \leq C_{16} P_1^{D+1}(\log\log X)/\log(P_1+1).$$

We establish now some consequences of Theorem 3 and 4. Let u_0, u_1, r and s be algebraic numbers such that

$$u_m = r u_{m-1} + s u_{m-2} \text{ for } m = 2, 3, \cdots.$$

We assume that the companion polynomial $X^2 - rX - s$ to the sequence $\{u_m\}_{m=0}^{\infty}$ has distinct non-zero roots α and β such that α/β is not a root of unity. Then, it is easy to see (cf. [13], Ch. B) that

$$(18) \qquad u_m = a\alpha^m + b\beta^m \text{ for } m = 0, 1, 2, \cdots$$

where

$$a = \frac{u_0\beta - u_1}{\beta - \alpha}, \; b = \frac{u_1 - u_0\alpha}{\beta - \alpha}.$$

Then $\{u_m\}_{m=0}^{\infty}$ is called a non-degenerate binary recursive sequence of algebraic numbers. There exists an effectively computable number C_{14} depending only on the sequence $\{u_m\}_{m=0}^{\infty}$ such that

$$u_m \neq 0 \text{ for } m \geq C_{17}.$$

Let $K = \mathbb{Q}(u_0, u_1, \alpha, \beta)$. Observe that $u_m \in K$ for $m \geq 0$. We write

$$N_{K/\mathbb{Q}}(u_m) = \frac{A_m}{B_m} \text{ for } m \geq C_{17}$$

where A_m and $B_m > 0$ are relatively prime rational integers. Then, as an immediate consequence of Theorem 4, we derive the following result which extends a result of Stewart [14].[1]

Corollary 3. *Let $\{u_m\}_{m=0}^{\infty}$ be a non-degenerate binary recursive sequence of algebraic numbers. Let α and β be roots of the companion polynomial of the sequence $\{u_m\}_{m=0}^{\infty}$. Let $K = \mathbb{Q}(u_0, u_1, \alpha, \beta)$ and let D be given by* (15). *Then, there exists an effectively computable number $C_{18} > 0$ depending only on the sequence $\{u_m\}_{m=0}^{\infty}$ such that*

$$P(A_m) \geq C_{18} m^{1/D+1} \text{ if } m \geq C_{18}. \tag{19}$$

Proof of Corollary 3. Let k be the least positive integer such that ka, kb, $k\alpha$ und $k\beta$ are algebraic integers. By considering the sequence $\{k^{m+1}u_m\}_{m=0}^{\infty}$, there is no loss of generality in assuming that a, b, α and β are elements of $\mathcal{O}_K$. We write

$$([\alpha^h], [\beta^h]) = [\pi] \text{ with } \pi \in \mathcal{O}_K$$

and

$$\alpha_1 = \pi^{-1}\alpha^h, \ \beta_1 = \pi^{-1}\beta^h$$

where h denotes the class number of K. Then α_1, $\beta_1 \in \mathcal{O}_K$ satisfy $([\alpha_1], [\beta_1]) = [1]$ and α_1/β_1 is not a root of unity. Putting $m = m_1 h + m_2$ with $m_1, m_2 \in \mathbb{Z}$, $0 \leq m_2 < h$ and $a_1 = \alpha^{m_2} a$, $b_1 = \beta^{m_2} b$ in (18), we see that

$$\pi^{-m_1} u_m = a_1 \alpha_1^{m_1} + b_1 \beta_1^{m_1}. \tag{20}$$

Now we apply (17) to the right hand side of (20) to complete the proof of Corollary 3. ◇

Remark. For a non-degenerate binary recursive sequence $\{u_m\}_{m=0}^{\infty}$ with u_0, u_1, r, $s \in \mathbb{Z}$, Shorey [11] showed that

$$\log Q(u_m) \geq C_{19} (\log m)^2 (\log\log m)^{-1} \text{ if } m \geq C_{20}, \tag{21}$$

[1] In the proofs of [14] and [12] on lower bounds for $P(u_m)$ and $P(u_m/u_n)$, we need to replace the assertions of van der Poorten by the theorems of Yu on p-adic linear forms in logarithms. In view of this, d should be replaced by D in these estimates. A similar remark applies to [13, Chapters 2,3].

where $C_{19} > 0$ and C_{20} are effectively computable numbers depending only on the sequence $\{u_m\}_{m=0}^{\infty}$. In fact, Shorey [11] proved the estimate (21) for $\frac{[u_m,u_n]}{(u_m,u_n)}$ with $m > n$ and $u_n \neq 0$. We note that our Theorem 3 above is an extension of (21).

Next, we derive from Theorems 3 and 4 the following result which is an effective and quantitive version of Corollary 2 with $m = 2$. Compare this with Theorem 5.2 of [13].

Corollary 4. *Let $\Delta > 0$ be a rational integer. Suppose that a, b, c are rational integers satisfying $ac \neq 0$ and $b^2 - 4ac \neq 0$. Let x and y be non-zero rational integers satisfying*

$$P(ax^2 + bxy + cy^2) \leq \Delta. \tag{22}$$

Then we have

(a) *There exists an effectively computable number $C_{21} > 0$ depending only on a, b, c and Δ such that*

$$P(x) \geq C_{21}(\log|x|)^{1/3}, \quad P(y) \geq C_{21}(\log|y|)^{1/3}. \tag{23}$$

(b) *There exists an effectively computable number $C_{22} > 0$ depending only on a, b, c and Δ such that*

$$\log Q(x) \geq C_{22}\frac{(\mathrm{loglog}x')^2}{\mathrm{logloglog}x'}, \ \log Q(y) \geq C_{22}\frac{(\mathrm{loglog}y')^2}{\mathrm{logloglog}y'}, \tag{24}$$

where $x' = \max(|x|, e^e)$ and $y' = \max(|y|, e^e)$.

Let α be a real algebraic number of degree 2. For $n \geq 0$, we write p_n/q_n for the n-th convergent in the continued fraction expansion of α. It is clear that the assumptions of Corollary 4 are satisfied with $x = p_n$, $y = q_n$. Therefore, the estimates (23) and (24) with $x = p_n$, $y = q_n$ are valid. In fact, this particular case of Corollary 4 is a consequence of the estimates (19) and (21) on the greatest prime factor and the greatest square free factor of a non-degenerate binary recursive sequence.

Proof of Corollary 4. There is no loss of generality in assuming that $a = 1$. Let α and β be non-zero distinct algebraic integers satisfying

$$x^2 + bxy + cy^2 = (x - \alpha y)(x - \beta y). \tag{25}$$

We set $K = \mathbb{Q}(\alpha)$. Then $D = 2$. Let $\wp_1, \cdots, \wp_t$ be the set of all prime ideals in K which divide rational primes not exceeding $N(\alpha\beta)\Delta$ and we write $\mathcal{L}$ for the set of all non-zero elements of $\mathcal{O}_K$ which have no prime ideal divisor different from $\wp_1, \cdots, \wp_t$. Then we observe from (22) and (25) that $\beta(x - \alpha y)$, $\alpha(x - \beta y)$, $(x - \alpha y)$ and $(-x + \beta y)$ are elements of $\mathcal{L}$. Furthermore, we observe that

$$(\beta - \alpha)x = \beta(x - \alpha y) + \alpha(-x + \beta y) \tag{26}$$

and

$$(\beta - \alpha)y = (x - \alpha y) + (-x + \beta y). \tag{27}$$

(a) We apply Theorem 4 with $\alpha_0 = \alpha_1 = 1$, $x_0 = \beta(x - \alpha y)$ and $x_1 = \alpha(-x + \beta y)$. For this, we observe from (26) that X given by (14) satisfies $2X \geq |(\beta - \alpha)x|$. Now, we derive from (16) that $P(x) \geq C_{21}(\log |x|)^{1/3}$. Similarly, the estimate for $P(y)$ follows from (27).
(b) We apply Theorem 3 with $x_0 = \beta(x - \alpha y)$, $x_1 = \alpha(-x + \beta y)$, as well as $x_0 = x - \alpha y$, $x_1 = -x + \beta y$, to obtain (24). ◊

3. Proofs of Theorems 3 and 4

We keep the notation of §2. In what follows, $C_{23}, C_{24}, \cdots$ will denote effectively computable positive numbers which, unless otherwise stated, depend only on K, S and $\underline{\alpha}$. First we prove Theorem 3. Suppose that $\beta \in \mathcal{O}_K \setminus \{0\}$ is represented in the form (9') with $\underline{\alpha} = (\alpha_1, \alpha_2) \in (\mathcal{O}_K \setminus \{0\})^2$, $x_0, x_1 \in \mathcal{L}$ and (12). We may assume that $|N(\beta)| > C_{23}$ with C_{23} sufficiently large. Further, we can write (cf. [13], Ch. A)

$$x_i = \rho_i \eta_1^{a_{i,1}} \cdots \eta_r^{a_{i,r}} \pi_1^{b_{i,1}} \cdots \pi_s^{b_{i,s}} \quad \text{for} \quad i = 0, 1, \tag{28}$$

where $a_{i,1}, \cdots, a_{i,r} \in \mathbb{Z}$, $b_{i,1}, \cdots, b_{i,s}$ are non-negative rational integers for $i = 0, 1$,

$$\max(\overline{|\rho_1|}, \overline{|\rho_2|}, \overline{|\eta_1|}, \cdots, \overline{|\eta_r|}, \overline{|\pi_1|}, \cdots, \overline{|\pi_s|}) \leq C_{24}(K, S), \tag{29}$$

$\{\eta_1, \cdots, \eta_r\}$ is a maximal system of independent units in $\mathcal{O}_K$ and the principal ideals $[\pi_1], \cdots, [\pi_s]$ are the h-th powers of the prime ideals in $\mathcal{O}_K$ corresponding to the finite places in S. Here h denotes the class number of K.

Theorem 3 is an immediate consequence of the following result.

Lemma 2. *Let $\beta \in \mathcal{O}_K \setminus \{0\}$ be represented in the form (9') with the properties (12), (28), (29) and $|N(\beta)| > C_{23}$. Further, suppose that*

$$\log P(N(\beta)) \leq (\text{loglog}|N(\beta)|)^2. \tag{30}$$

Then, there exists $C_{25} > 0$ such that

$$\sum_{\substack{p|N(\beta) \\ p \geq (\log|N(\beta)|)^{C_{25}}}} 1 \geq C_{25} \frac{\text{loglog}|N(\beta)|}{\text{logloglog}|N(\beta)|}$$

where p runs through rational primes.

Proof. We may assume that

$$\sum_{\substack{p|N(\beta) \\ p \geq (\log|N(\beta)|)^{\epsilon}}} 1 < \epsilon \frac{\text{loglog}|N(\beta)|}{\text{logloglog}|N(\beta)|}, \tag{31}$$

where ϵ is an effectively computable positive number with $\epsilon \leq 1$ which depends only on K, S and $\underline{\alpha}$ and which will be chosen suitably later. Thus, we allow C_{23} to depend also on ϵ.

Denote by $\mathcal{P}$ the set of all prime ideals in $\mathcal{O}_K$, and put

$$\mathcal{P}_1 = \{\wp \in \mathcal{P} | \wp | p \text{ for some positive rational prime } p < (\log|N(\beta)|)^{\epsilon}\} \tag{32}$$

and

$$\mathcal{P}_2 = \{\wp \in \mathcal{P} | \wp | p \text{ for some rational prime } p \text{ with } (\log|N(\beta)|)^{\epsilon} \leq p \leq \exp\{(\text{loglog}|N(\beta)|)^2\}\}. \tag{33}$$

Then $\wp \in \mathcal{P}_1 \cup \mathcal{P}_2$ for each prime ideal divisor $\wp$ of β. The product of h ideals from any fixed ideal class (modulo the group of principal ideals) is a principal ideal. Hence β can be written in the form

$$\beta = \beta_1 \cdot \beta_2 \quad \text{with} \quad \beta_1, \beta_2 \in \mathcal{O}_K \tag{34}$$

so that all prime ideal divisors of β_1 belong to $\mathcal{P}_1$ and β_2 is divisible by at most $h(h-1)$ prime ideals (with multiplicities) from $\mathcal{P}_1$. Further, this, together with (30) and (31), implies

$$\beta = \rho_2' \gamma_1'^{d_1} \cdots \gamma_t'^{d_t}$$

where ρ_2' is a unit in $\mathcal{O}_K, \gamma_1', \cdots, \gamma_t'$ are non-units in $\mathcal{O}_K$ and $d_1, \cdots, d_t$ are non-negative rational integers such that

$$t \le C_{26} \epsilon \frac{\mathrm{loglog}|N(\beta)|}{\mathrm{logloglog}|N(\beta)|} + C_{27} \tag{35}$$

and

$$\log |N(\gamma_j')| \le C_{28} (\mathrm{loglog}|N(\beta)|)^2 \quad \text{for} \quad j = 1, \cdots, t.$$

Consequently, we apply Lemma A.15 of [13] to find associates $\gamma_1, \cdots, \gamma_t$ of $\gamma_1', \cdots, \gamma_t'$, respectively, such that

$$\log \overline{|\gamma_j|} \le C_{29} (\mathrm{loglog}|N(\beta)|)^2 \quad \text{for} \quad j = 1, \cdots, t. \tag{36}$$

Further, on multiplying both sides of (9') by an appropriate unit and applying again Lemma A.15 of [13] to x_0 and x_1, there is no loss of generality in assuming that

$$\beta_2 = \gamma_1^{d_1} \cdots \gamma_t^{d_t}, \tag{37}$$

$$\log \overline{|\beta_1|} \le C_{30} \log |N(\beta_1)| \tag{38}$$

and (12), (28), (29) hold. Also, observe that

$$d_j \le (\log |N(\beta_2)|) / \log 2 \le 2 \log |N(\beta)| \quad \text{for} \quad j = 1, \cdots, t. \tag{39}$$

Let in (28),

$$V =: \max_{\substack{1\le j\le r\\ i=0,1}} |a_{i,j}|,\ W =: \max_{\substack{1\le j\le r\\ i=0,1}} b_{i,j}, \tag{40}$$

and we put

$$U =: \max(V, W). \tag{41}$$

In view of $|N(\beta)| > C_{23}$, we have $U > C_{31}$ with some C_{31} sufficiently large. We apply an estimate of Yu ([16], Theorem 1') on p-adic linear forms in logarithms to derive from (9'), (28), (29), (12), (40) and (41) that

$$\mathrm{ord}_{\wp}(\beta) \le C_{32} P^D (\log U)/\log p, \tag{42}$$

where $\wp$ is a prime ideal in $\mathcal{P}_1 \cup \mathcal{P}_2$ dividing a rational prime p. Now, we apply (42), (32) and Theorem 9 of [9] to derive that

$$\log|N(\beta_1)| \le (\log|N(\beta)|)^{C_{33}\epsilon} \log U$$

whence, by (38),

$$\log \overline{|\beta_1|} \le C_{30}(\log|N(\beta)|)^{C_{33}\epsilon} \log U. \tag{43}$$

Let $\wp$ be a prime ideal divisor of π_1 in $\mathcal{O}_K$. We apply again Theorem 1' of Yu [16] on p-adic linear forms in logarithms to $\beta - \alpha_1 x_1$ to derive from (9'), (28), (29), (12), (40), (41), (39), (36) and (35) that

$$b_{0,1} \le (\log|N(\beta)|)^{C_{34}\epsilon} (\log U)^2. \tag{44}$$

Repeated applications of estimates for p-adic linear forms in logarithms provide the estimate (44) for all $b_{i,j}$ with $i = 0, 1$ and $j = 1, \cdots, s$. Thus

$$W \le (\log|N(\beta)|)^{C_{34}\epsilon} (\log U)^2. \tag{45}$$

If $U \le W^2$, then we observe from (45) that

$$W \le (\log|N(\beta)|)^{2C_{34}\epsilon}, U \le (\log|N(\beta)|)^{4C_{34}\epsilon}$$

which, together with (9'), (28), (29) and $|N(\beta)| > C_{23}$, implies that $\log|N(\beta)| \le (\log|N(\beta)|)^{8C_{34}\epsilon}$ which is not possible if $\epsilon < (8C_{34})^{-1}$. Thus, we assume that

$$U > W^2. \tag{46}$$

Then, by (41), (9'), (28), (29) and $|N(\beta)| > C_{23}$,

$$U = V \text{ and } U \ge (\log|N(\beta)|)^{1/2}. \tag{47}$$

There is no loss of generality in assuming that $|a_{0,1}| = V$. We write from (28) that, for each embedding σ of K in $\mathbb{C}$,

$$\sum_{j=1}^{r} a_{0,j} \log|\eta_j^{(\sigma)}| = -\log|\rho_0^{(\sigma)}| + \log|x_0^{(\sigma)}| - \sum_{j=1}^{t} b_{0,j}|\pi_j^{(\sigma)}|.$$

This, together with (47), (29) and (46), implies (cf. also [13], Ch. A) that

$$U = V \le C_{35}(\log\overline{|x_0|} + U^{1/2}).$$

Therefore, in view of (47) and $|N(\beta)| > C_{23}$

$$\log\overline{|x_0|} \ge C_{36}U. \tag{48}$$

On the other hand, we see from (28) and (46) that

$$\log|N(x_0)| \le C_{37}W < C_{37}U^{1/2}. \tag{49}$$

By (46), we have $d \ge 2$. Further, in view of (48), (49), (47) and $|N(\beta)| > C_{23}$ we may assume that there exists an embedding σ of K in C such that

$$\log|x_0^{(\sigma)}| \le -\frac{C_{36}}{d}U. \tag{50}$$

Now, apply Theorem 2 of Baker [1] on linear forms in logarithms to obtain from (9'), (28), (29), (34), (37), (43), (36), (39), (35), (41) and (47) that

$$\log|(\alpha_0 x_0)^{(\sigma)}| = \log|\beta^{(\sigma)} - (\alpha_1 x_1)^{(\sigma)}| \ge \tag{51}$$

$$\geq (\log |N(\beta)|)^{C_{38}\epsilon} (\log U)^2.$$

Finally, we combine (50) and (51) to derive that

$$U \leq (\log |N(\beta)|)^{2C_{38}\epsilon}$$

which, in view of (47), is not possible if $\epsilon < (4C_{38})^{-1}$. Finally, we set $\epsilon = \min((8C_{34})^{-1}, (4C_{38})^{-1}, 1)$ and $C_{25} = \epsilon/2$ to complete the proof of Lemma 2. ◇

Proof of Theorem 4. Suppose that $\beta \in O_K \setminus \{0\}$ and $x_0, x_1 \in \mathcal{L}$ satisfying (9') and (12). Then, as we have seen above, we may also assume that (28) and (29) hold. Let V, W, U and X be defined by (40), (41) and (14), respectively. Then, using some arguments from the above proof, it is easy to see that

$$U = \max(V, W) \leq C_{39} \log X. \tag{52}$$

We apply Theorem 2 of Baker [1] on linear forms in logarithms to derive form (9'), (12), (28), (29) and (52) that

$$|N(\beta)| \geq C_{40}(\prod_{\sigma} \max(|\alpha_0 x_0)^{(\sigma)}|, |(\alpha_1 x_1)^{(\sigma)}|)(\log X)^{-C_{41}}, \tag{53}$$

where the product is taken over all the embeddings σ of K in $\mathbb{C}$. On the other hand, it follows from (42), (52) and Theorem 9 of [9] that

$$\log |N(\beta)| \leq C_{42} P_1^{D+1} (\mathrm{loglog} X)/\log(P_1 + 1). \tag{54}$$

We combine (53) and (54) to derive (16). Finally, (17) follows from (16) and Lemma C of [10]. ◇

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ON RINGS SATISFYING CERTAIN POLYNOMIAL IDENTITIES*

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Abstract: Let $m>n\geq 1$ be natural numbers such that m-n is odd; we prove that the identity $x^m = x^n$ implies $x^{m-n+1} = x$ in rings with unity. Moreover we describe the free ring corresponding to $x^n = x$, where n=2^t.

1. Preliminaries

During the last forty years the investigation of rings with polynomial identitities became a very important branch of ring theory. The

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pioneering papers are due to Jacobson ([3], [4]). He proved that a ring satisfying $x^n = x$ $(n \geq 2)$ is commutative (in fact he proved a stronger version of this result). In the present note we introduce the notion of (m,n)-Boolean rings by generalizing Jacobson's above identity. The structure of (m,n)-Boolean rings heavily depends on the parity of the difference $m-n$. Our main result is a reduction theorem for the odd case. Another reduction theorem for the $x^n = x$ $(n \geq 2)$ case will be also stated. Finally, in the $n = 2^t$ case we describe the free ring statisfying $x^n = x$.

2. Reduction theorems for (m,n)-Boolean rings

Given two natural numbers $m > n \geq 1$, a ring R is said to be (m,n)-*Boolean* if $x^m = x^n$ for all $x \in R$.

Theorem 2.1. *Let R be an (m,n)-Boolean ring with unity, where $m-n$ is odd. Then R is $(m-n+1,1)$-Boolean (and by Jacobson's well-known theorem we also get the commutativy of R).*

Proof. On applying $x^m = x^n$ to $x = -1_R$ we obtain $1_R + 1_R = 0$, i.e. that $2x = 0$ for all $x \in R$. Now we prove that R has no nilpotent element. Let $k \geq 2$ be an integer and suppose that $x^k = 0$ and $x^{k-1} \neq 0$ for a nilpotent $x \in R$. Using the binomial theorem, $(1_R + x^{k-1})^m = (1_R + x^{k-1})^n$ gives that $1_R + mx^{k-1} = 1_R + nx^{k-1}$, whence we get $(m-n)x^{k-1} = 0$. The odd parity of $m-n$ gives that $x^{k-1} = (m-n)x^{k-1} = 0$, a contradiction. The absence of nilpotent elements enables us to use a theorem of Andrunakievich and Rjabuhin (see [1]). According to this theorem R is a subdirect product of *domains* (i.e. not necessarily commutative rings without zero divisors) R_i $(i \in I)$. Since R_i is a factor of R, the identity $x^m = x^n$ remains true in R_i. But it can easily be seen that in a domain $x^m = x^n$ implies $x^{m-n+1} = x$. Hence any subdirect product of the rings R_i $(i \in I)$ will also satisfy $x^{m-n+1} = x$. ◊

Remark. In the case of even $m-n$ we cannot expect such a reduction theorem. For instance $\mathbf{Z}_{12}$ and the ring of 2×2 upper triangular matrices over a Boolean ring are examples of (4,2)-Boolean rings, the former has a nilpotent element and the latter is non-commutative.

Theorem 2.2. *An* $(n, 1)$*-Boolean ring* R *is* $(n^*, 1)$*-Boolean, where* $n^* - 1 = l.c.m.\{p^k - 1 | p$ *is prime,* $p^k - 1$ *is a divisor of* $n - 1\}$.

Remark. The authors believe that this result is not essentially new, however we were not able to find a reference. Related investigations can be found in [2], [6] and [7].

Proof. We can proceed similarly to the proof of Th. 2.1. A domain satisfies $x^n = x$ if and only if it is a finite field of the form GF(p^k), where $p^k - 1$ is a divisor of $n - 1$. This result is explicit in [6] and in [5]. Since each subdirect factor R_i of R satisfies $x^{n^*} = x$, we get that their subdirect product R will also satisfy the same identity.◇

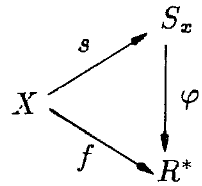
Remark. An immediate application of Th. 2.1. and Th. 2.2. can give the following reduction result. *Let* R *be a* $(16, 11)$*-Boolean ring with unity, then* Th. 2.1. *gives* $(16, 11) \Rightarrow (6, 1)$, *and* Th. 2.2. *gives* $(6, 1) \Rightarrow (2, 1)$, *where* $2 = 6^*$. *Thus we get that* R *is a Boolean ring in the classical sense.*

3. The free (2^t,1)-Boolean ring

Theorem 3.1. *Let* $n = 2^t$, *then the free* $(n, 1)$*-Boolean ring generated by a non-void set* X *can be obtained as the semigroup ring* $\mathbb{Z}_2(S_x)$, *where* S_x *is the free semigroup on* X *with defining relations* $x^n = x$ *and* $xy = yx$.

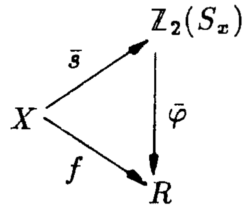
Proof. Using the polynomial theorem and the well known fact that polynomial coefficients of the form $\frac{n!}{i_1!i_2!\ldots i_k!}$ (where $n = 2^t = i_1 + i_2 + \ldots + i_k$ and $1 \leq i_\nu \leq n - 1$ for some ν) are even integers, we obtain that $\mathbb{Z}_2(S_x)$ satisfies $x^n = x$.
In order to prove universality let $f : X \to R$ be a set mapping with R an $(n, 1)$-Boolean ring. Since the multiplicative semigroup R^* of R satisfies $x^n = x$ and $xy = yx$ (by Jacobson's theorem) there is unique semigroup-homomorphic extension φ of f making the diagram (3.1) commute

(3.1)



Now it is easy to see that the definiton $\bar{\varphi}(\sum_{\sigma\in S_x} \bar{n}_\sigma \sigma) = \sum_{\sigma\in S_x} n_\sigma \varphi(\sigma)$ with $\bar{n}_\sigma = n_\sigma + (2) \in \mathbb{Z}_2$ is correct and gives a $\mathbb{Z}_2(S_x) \to R$ ring-homomorphism making (3.2) commute (we need $2R = 0$!)

(3.2)



$\bar{s}(x) = 1.s(x)$

Since the subset $\bar{s}(X) \subseteq \mathbb{Z}_2(S_x)$ generates $\mathbb{Z}_2(S_x)$ as a ring, the unicity of $\bar{\varphi}$ is clear. $\diamond$

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SEQUENCES OF DOMINATING SETS

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Abstract: It is proved for any $0 < \beta < 1$ and any graph $G = (V, E)$ there exists an ordering $v_1, v_2, \ldots, v_{|V|}$ of vertices of G such that either for every $i \in \{1, \ldots, |V|\}$ the set $\{v_1, \ldots, v_i\}$ dominates in G all but at most $|V| \cdot \beta^i$ vertices, or for every $j \in \{1, \ldots, |V|\}$ the set $\{v_1, \ldots, v_j\}$ dominates in the complement $\bar{G}$ of G all but at most $|V|(1 - \beta)^j$ vertices.

Let X be a subset of the vertex-set of a graph $G = (V, E)$ and $N_G(X) = X \cup \{y \in V | \exists x \in X : (x, y) \in E\}$. Let us say that X is β-dominating in G, if $|V \setminus N_G(X)| \leq |V| \cdot \beta^{|X|}$. By $\bar{G}$ we denote the complement of G.

Erdős and Hajnal [1] conjectured that for any positive integer t and any graph $G = (V, E)$ with $|V| \geq t$ either there exists a 0.5-dominating set X in G with $|X| = t$ or there exists a (1-0.5)-dominating set Y in $\bar{G}$ with $|Y| = t$. Erdős, Faudree, Gyárfás and Schelp [2] proved that this conjecture remains true even if we put any $0 < \beta < 1$ instead of 0.5.

The aim of the present note is to prove the following somewhat stronger statement, which was obtained independently of [2].

Proposition 1. *For any $0 < \beta < 1$ and any graph $G = (V, E)$ either there exists a numbering $v_1, v_2, \dots, v_{|V|}$ of the vertices of G such that the set $\{v_1, v_2, \dots, v_i\}$ is β-dominating in G for every $i \in \{1, \dots, |V|\}$, or there exists a numbering $u_1, u_2, \dots, u_{|V|}$ of the vertices of G such that the set $\{u_1, u_2, \dots, u_j\}$ is $(1-\beta)$-dominating in $\bar{G}$ for every $j \in$ $\in \{1, \dots, |V|\}$.*

Proposition 1 is a consequence of the following proposition. (To see this, apply Proposition 2 to the bipartite graph $\widetilde{G} = (X, Y; \widetilde{E})$, where $\widetilde{G}$ is obtained from $G = (V, E)$ as follows: $|X| = |Y| = |V|$ and $(x_i, y_j) \in \widetilde{E}$ iff $(v_i, v_j) \in E$).

Proposition 2. *Let $G = (X, Y; E)$ be a bipartite graph with parts X and Y, and $0 < \beta < 1$. Then at least one of the following assertions is true:*

(a) *there is a numbering $x_1, x_2, \dots, x_{|X|}$ of the vertices of X such that*
$$|Y \setminus N_G(\{x_1, x_2, \dots, x_i\})| \le |Y| \cdot \beta^i$$
for every $i \in \{1, \dots, |X|\}$;

(b) *there is a numbering $y_1, y_2, \dots, y_{|Y|}$ of the vertices of Y such that*

$$|X \setminus N_{\bar{G}}(\{y_1, y_2, \dots, y_j\})| < |X|(1-\beta)^j \tag{1}$$

for every $j \in \{1, \dots, |Y|\}$.

Proof. We try to construct the proper numbering of vertices of X, using the following *Procedure 1.*
BEGIN. Let $X_0 := \emptyset$; $i := 1$;
Step i. If $i = |X| + 1$, then END. If there is $x \in X \setminus X_{i-1}$ such that $|(Y \setminus N_G(X_{i-1})) \setminus N_G(\{x\})| \le \beta |Y \setminus N_G(X_{i-1})|$ (in particular, if $Y \setminus N_G(X_{i-1}) = \emptyset$), then set $x_i := x$, $X_i := X_{i-1} \cup \{x_i\}$ and go to Step $i + 1$. Else END.

If the Procedure stops on Step t and $t = |X| + 1$, then Assertion (a) of our Proposition 2 is true. Let $t \le |X|$ and $Y_0 = Y \setminus N_G(X_{t-1})$. Then $Y_0 \ne \emptyset$ and, by the construction, for $x \in X$ we have

$$|Y_0 \cap N_G(x)| < (1-\beta)|Y_0|. \tag{2}$$

The following *Procedure 2* will make it possible to number the vertices of Y_0 properly.
BEGIN. *Step* $k(1 \leq k \leq |Y_0|)$. Before Step k the vertices $y_1, \dots, y_{k-1} \in$ $\in Y_0$ are chosen that the inequality (1) is fullfilled for $j = 1, 2, \dots, k-1$ and, denoting $Y_{k-1} := Y_0 \setminus \{y_1, \dots, y_{k-1}\}$, for any $x \in$ $\in X \setminus N_{\bar{G}}(\{y_1, \dots, y_{k-1}\}) = \bigcap_{j=1}^{k-1} N_G(y_j) \cap X$ the inequality

$$(3) \qquad |N_G(\{x\}) \cap Y_{k-1}| < (1-\beta)|Y_{k-1}|$$

holds. Note that for $k = 1$, (3) follows from (2). If $X \subset N_{\bar{G}}(\{y_1, \dots, y_{k-1}\})$, then choose an arbitrary $y \in Y_{k-1}$, $y_k := y$, $Y_k := Y_{k-1} \setminus \{y_k\}$ and go to Step $k+1$. Suppose $\bigcap_{j=1}^{k-1} N_G(\{y_j\}) \cap X \neq \emptyset$. Due to (3), there exists $y \in Y_{k-1}$ such that

$$(4) \qquad |\bigcap_{j=1}^{k-1} N_G(\{y_j\}) \cap X \cap N_G(\{y\})| < (1-\beta)|\bigcap_{j=1}^{k-1} N_G(\{y_j\}) \cap X|.$$

Set $y_k := y$, $Y_k := Y_{k-1} \setminus \{y_k\}$. Notice that (4) implies the validity of (1) for $j = k$. Because of (3), we have

$$(5) \qquad |N_G(\{x\}) \cap Y_k| = |N_G(\{x\}) \cap Y_{k-1}| - 1 < (1-\beta)|Y_{k-1}| - 1 < < (1-\beta)|Y_k|$$

for every $x \in \bigcap_{j=1}^{k} N_G(\{y_j\}) \cap X$. To to Step $k+1$, knowing that for that Step inequality (3) holds, since now (5) holds. END.

Thus, on completion of Procedure 2 the vertices of Y_0 will be numbered properly. But, according to (2), $N_{\bar{G}}(Y_0) \supset X$. Thus, the vertices of $Y \setminus Y_0$ we can number by $|Y_0| + 1, \dots, |Y|$ in an arbitrary order.

Remark. Evidently, a polynomial time via $|X| + |Y|$ is sufficient for numbering X or Y.

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WELL COVERED AND WELL DOMINATED BLOCK GRAPHS AND UNICYCLIC GRAPHS

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Abstract: A graph is called well covered if every maximal independent set is a maximum independent set. Analogously, a well dominated graph is one in which every minimal dominating set is a minimum dominating set. In this paper, characterizations of well dominated and well covered block graphs and unicyclic graphs are given.

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In this paper, we discuss finite undirected simple graphs. For any undefined term see [2] and [10]. For a graph G, we denote by $V(G)$ and $E(G)$ the vertex set and edge set of G, respectively. For $v \in V(G)$, let $N_G(v)$ be the set of vertices (neighbours) adjacent to v in G and, more generally, $N_G(S) = \bigcup_{v \in S} N_G(v)$ and $\overline{N}_G(S) = N_G(S) \cup S$ for $S \subseteq V(G)$. If $X \subseteq V(G)$, then $[X]$ (resp., $G - X$) denotes the subgraph of G induced by X (resp., $V(G) - X$). We write $G - x$ instead of $G - \{x\}$ if $x \in V(G)$.

The vertex v of G is an end vertex of G if $d_G(v) = 1$, where $d_G(x) = |N_G(x)|$ is the degree of $x \in V(G)$. An edge incident with an end vertex of G is called an end edge of G. For a graph G, let $\Omega(G)$ ($E_e(G)$, resp.) be the set of end vertices (end edges, resp.) of G. A vertex v of a connected graph G is called a cut vertex of G if $G - v$ contains more components than G. Let $C(G)$ be the set of cut vertices of G. For $v \in V(G)$, let $NC_G(v) = N_G(v) - C(G)$. A connected graph with no cut vertices is called a block. A block of a graph G is a subgraph of G which is itself a block and which is maximal with respect to that property. A graph G is called a block graph if every block of G is a complete graph. In this paper, we define an exterior block of a graph G as a block containing at least one non-cut vertex of G. For a graph G, the corona $G \circ K_1$ of G and K_1 is the supergraph of G obtained from G by adding, for every vertex x of G, exactly one new vertex adjacent to x only. Note that a graph H is the corona of some graph G and K_1 if and only if $E_e(H)$ is a perfect matching of H.

A set $D \subseteq V(G)$ is a dominating set of G if $N_G(v) \cap D \neq \emptyset$ for every $v \in V(G) - D$, and is an independent set of G if $N_G(v) \cap D = \emptyset$ for every $v \in D$. Let $i(G)$ and $\alpha(G)$ ($\gamma(G)$ and $\Gamma(G)$, resp.) denote the minimum and maximum cardinalities of a maximal independent set (a minimal dominating set, resp.) in G. A graph G is said to be well covered if every maximal independent set in G is a maximum independent set. A graph G is said to be well dominated [7] if every minimal dominating set in G is a minimum dominating set. Equivalently, G is a well covered (dominated, resp.) graph if $i(G) = \alpha(G)$ ($\gamma(G) = \Gamma(G)$, resp.).

Well covered graphs were introduced by Plummer in 1970 [11]. Until now, however, only a few classes of well covered graphs have been characterized. For example, Ravindra [12] gave a characterization of well covered bipartite graphs. Recently Finbow, Hartnell, and Nowakowski in [7], [8], and [9] have completely described well covered and well dominated graphs of girth at least 5, well dominated bipartite graphs, and well covered graphs containing neither a cycle C_4 nor a cycle C_5 as a subgraph. For related results the reader is referred to [1], [3-6], and [13-15]. In this paper, it is shown that for a block graph G one of the four equations $\gamma(G) = \alpha(G)$, $\gamma(G) = \Gamma(G)$, $i(G) = \alpha(G)$, $i(G) = \Gamma(G)$ holds if and only if the other three hold. Structural characterizations of well covered and well dominated block graphs are given. Similar results for unicyclic graphs are presented.

In the sequel, we will need the following simple results and observations.

Proposition 1. *For any graph G,*

$$\gamma(G) \leq i(G) \leq \alpha(G) \leq \Gamma(G).$$

Proof. It follows at once from the simple observation that every maximal independent set in G is a minimal dominating set in G. $\diamondsuit$

This proposition implies that every well dominated graph is well covered. The converse implication is not necessarily true (see, for example, Theorems 2 and 3 below). The next proposition implies that the corona of any graph G (and K_1) is a well dominated graph. Theorem 1, among other things, proves that every well covered block graph is well dominated.

Proposition 2. *For any graph G,*

$$\gamma(G \circ K_1) = i(G \circ K_1) = \alpha(G \circ K_1) = \Gamma(G \circ K_1) = |V(G)|.$$

Proof. After Proposition 1, it is enough to show that every minimal dominating set in $G \circ K_1$ has exactly $|V(G)|$ vertices. Let D be a minimal dominating set in $G \circ K_1$. For a vertex $x \in V(G)$, let $\overline{x}$ be the only neighbour of x in $\Omega(G \circ K_1)$. It is clear from the definition of $G \circ K_1$ that the sets $\{x, \overline{x}\}$, $x \in V(G)$, form a disjoint partition of $V(G \circ K_1)$. Therefore the minimality of D implies that $|D \cap \{x, \overline{x}\}| = 1$ for every $x \in V(G)$. Hence $|D| = |V(G)|$. $\diamondsuit$

Proposition 3. [2]. *An independent set I of a graph G is maximum if and only if*

$$|N_G(J) \cap I| \geq |J|$$

for every independent subset J of $V(G) - I$. $\diamondsuit$

Corollary 1. *Every vertex of a well covered block graph G belongs to at most one exterior block of G.*

Proof. Suppose, to the contrary, that a vertex v belongs to at least two exterior blocks of G, say B_1 and B_2. Let I be a maximal independent set which contains v, and let $v_i \in V(B_i) - C(G)$ for $i = 1, 2$. Then $|N_G(\{v_1, v_2\}) \cap I| = |\{v\}| < |\{v_1, v_2\}|$ and therefore, by Proposition 3, I is not a maximum independent set in G which is impossible in a well covered graph. $\diamondsuit$

Corollary 2. *Let v be a cut vertex in a well covered block graph G. If the set $NC_G(v)$ is not empty, then every two vertices of $NC_G(v)$ are adjacent.*

Proof. Suppose, to the contrary, that two vertices v_1 and v_2 of $NC_G(v)$ are not adjacent. Then they belong to different exterior blocks of G, say B_1 and B_2. Clearly, v belongs to B_1 and B_2 which (according to Corollary 1) is impossible in a well covered graph. $\diamondsuit$

Proposition 4. *If G is a well covered graph and I is an independent set in G, then $G - \overline{N}(I)$ is well covered.*

Proof. Immediate by contradiction. $\diamondsuit$

Now we are prepared to give characterizations of well covered and well dominated block graphs.

Theorem 1. *For a block graph G, the following statements are equivalent:*

(i) $\gamma(G) = \Gamma(G)$;

(ii) $\gamma(G) = \alpha(G)$;

(iii) $i(G) = \alpha(G)$;

(iv) $i(G) = \Gamma(G)$;

(v) *The vertex sets $V(G_1), \ldots, V(G_k)$ of the exterior blocks of G form a disjoint partition of $V(G)$;*

(vi) *The induced subgraph $[NC_G(v)]$ of G is nonempty and complete every cut vertex v of G.*

Proof. The implications (i) $\Rightarrow$ (ii), (ii) $\Rightarrow$ (iii), (i) $\Rightarrow$ (iv), and (iv) $\Rightarrow$ $\Rightarrow$ (iii) follow at once from Proposition 1. We will show the implications (iii) $\Rightarrow$ (v), (v) $\Rightarrow$ (i), (v) $\Rightarrow$ (vi), and (vi) $\Rightarrow$ (v).

(iii) $\Rightarrow$ (v). Suppose that the implication (iii) $\Rightarrow$ (v) is false and let G be a well covered block graph with minimum number of vertices in which the vertex sets $V(G_1), \ldots, V(G_k)$ of the exterior blocks of G do not form a disjoint partition of $V(G)$. According to Corollary 1, the sets $V(G_1), \ldots, V(G_k)$ are mutually disjoint. The choice of G implies that G is connected and its diameter d is greater than three. Let $P = (v_0, v_1, \ldots, v_d)$ be any longest path without triangular chords in G, and let B_i be that block of G which contains the vertices v_{i-1} and v_i of $P (i = 1, \ldots, d)$. From Corollary 1 and the choice of P it follows that the blocks $B_1, \ldots, B_d$ are different, $\{v_1, \ldots, v_{d-1}\} \subseteq C(G)$, B_1 is an exterior block of G, and B_1 and B_2 are the only blocks of G which contain the vertex v_1. In addition, the choice of G makes it obvious that v_1 and v_2 are the only vertices of B_2. Let us consider the connected block graph $H = G - \overline{N}_G(v_0) = G - V(B_1)$. Since H is well covered (by Proposition 4) and has fewer vertices than G, the vertex sets $V(H_1), \ldots, V(H_l)$ of the exterior blocks $H_1, \ldots, H_l$ of H form a disjoint partition of $V(H)$.

We now claim that v_2 is not a cut vertex of H. For if not, then $B_1, H_1, \ldots, H_l$ are the exterior blocks of G and their vertex sets $V(B_1)$, $V(H_1), \ldots, V(H_l)$ form a disjoint partition of $V(G)$, a contradiction. This implies the desired claim. In a similar manner, we find that every vertex of $B_3 - v_2$ is a cut vertex of H. From the above it follows that B_3 is one of the exterior blocks of H, say $B_3 = H_l$, and B_3 is not an exterior block of G. Hence $B_1, H_1, \ldots, H_{l-1}$ are the exterior blocks of G and the sets $V(B_1)$, $V(H_1), \ldots, V(H_{l-1})$ form a disjoint partition of $V(G) - V(B_3)$.

We now show that the graph G has maximal independent sets of different cardinalities. Take exactly one vertex u_i from the set $V(H_i) - C(H)$ $(i = 1, \ldots, l)$. From the properties of the blocks $B_1, H_1, \ldots, H_l$ it follows that $u_l = v_2$ and $I = \{v_0, v_2, u_1, \ldots, u_{l-1}\}$ is a maximal independent set in G. On the other hand, let $V(B_3) - \{v_2\} = \{x_1, \ldots, x_p\}$. Since B_3 is an exterior block of H and each x_i is a cut vertex of H, there exists a nonexterior block F_i of H that contains $x_i (i = 1, \ldots, p)$. Let z_i be any vertex of $F_i - x_i$ and let H_{j_i} be the exterior block of H that contains $z_i (i = 1, \ldots, p)$. Without loss of generality, we may assume that $\{i_1, \ldots, i_p\} = \{1, \ldots, p\}$. It is not hard to observe that the set $I' = \{v_1, z_1, \ldots, z_p, u_{p+1}, \ldots, u_{l-1}\}$ is a maximal independent set in G. (The graph in Figur 1 illustrates these constructions.) Since $|I'| \neq |I|$, G is not a well covered graph, a contradiction. This proves the implication (iii) $\Rightarrow$ (v).

(v) $\Rightarrow$ (i). Assume that (v) holds. Since $V(G_i) - C(G) \neq \emptyset$, we may choose exactly one vertex x_i from the set $V(G_i) - C(G)$ $(i = 1, \ldots, k)$ and form the set $D = \{x_1, \ldots, x_k\}$. (v) implies that D is a dominating set in G. We claim that $\gamma(G) = |D| = k$. Suppose, to the contrary, that there exists a dominating set D_1 in G such that $|D_1| < k$. Then it follows from (v) that $D_1 \cap V(G_{i_0}) = \emptyset$ for some $i_0 \in \{1, \ldots, k\}$, which implies that $x_{i_0} \notin D_1$ and $N_G(x_{i_0}) \cap D_1 = \emptyset$ (since $N_G(x_{i_0}) \subset V(G_{i_0})$), a contradiction. This proves that $\gamma(G) = k$. Similarly, we claim that $\Gamma(G) = |D| = k$. Suppose indirectly that there is a minimal dominating set D_2 in G such that $|D_2| > k$. Then (v) implies that $|D_2 \cap V(G_{j_0})| \geq 2$ for some $j_0 \in \{1, \ldots, k\}$ and, in addition, $D_2 \cap V(G_i) \neq \emptyset$ for each $i \in \{1, \ldots, k\}$. Let v be any vertex of $D_2 \cap V(G_{j_0})$, and let $D_2' = D_2 - \{v\}$. Clearly, D_2' is a dominating set in G and contains one vertex less than D_2 which is impossible since D_2 was a minimal dominating set in G. Therefore $\Gamma(G) = k$. Consequently $\gamma(G) = \Gamma(G)$.

(v) $\Rightarrow$ (vi). Assume that (v) holds and let v be a cut vertex of G. By (v), the vertex v belongs to $V(G_i)$ for some $i \in \{1, \ldots, k\}$. Since the set $V(G_i) - C(G)$ is nonempty and v is adjacent to every vertex of $V(G_i) - C(G)$, the set $NC_G(v)$ is nonempty. Hence, the subgraph $[NC_G(v)]$ is nonempty and complete (by Corollary 2 and the equivalence of (v) and (iii)).

(vi) $\Rightarrow$ (v). Assume that (vi) holds. First let us observe that the sets $V(G_1), \ldots, V(G_k)$ are disjoint. For if not, then there exist $i, j \in \{1, \ldots, k\}$, $i \neq j$, and a vertex v such that $v \in V(G_i) \cap V(G_j)$. Certainly, v is a cut vertex of G and since the sets $NC_G(v) \cap V(G_i)$, $NC_G(v) \cap V(G_j)$ are nonempty, the subgraph $[NC_G(v)]$ is not complete. This contradicts our assumption. Hence, the sets $V(G_1), \ldots, V(G_k)$ are disjoint and it remains to show that $V(G) = \bigcup_{i=1}^{k} V(G_i)$. To prove this it is sufficient to show that $C(G) \subset \bigcup_{i=1}^{k} V(G_i)$, since $\bigcup_{i=1}^{k} V(G_i) \subset V(G)$ and $V(G) - C(G) \subset \bigcup_{i=1}^{k} V(G_i)$ from the definition of the graphs $G_1, \ldots, G_k$. It follows from (vi) that for every $v \in C(G)$, $[NC_G(v)]$ is a subgraph of exactly one of the graphs $G_1, \ldots, G_k$. This implies that every $v \in C(G)$ belongs to exactly one of the graphs $G_1, \ldots, G_k$ and therefore $C(G) \subset \bigcup_{i=1}^{k} V(G_i)$. This proves the implication (vi) $\Rightarrow$ (v) and completes the proof of the theorem. $\diamondsuit$

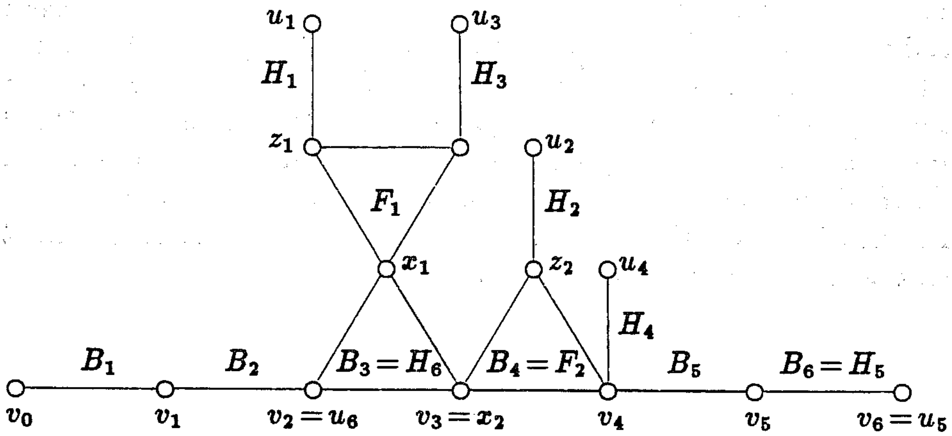

Figure 1

From Theorem 1 we can immediately deduce the following corollary for trees.

Corollary 3. *For a tree T, each of the statements* (i) – (vi) *of Theorem 1 is equivalent to the statement*

(vii) $T = K_1$ *or* $T = R \circ K_1$ *for some tree* R. $\diamondsuit$

Theorem 1 and Proposition 1 imply that for a block graph G, each of the equations $\gamma(G) = \Gamma(G)$, $\gamma(G) = \alpha(G)$, $i(G) = \alpha(G)$, $i(G) = \Gamma(G)$ implies each of the equations $\gamma(G) = i(G)$ and $\alpha(G) = \Gamma(G)$. The converse is not true. This can be seen with the aid of the graph $K_{1,2}$.

The final section of this paper is devoted to characterizations of well dominated and well convered unicyclic graphs. Let us recall that a unicyclic graph is a connected graph with exactly one cycle. Let $\mathcal{U}$ denote the set of all unicyclic graphs. For $G \in \mathcal{U}$, we denote by C_G the unique cycle of G, and by $g(G)$ the length of C_G, i.e., $g(G)$ is the girth of G. Let $\mathcal{KU}$ be the subfamily of $\mathcal{U}$, where $G \in \mathcal{KU}$ if and only if $G = H \circ K_1$ for some $H \in \mathcal{U}$. In what follows, it is helpful to note that a graph G belongs to the set $\mathcal{KU}$ if and only if G is a unicyclic graph and the sets of the family $\{\{v, u\} : vu \in E_e(G)\}$ form a disjoint partition of the set $V(G)$. Similarly we define the subfamilies $\mathcal{S}_3, \mathcal{S}_4$, and $\mathcal{S}_5$ of $\mathcal{U}$. A graph G is in the family of $\mathcal{S}_3$ if G is a unicyclic graph of girth 3 in which the unique cycle C_G has 1 or 2 vertices of degree three or more and the sets of the family $\{V(C_G)\} \cup \{\{v,u\} : vu \in E_e(G)\}$ form a disjoint partition of the set $V(G)$. A graph G is in the family $\mathcal{S}_4$ if G is a unicyclic graph of girth 4, the unique cycle C_G of G contains exactly two adjacent vertices of degree two (in G), say a and b, and the set $\{ab\} \cup E_e(G)$ is a perfect matching of G. Finally, a graph G is in the family $\mathcal{S}_5$ if G is a unicyclic graph of girth 5, the unique cycle C_G of G does not contain two adjacent vertices of degree three or more, and the sets of the family $\{V(C_G)\} \cup \{\{v,u\} : vu \in E_e(G)\}$ form a disjoint partition of the set $V(G)$. (The graphs G_2, G_3, and G_4 in Figure 2 belong to $\mathcal{S}_3$, $\mathcal{S}_4$, and $\mathcal{S}_5$, respectively.)

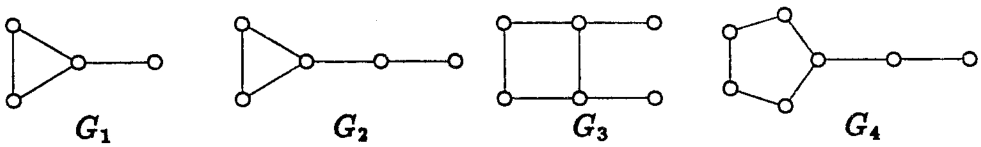

Figure 2

Proposition 5. *For any* $G \in \mathcal{S}_3$, $\gamma(G) = \Gamma(G)$.

Proof. Since every graph G in $\mathcal{S}_3$ is a block graph in which the unique cycle $C_G = C_3$ of G and the subgraphs generated by the end edges of G are the exterior blocks of G and their vertex sets form a disjoint partition of $V(G)$, the result follows from Theorem 1. $\diamondsuit$

Proposition 6. *For any* $G \in \mathcal{S}_4$, $\gamma(G) = |E_e(G)|$ *and* $i(G) = \alpha(G) = = \Gamma(G) = |E_e(G)| + 1$.

Proof. For $G \in \mathcal{S}_4$, let C_G be the unique cycle of G, and let a and b be the adjacent vertices of degree two (in G). Let D be any minimal dominating set in G. It follows from the minimality of D that $|D \cap \cap \{v,u\}| = 1$ for each $vu \in E_e(G)$ and $|D \cap \{a,b\}| \leq 1$. Therefore, since the sets of the family $\{\{a,b\}\} \cup \{\{v,u\} : vu \in E_e(G)\}$ form a disjoint partition of $V(G)$, $|E_e(G)| \leq \gamma(G) \leq |D| \leq \Gamma(G) \leq |E_e(G)| + 1$. From this and from the fact that the sets $N_G(\Omega(G))$ (of cardinality $|E_e(G)|$) and $\Omega(G) \cup \{a\}$ (of cardinality $|E_e(G)| + 1$) are minimal dominating sets in G, we obtain $\gamma(G) = |E_e(G)|$ and $\Gamma(G) = |E_e(G)| + 1$. Similar analysis shows that every maximal independent set of G has exactly $|E_e(G)| + 1$ vertices. Thus, $i(G) = \alpha(G) = |E_e(G)| + 1$. ◇

Proposition 7. *For any* $G \in \mathcal{S}_5$, $\gamma(G) = \Gamma(G) = |E_e(G)| + 2$.

Proof. For $G \in \mathcal{S}_5$, let C_G be the unique cycle of G, and let D be any minimal dominating set in G. We need only observe that $|D| = |E_e(G)| + 2$. Because the sets of the family $\{V(C_G)\} \cup \{\{v, u\} : vu \in E_e(G)\}$ form a disjoint partition of $V(G)$ and D is a minimal dominating set in G, we find $|D| = |D \cap V(C_G)| + \Sigma_{vu \in E_e(G)} |D \cap \{v, u\}| = |D \cap V(C_G)| + |E_e(G)|$. Simple observations show that $|D \cap V(C_G)| = 2$, and so, $|D| = |E_e(G)| + 2$, as required. ◇

Proposition 8. *Let G be a unicyclic graph with $g(G) \geq 5$. Then the following statements are equivalent:*

(i) $\gamma(G) = \Gamma(G)$;
(ii) $\gamma(G) = \alpha(G)$;
(iii) $i(G) = \Gamma(G)$;
(iv) $i(G) = \alpha(G)$;
(v) $G \in \{C_5, C_7\} \cup \mathcal{S}_5 \cup \{H \circ K_1 : H \in \mathcal{U}$ *and* $g(H) \geq 5\}$.

Proof. The implications (i) $\Rightarrow$ (ii), (i) $\Rightarrow$ (iii), (ii) $\Rightarrow$ (iv) and (iii) $\Rightarrow$ (iv) immediately follow from Proposition 1. The implication (v) $\Rightarrow$ (i) is obvious if $G \in \{C_5, C_7\}$ and follows from Propositions 7 and 2 if $G \in \mathcal{S}_5 \cup \{H \circ K_1 : H \in \mathcal{U}$ with $g(H) \geq 5\}$. Finally, it is a simple matter to obtain the implication (iv) $\Rightarrow$ (v) from [8, Corollary 4] (see also [7]). ◇

Theorem 2. *For a unicyclic graph G, the following statements are equivalent:*

(i) $\gamma(G) = \Gamma(G)$;
(ii) $\gamma(G) = \alpha(G)$;
(iii) $G \in \{C_3, C_4, C_5, C_7\} \cup \mathcal{K}\mathcal{U} \cup \mathcal{S}_3 \cup \mathcal{S}_5$.

Proof. The implication (i) $\Rightarrow$ (ii) follows from Proposition 1. The equivalence of (ii) and (iii) has been proved in [15]. By Propositions 2, 5 and 7, the implication (iii) $\Rightarrow$ (i) is true for every graph $G \in \mathcal{KU} \cup \mathcal{S}_3 \cup \mathcal{S}_5$. Finally, it is straightforward to verify that the cycles C_3, C_4, C_5 and C_7 are well dominated. $\diamondsuit$

As a consequence of Theorem 2 and Proposition 1 we see that for a unicyclic graph G, each of the equations $\gamma(G) = i(G)$, $i(G) = \alpha(G)$, $i(G) = \Gamma(G)$, $\alpha(G) = \Gamma(G)$ follow from each of the equations $\gamma(G) = \Gamma G)$ and $\gamma(G) = \alpha(G)$. The graphs G_1 and G_3 (shown in Figure 2) prove that the converse is not necessarily true.

The next theorem presents necessary and sufficient conditions for a unicyclic graph to be well covered. The proof is based on the following proposition.

Proposition 9 [12]. *A bipartite graph G without isolated vertices is well covered if and only if G has a perfect matching M and, for every edge $vu \in M$, the subgraph induced by the set $N_G(\{v,u\})$ is a complete bipartite graph.* $\diamondsuit$

Theorem 3. *For a unicyclic graph G, the following statements are equivalent:*

(i) $i(G) = \Gamma(G)$;
(ii) $i(G) = \alpha(G)$;
(iii) $G \in \{C_3, C_4, C_5, C_7\} \cup \mathcal{KU} \cup \mathcal{S}_3 \cup \mathcal{S}_4 \cup \mathcal{S}_5$.

Proof. The implications (i) $\Rightarrow$ (ii) and (iii) $\Rightarrow$ (i) easily follow from Theorem 2 and Propositions 1 and 6. Thus it remains to prove the implication (ii) $\Rightarrow$ (iii).

Assume that G is a unicyclic graph and $i(G) = \alpha(G)$. Let C_G be the unique cycle of G, and let $E_e(G)$ be the set of end edges of G. We split the proof into three parts, based on the girth $g(G)$ of G.

Case 1. If $g(G) \geq 5$, then $G \in \{C_5, C_7\} \cup \mathcal{S}_5 \cup \mathcal{KU}$ (by Proposition 8).

Case 2. Assume that $g(G) = 4$. Then G is bipartite and therefore by Proposition 9, G has a perfect matching M such that for every edge $vu \in M$, the subgraph induced by $N_G(\{v,u\})$ is a complete bipartite graph. We will show that either $G = C_4$ of $G \in \mathcal{KU} \cup \mathcal{S}_4$. In order to prove this, let us assume that $G \neq C_4$. It is clear that $E_e(G) \subseteq M$ and, in addition, $G \in \mathcal{KU}$ if (and only if) $M = E_e(G)$. Thus assume

that $M \neq E_e(G)$. We claim that $M \subseteq E(C_G) \cup E_e(G)$. For if not, then $M - (E(C_G) \cup E_e(G)) \neq \emptyset$ and for any edge $vu \in M - (E(C_G) \cup E_e(G))$, the sets $N_G(v) - \{u\}$ and $N_G(u) - \{v\}$ are not empty and no vertex of $N_G(v) - \{u\}$ is adjacent to a vertex of $N_G(u) - \{v\}$, so $[N_G(\{v,u\})]$ is not a complete bipartite graph, a contradiction. We therefore henceforth suppose that $M \subseteq E(C_G) \cup E_e(G)$ and $M \cap E(C_G) \neq \emptyset$. Certainly, $|M \cap E(C_G)| = 1$; otherwise $|M \cap E(C_G)| = 2$, say $M \cap E(C_G) = \{xy, wz\}$, and then, since $G \neq C_4$, at least one of the subgraphs $[N_G(\{x,y\})]$ and $[N_G(\{w,z\})]$ is not a complete bipartite graph, a contradiction. Let vu be the only edge of $M \cap E(C_G)$. Then $M = \{vu\} \cup E_e(G)$ and, moreover, $|N_G(v)| = |N_G(u)| = 2$; otherwise $|N_G(v)| \geq 3$ or $|N_G(u)| \geq 3$ and then $[N_G(\{v,u\})]$ would not be a complete bipartite graph, a contradiction. This implies that $G \in \mathcal{S}_4$.
Case 3. If $g(G) = 3$, then G is a well covered block graph. We will show that either $G = C_3$ or $G \in \mathcal{KU} \cup \mathcal{S}_3$. Assume that $G \neq C_3$, and let $G_1, \ldots, G_k$ be the exterior blocks of G. By Theorem 1, the vertex sets $V(G_1), \ldots, V(G_k)$ form a disjoint partition of $V(G)$. If C_G is one of the blocks $G_1, \ldots, G_k$, say $C_G = G_1$, then $\{V(G_1), \ldots, V(G_k)\} = \{V(C_G)\} \cup \{\{v,u\} : vu \in E_e(G)\}$ and $G \in \mathcal{S}_3$. If C_G is not an exterior block of G, then $\{V(G_1), \ldots, V(G_k)\} = \{\{v,u\} : vu \in E_e(G)\}$ and $G \in \mathcal{KU}$. This proves the implication (ii) $\Rightarrow$ (iii) and completes the proof of the theorem. $\Diamond$

In conclusion, let us note the according to Theorems 2 and 3, the well covered unicyclic graphs which are not well dominated are precisely those which belong to the family $\mathcal{S}_4$.

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SIMULTANEOUS EXTENSIONS OF PROXIMITIES, SEMI-UNIFORMITIES, CONTIGUITIES AND MEROTOPIES I*

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Abstract: Given compatible proximities (in the sense of Čech) on some subspaces of a closure space, we are looking for a common compatible extension of these proximities. In Part II, proximities will be replaced by semi-uniformities, contiguities or merotopies. In Parts III and IV, we shall consider similar extension problems in proximity, semi-uniform and contiguity spaces.

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We are going to investigate problems of the following type: Let X be a set, σ a topological structure (e.g. a closure) on $X, \{X_i : i \in I\}$ a system of subsets of X; assume that a richer structure (e.g. a proximity) Σ_i is given on each X_i; we aim at finding a common *extension* of these structures, i.e. a structure Σ compatible with σ such that $\Sigma|X_i = \Sigma_i$ $(i \in I)$, where $\Sigma|X_i$ denotes the restriction of Σ to X_i. Two natural necessary conditions for the existence of such an extension: (i) Σ_i has to be compatible with $\sigma|X_i$, and (ii) $\Sigma_i|X_i \cap X_j = \Sigma_j|X_i \cap X_j$ $(i, j \in I)$ [assuming, of course, that for arbitrary structures σ and Σ on X, and for $B \subset A \subset X$, (i) $\Sigma|A$ is compatible with $\sigma|A$ whenever Σ is compatible with σ, and (ii) $\Sigma|B = (\Sigma|A)|B$; these conditions will be evidently satisfied in each particular case we are going to consider].

See [13] for a survey of the classical extension problem when $|I| = 1$.

§0 contains all the necessary definitions and notation (including those needed only in Parts II to IV). §1 deals with the case when σ is a closure and Σ a proximity.

In Part II, σ will be again a closure, and Σ a semi-uniformity, a contiguity or a merotopy.

In Part III, σ will be a proximity, and Σ a contiguity or a merotopy.

The following cases will be investigated in Part IV: a) σ is a proximity, Σ a semi-uniformity, b) σ is a semi-uniformity or a contiguity, Σ a merotopy.

Each of the above mentioned questions will be considered in three variants: a) without separation axioms; b) for Riesz-type structures; c) for Lodato-type structures.

These problems clearly have category theoretical aspects, which will not be investigated here. It would be interesting to find out the category theoretical reasons for the similarity of some results, and for the dissimilarity of others, cf. [13] Problem 72.

0. Preliminaries

All the unproved statements in this section are either well-known or trivial (usually both).

0.1 Closures. A *closure* [2] on X is a function $c : \exp X \to \exp X$ such that, for $A, B \subset X$,

C1. $c(\emptyset) = \emptyset$,
C2. $A \subset c(A)$,
C3. $A \subset B$ implies $c(A) \subset c(B)$,
C4. $c(A \cup B) \subset c(A) \cup c(B)$.

If, in additon, $c(c(A)) = c(A)$ for every $A \subset X$ then c is a *topology*.

The closure c is said to be *symmetric* [27] (semi-uniformizable in [2]) if $y \in c(\{x\})$ implies $x \in c(\{y\})$ for $x, y \in X$; it is *separated* [7] (semi-separated in [2], D_1 in [27]) if $c(\{x\}) = \{x\}$ for $x \in X$, and *weakly separated* [8] if $x \notin c(A)$ implies $c(\{x\}) \cap c(A) = \emptyset$. A symmetric closure is weakly separated iff $x \in c(A)$ implies $c(\{x\}) \subset c(A)$; this condition is Axiom H_2 in [27]. Separated implies weakly separated, which in turn implies symmetric. A topology is separated iff it is T_1, and weakly separated iff it is symmetric iff it is S_1 in the sense of [6] (better known as R_0, but we shall use the term *S_1-topology*).

If c is a closure on X , and $x \in X$ then a *c-neighbourhood* [2] of x is a set $V \subset X$ such that $x \notin c(X \setminus V)$; the c-neighbourhoods of x constitute the *c-neighbourhood filter* of x; a *c-neighbourhood (sub)base* of x is a (sub)base for the c-neighbourhood filter of x. (Occasionally, when there is no danger of confusion, the letter c will be dropped from these names; the same convention applies to other notions depending on some structure.) For $A \subset X$, $\mathrm{int}_c A$ denotes the set of all $x \in X$ such that A is a neighbourhood of x. $\mathrm{int}\, A = X \setminus c(X \setminus A)$.

If c and c' are closures on X then c is said to be *coarser* than c' (c' *finer* than c) if $c'(A) \subset c(A)$ for $A \subset X$.

For $X_0 \subset X$, the *restriction* to X_0 of the closure c, denoted by $c|X_0$, is defined by $c_0(A) = c(A) \cap X_0$ $(A \subset X_0)$, where $c_0 = c|X_0$; c_0 is a closure on X_0, symmetric, (weakly) separated or topological if c is so. If c' is finer than c then $c'|X_0$ is finer than $c|X_0$.

Denoting the c-neighbourhood filter of $x \in X$ by $\mathsf{v}(x)$, we say that $\mathsf{s}_0(x) = \mathsf{v}(x)|X_0$ is the *trace filter* (on X_0) of the point x, where, for $\mathsf{s} \subset \exp X$,

$$\mathsf{s}|X_0 = \{S \cap X_0 : S \in \mathsf{s}\},$$

called the *trace* (on X_0) of s. For $x \in X_0$, $\mathsf{s}_0(X)$ coincides with the c_0-neighbourhood filter of x, while $\mathsf{s}_0(x) = \exp X_0$ (the *zero filter* on

X_0) whenever $x \notin c(X_0)$. This means that, in general, only the trace filters of the points in $c(X_0) \setminus X_0$ will be of interest.

0.2 Proximities. A *proximity* [2] (called basic proximity or Čech proximity when the shorter term is reserved for proximities in the sense of Efremovich) on X is a relation $\delta \subset \exp X \times \exp X$ such that, for $A, B, C, A', B' \subset X$

P1. $A\delta B$ implies $B\delta A$,
P2. $A\delta X$ implies $A \neq \emptyset$,
P3. $A \cap B \neq \emptyset$ implies $A\delta B$,
P4. $A\delta B$, $A \subset A'$, $B \subset B'$ imply $A'\delta B'$,
P5. $(A \cup B)\delta C$ implies that either $A\delta C$ or $B\delta C$.

We write $\bar{\delta}$ for non-δ. Parantheses will often be omitted, e.g.: $A \cup B\delta C$.

The relation β is a *base* for the relation δ (this is in fact a sub-base-like notion) provided that

$$A\bar{\delta}B \text{ iff there are } n, m \in \mathrm{N} \text{ and sets } A_i, B_j \subset X$$
$$(1 \leq i \leq n,\ 1 \leq j \leq m) \text{ such that } A_i\bar{\beta}B_j$$
$$\text{for each } i \text{ and } j,\ A = \bigcup_1^n A_i,\ B = \bigcup_1^m B_j.$$

(N denotes the set of the positive integers.) Clearly, $\delta \subset \beta$. If β is a base for δ, and β satisfies Axioms P1 to P4 then δ is a proximity; any proximity is a base for itself.

A proximity δ *induces* a symmetric closure $c = c(\delta)$ defined by

$$x \in c(A) \text{ iff } \{x\}\delta A.$$

The proximity δ is said to be *Riesz* [26] (SP″ in [7], weakly Lodato in [8]) if, with $c = c(\delta)$

PRi. $A\bar{\delta}B$ implies $c(A) \cap c(B) = \emptyset$,

and *Lodato* [25] (P_s-relation in [23]) if

PLo. $A\bar{\delta}B$ implies $c(A)\bar{\delta}c(B)$.

PLo implies PRi. δ is Riesz or Lodato iff there is a base β for δ such that $A\bar{\beta}B$ implies $c(A) \cap c(B) = \emptyset$, respectively $c(A)\bar{\beta}c(B)$ [$c(A)\bar{\delta}c(B)$]. If δ is Riesz (Lodato) then $c(\delta)$ is weakly separated (it is an S_1-topology).

For proximities δ and δ' on X, δ is said to be *coarser* than δ' (δ' *finer* than δ) if $\delta \supset \delta'$. If β is a base for δ, β' for δ', and $\beta \supset \beta'$ then $\delta \supset \delta'$; in particular, if β is a base for δ, δ' is a proximity, and $\bar{\beta} \subset \bar{\delta}'$ then δ is coarser than δ'. The finest proximity on X is called *discrete* ($A\delta B$ iff $A \cap B \neq \emptyset$); the coarsest one is called *indiscrete* ($A\delta B$ iff $A \neq \emptyset \neq B$). A finer proximity induces a finer closure.

If $X_0 \subset X$, the *restriction* $\beta_0 = \beta|X_0$ of the relation β is defined for $A, B \subset X_0$ by $A\beta_0 B$ iff $A\beta B$. If β is a base for δ then $\beta|X_0$ is a base for $\delta|X_0$. The restriction of a (Riesz/Lodato) proximity is again a (Riesz/Lodato) proximity. For a proximity δ, $\delta|X_0$ induces $c(\delta)|X_0$. The restriction of a finer proximity is finer.

[If $\beta_0 = \beta|X_0$, we write $\bar{\beta}_0$ for non-β_0 in X_0; this notation cannot be misunderstood if our attention is restricted to relations β satisfying axioms P1 to P4 (or just P2 and $X\beta X$ if $X \neq \emptyset$), because then β, as well as $\bar{\beta}$, determines the fundamental set: it is $\bigcup \operatorname{dom}\beta = \bigcup \operatorname{dom}\bar{\beta}$].

A filter $\mathfrak{s}$ on X is said to be *δ-compressed* [6,7] (or: $\mathfrak{s}$ is a compressed filter in the proximity space (X, δ)) if $A, B \subset X$, $A, B \in \sec \mathfrak{s}$ imply $A\delta B$, where

$$\sec \mathfrak{s} = \sec_X \mathfrak{s} = \{A \subset X : \; A \cap S \neq \emptyset \; (S \in \mathfrak{s})\}.$$

The zero filter is compressed. A proximity δ is Riesz iff each $c(\delta)$-neighbourhood filter is δ-compressed. If $\mathfrak{s}$ is δ-compressed then $\mathfrak{s}|X_0$ is $\delta|X_0$-compressed.

0.3 Semi-uniformities. A *semi-uniformity* [2] on X is a filter $\mathcal{U}$ on $X \times X$ such that

U1. each $U \in \mathcal{U}$ is an *entourage*, i.e. $\Delta \subset U$,
U2. $U^{-1} \in \mathcal{U}$ for $U \in \mathcal{U}$,

where $\Delta = \Delta_X$ is the diagonal of X, and U^{-1} is the inverse of U:

$$\Delta_X = \{(x, x) : x \in X\}, \quad U^{-1} = \{(x, y) : yUx\},$$

and xUy means $(x, y) \in U$. For $x \in X$ and $A \subset X$ we write

$$U[A] = \{y : \exists x \in A, xUy\}, \quad Ux = U[\{x\}].$$

A *(sub)base* for a semi-uniformity is to be understood as a filter (sub)base on $X \times X$. The symmetric entourages contained by the semi-uniformity $\mathcal{U}$ form a base for $\mathcal{U}$. Any non-empty collection $\mathcal{S}$ of entourages is a subbase for a semi-uniformity, provided that for each $U \in \mathcal{U}$, U^{-1} contains some $V \in \mathcal{S}$; in particular, any non-empty collection of symmetric entourages is a subbase for some semi-uniformity.

A semi-uniformity $\mathcal{U}$ *induces* a proximity $\delta = \delta(\mathcal{U})$ defined by

$$(1) \qquad A\delta B \text{ iff } (A \times B) \cap U \neq \emptyset \quad (U \in \mathcal{U});$$

equivalently:

$$(2) \qquad A\bar{\delta}B \text{ iff } U[A] \cap B = \emptyset \text{ for some } U \in \mathcal{U}.$$

Hence $\mathcal{U}$ *induces* a closure $c(\mathcal{U}) = c(\delta(\mathcal{U}))$. $\{Ux : U \in \mathcal{U}\}$ is the $c(\mathcal{U})$-neighbourhood filter of $x \in X$. In (1) and (2), $\mathcal{U}$ can be replaced by any base for $\mathcal{U}$. If $\mathcal{S}$ is a (sub)base for $\mathcal{U}$ then $\{Ux : U \in \mathcal{S}\}$ is a (sub)base for $\mathsf{v}(x)$ in $c(\mathcal{U})$.

The semi-uniformity $\mathcal{U}$ is said to be *Riesz* if

URi. $U \in \mathcal{U}$ implies $\Delta \subset \operatorname{int}_{c\times c} U$,

where the $(c \times c)$-neighbourhood filter of $(x, y) \in X \times X$ is generated by the filter base

$$\{G \times H : G \in \mathsf{v}(x), \quad H \in \mathsf{v}(y)\},$$

and $c = c(\mathcal{U})$. $\mathcal{U}$ is said to be *Lodato* if

ULo. $U \in \mathcal{U}$ implies $\operatorname{int}_{c\times c} U \in \mathcal{U}$.

$\mathcal{U}$ is Riesz (Lodato) iff URi (ULo) holds with $\mathcal{U}$ replaced by a subbase; $\mathcal{U}$ is Lodato iff it has a (sub)base consisting of open entourages. (A set A is *c-open* if $A = \operatorname{int} A$; an *open entourage* is meant to be $(c(\mathcal{U}) \times c(\mathcal{U}))$-open.) ULo implies URi. If $\mathcal{U}$ is Riesz (Lodato) then so is $\delta(\mathcal{U})$. URi and ULo fit naturally between the corresponding axioms for proximities and merotopies, so they are probably known; nevertheless, we are unable to cite a source.

For two semi-uniformities $\mathcal{U}$ and $\mathcal{U}'$ on X, $\mathcal{U}$ is said to be *coarser* than $\mathcal{U}'$ ($\mathcal{U}'$ *finer* than $\mathcal{U}$) if $\mathcal{U} \subset \mathcal{U}'$; in this case $\delta(\mathcal{U})$ is coarser than $\delta(\mathcal{U}')$.

For $X_0 \subset X$, the *restriction* $\mathcal{U}|X_0$ of the semi-uniformity $\mathcal{U}$ to X_0 is defined by

$$\mathcal{U}|X_0 = \{U|X_0 : U \in \mathcal{U}\}, \quad U|X_0 = U \cap (X_0 \times X_0).$$

$\mathcal{U}|X_0$ is a semi-uniformity on X_0 satisfying $\delta(\mathcal{U}|X_0) = \delta(\mathcal{U})|X_0$. If $\mathcal{U}$ is Riesz or Lodato then so is $\mathcal{U}|X_0$. The restriction of a finer semi-uniformity is finer.

A filter s on X is *$\mathcal{U}$-Cauchy* if $U \in \mathcal{U}$ implies $S \times S \subset U$ for some $S \in \mathsf{s}$. ($\mathcal{U}$ can be replaced by a subbase in this definition.) If s is $\mathcal{U}$-Cauchy then it is $\delta(\mathcal{U})$-compressed, and $\mathsf{s}|X_0$ is $\mathcal{U}|X_0$-Cauchy for $X_0 \subset X$. $\mathcal{U}$ is Riesz iff every $c(\mathcal{U})$-neighbourhood filter is Cauchy.

0.4 Merotopies. A *merotopy* [21] (quasi-uniformity in [19], Čech nearness in [24]) on X is a non-empty collection M of covers of X such that

M1. if $\mathsf{c} \in \mathsf{M}$ and c refines d then $\mathsf{d} \in \mathsf{M}$,
M2. if $\mathsf{c}, \mathsf{d} \in \mathsf{M}$ then $\mathsf{c}(\cap)\mathsf{d} \in \mathsf{M}$,

where

$$\mathsf{c}(\cap)\mathsf{d} = \{C \cap D : C \in \mathsf{c},\ D \in \mathsf{d}\}.$$

($\{\emptyset\}$ is a cover of $X = \emptyset$; $\emptyset$ is not a cover of it. c *refines* d, or c is a *refinement* of d, if for any $C \in \mathsf{c}$ there is a $D \in \mathsf{d}$ with $C \subset D$.) M2 can be replaced by

M2$'$. any two elements of M have a common refinement in M.

A subset B of a merotopy M is a *base* for M if every element of M has a refinement in B; B satisfies Axiom M2$'$. Conversely, any non-empty collection B of covers that satisfies M2$'$ is a base for exactly one merotopy M; a cover c belongs to M iff it has a refinement in B.

For a finite non-empty family F of covers, we define $(\cap)\mathsf{F}$ as follows:

$$A \in (\cap)\mathsf{F} \ \text{ iff } \ \exists A(\mathsf{c}) \in \mathsf{c} \ \ (\mathsf{c} \in \mathsf{F}), \ \ A = \cap\{A(\mathsf{c}) : \mathsf{c} \in \mathsf{F}\}.$$

(If $\mathsf{F} = \{\mathsf{c}, \mathsf{d}\}$ and $\mathsf{c} \neq \mathsf{d}$ then $(\cap)\mathsf{F} = \mathsf{c}(\cap)\mathsf{d}$.) A subset S of a merotopy M is a *subbase* for M if

$$\{(\cap)\mathsf{F} : \emptyset \neq \mathsf{F} \subset \mathsf{S},\ \mathsf{F} \text{ is finite}\}$$

is a base for M. Any non-empty collection of covers of X is a subbase for exactly one merotopy on X.

A merotopy M *induces* a semi-uniformity $\mathcal{U}(\mathsf{M})$, for which a base $\mathcal{B}$ (the one consisting of all the symmetric elements of $\mathcal{U}(\mathsf{M})$) is defined by

$$\mathcal{B} = \{U(\mathsf{c}) : \mathsf{c} \in \mathsf{M}\}, \quad U(\mathsf{c}) = \bigcup \{C \times C : C \in \mathsf{c}\}.$$

(Taking c from a (sub)base only, we obtain a (sub)base for $\mathcal{U}(\mathsf{M})$.) Hence M induces a proximity $\delta(\mathsf{M}) = \delta(\mathcal{U}(\mathsf{M}))$ and a closure $c(\mathsf{M}) = c(\delta(\mathsf{M}))$. For $\delta = \delta(\mathsf{M})$,

$$A \delta B \text{ iff } \operatorname{St}(A, \mathsf{c}) \cap B \neq \emptyset \quad (\mathsf{c} \in \mathsf{M}), \tag{1}$$

where

$$\operatorname{St}(A, \mathsf{c}) = \bigcup \{C \in \mathsf{c} : A \cap C \neq \emptyset\}.$$

$\{\operatorname{St}(x, \mathsf{c}) : \mathsf{c} \in \mathsf{M}\}$ is the $c(\mathsf{M})$ neighbourhood filter of x, where $\operatorname{St}(x, \mathsf{c}) = \operatorname{St}(\{x\}, \mathsf{c})$. M can be replaced by a base in (1). If S is a (sub)base for M then $\{\operatorname{St}(x, \mathsf{c}) : \mathsf{c} \in \mathsf{S}\}$ is a (sub)base for $\mathsf{v}(x)$ in $c(\mathsf{M})$.

A merotopy M on X is said to be *Riesz* (Riesz nearness in [3]) if

MRi. for each $\mathsf{c} \in \mathsf{M}$, $\operatorname{int} \mathsf{c}$ is a cover of X,

where

$$\operatorname{int} \mathsf{c} = \operatorname{int}_c \mathsf{c} = \{\operatorname{int}_c C : C \in \mathsf{c}\},$$

and $c = c(\mathsf{M})$. M is said to be *Lodato* (nearness in [16], Lodato nearness in [24]) if

MLo. $\mathsf{c} \in \mathsf{M}$ implies $\operatorname{int} \mathsf{c} \in \mathsf{M}$.

MLo implies MRi. M is Riesz (Lodato) iff MRi (MLo) holds for some subbase for M; M is Lodato iff it has a subbase consisting of $c(\mathsf{M})$-open covers. If M is Riesz (Lodato) then so is $\mathcal{U}(\mathsf{M})$.

For two merotopies M and M′ on X, M is said to be *coarser* than M′ (M′ *finer* than M) if $\mathsf{M} \subset \mathsf{M}'$. If S is a subbase for M and $\mathsf{S} \subset \mathsf{M}'$ then M is coarser than M′. $\{\{X\}\}$ is a base for the *indiscrete* (coarsest) merotopy on X; the *discrete* (finest) merotopy on X consists of all the covers of X. A finer merotopy induces a finer semi-uniformity.

For $X_0 \subset X$, the *restriction* $\mathsf{M}|X_0$ of the merotopy M to X_0 is defined by

$$\mathsf{M}|X_0 = \{\mathsf{c}|X_0 : \mathsf{c} \in \mathsf{M}\}. \tag{2}$$

$\mathsf{M}|X_0$ is a merotopy on X_0 satisfying $\mathcal{U}(\mathsf{M}|X_0) = \mathcal{U}(\mathsf{M})|X_0$. If M is replaced by a (sub)base then (2) yields a (sub)base for $\mathsf{M}|X_0$. If M is Riesz or Lodato then so is $\mathsf{M}|X_0$. The restriction of a finer merotopy is finer.

A filter s on X is M-*Cauchy* [19] if $\mathsf{s} \cap \mathsf{c} \neq \emptyset$ for $\mathsf{c} \in \mathsf{M}$ (equivalently: for $c \in \mathsf{S}$, where S is a subbase for M). M-Cauchy filters are $\mathcal{U}(M)$-Cauchy as well. If s is M-Cauchy then $\mathsf{s}|X_0$ is $\mathsf{M}|X_0$-Cauchy. M is Riesz iff every $c(\mathsf{M})$-neighbourhood filter is M-Cauchy.

0.5 Contiguities. A *contiguity* (essentially [20,17]) on X is a non-empty collection Γ of finite covers of X such that

Co1. if $\mathsf{c} \in \Gamma$, c refines d, and d is finite then $\mathsf{d} \in \Gamma$,
Co2. if $\mathsf{c}, \mathsf{d} \in \Gamma$ then $\mathsf{c}(\cap)\mathsf{d} \in \Gamma$.

Base and *subbase* for a contiguity, *Riesz* and *Lodato* contiguities, *finer* and *coarser* contiguities, the *restriction* $\Gamma|X_0$ of a contiguity, and Γ-*Cauchy* filters are defined in the same way as for merotopies. ("Contiguity" means a Lodato contiguity in [16].) The proximity $\delta = \delta(\Gamma)$ *induced* by Γ is defined by 0.4 (1) (with Γ substituted for M); $c(\Gamma) = c(\delta(\Gamma))$ is the closure *induced* by Γ. (It is superfluous to define $\mathcal{U}(\Gamma)$ in the same way as $\mathcal{U}(\mathsf{M})$, because $\mathcal{U}(\Gamma)$ is then uniquely determined by $\delta(\Gamma)$.) The analogues of all the statements for merotopies listed in 0.4 are valid for contiguities, too. In addition, if S is a subbase for Γ then

$$A\beta B \text{ iff } \mathrm{St}(A,\mathsf{c}) \cap B \neq \emptyset \quad (c \in \mathsf{S})$$

defines a base β for $\delta(\Gamma)$.

For a merotopy M, the contiguity $\Gamma(\mathsf{M})$ *induced* by M consists of the finite elements of M. If M is Riesz or Lodato then so is $\Gamma(\mathsf{M})$. $\delta(\Gamma(\mathsf{M})) = \delta(\mathsf{M})$ and $\Gamma(\mathsf{M})|X_0 = \Gamma(\mathsf{M}|X_0)$. A finer merotopy induces a finer contiguity. Any M-Cauchy filter is $\Gamma(\mathsf{M})$-Cauchy.

Contiguities as well as semi-uniformities are structures lying between merotopies and proximities. Neither of the structures $\Gamma(\mathsf{M})$ and $\mathcal{U}(\mathsf{M})$ determines the other, and they together do not determine M.

0.6 Conventions. A *family of proximities* in the closure space (X, c) is a system $\{\delta_i : i \in I\}$, where I is a (possibly empty) set of indices, such that δ_i is a proximity on some $X_i \subset X$, $X \neq \emptyset$, $X_i \neq \emptyset$ $(i \in I)$, and the two conditions mentioned in the introduction are fullfilled, i.e.

(1) $$c(\delta_i) = c|X_i \qquad (i \in I),$$

(2) $$\delta_i|X_i \cap X_j = \delta_j|X_i \cap X_j \qquad (i, j \in I).$$

Where these conditions have to be referred to, we shall say that the family of proximities is (the proximities δ_i are) (1) *compatible* and (2) *accordant.* When speaking about a *family of proximities in a closure space*, it will be understood that the closure space is denoted by (X, c), and I, δ_i and X_i are used as above; moreover, $c_i = c|X_i$, $\mathrm{int} = \mathrm{int}_c$, $\mathrm{int}_i = \mathrm{int}_{c_i}$, $\mathrm{Int} = \mathrm{int}_{c\times c}$, $\mathrm{Int}_i = \mathrm{int}_{c_i\times c_i}$, $X_{ij} = X_i \cap X_j$, $\mathsf{v}(x)$ is the c-neighbourhood filter of $x \in X$, $\mathsf{s}_i(x)$ the c-trace filter of $x \in X$ on X_i. The expression *"the trace filters are compressed"* means that $\mathsf{s}_i(x)$ is δ_i-compressed for each $x \in X$ and each $i \in I$. An *extension* of $\{\delta_i : i \in I\}$ (or of the proximities δ_i) is a proximity δ on X such that $c = c(\delta)$ and $\delta_i = \delta|X_i$ $(i \in I)$. In case the proximities δ_i have an extension, we shall also say that they can be extended.

Analogous terminology, notations and conventions will be used for other kinds of structures, with $\mathcal{U}$ and $\mathcal{U}_i$ standing for semi-uniformities, Γ and Γ_i for contiguities, M and M_i for merotopies; "compressed" will be replaced by "Cauchy". If the structure given on X is not a closure then c denotes the closure induced by it, and the notations derived from c (int, $\mathsf{s}_i(x)$, etc.) will be used as above.

1. Extending a family of proximities in a closure space

A. WITHOUT SEPARATION AXIOMS

1.1 If a family of proximities can be extended in a closure space then the closure clearly has to be symmetric. We are going to show that this condition is sufficient, too. In fact, we construct the finest and the coarsest extension.

Definition. Given a family of proximities in a closure space, define $\delta^1 \subset \exp X \times \exp X$ as follows. $A\,\delta^1 B$ iff one of the following conditions holds:

$$A \cap c(B) \neq \emptyset, \tag{1}$$

$$c(A) \cap B \neq \emptyset, \tag{2}$$

$$A \cap X_i \delta_i B \cap X_i \text{ for some} \tag{3}$$

In case confusion seems to be possible, we write $\delta^1(c, \delta_i)$, or, more precisely, $\delta^1(c, \{\delta_i : i \in I\})$; in particular, $\delta^1(c) = \delta^1(c, \emptyset)$. ◇

Theorem. *A family of proximities in a symmetric closure space always has extensions; δ^1 is the finest one.*

Proof. It is easy to see that δ^1 is a proximity on X.

1° *$\delta^1|X_i$ is coarser than δ_i.* If $A\delta_i B$ then (3) holds, and therefore $A\delta^1 B$.

2° *$\delta^1|X_i$ is finer than δ_i.* Assume $A(\delta^1|X_i)B$; this means that $A\delta^1 B$ and $A, B \subset X_i$. If (1) holds then $A \cap c_i(B) \neq \emptyset$, so there is a point $x \in A \cap c_i(B)$; hence $\{x\}\delta_i B$ by the compatibility of δ_i, thus $A\delta_i B$. The case of (2) is analogous. Finally, if (3) holds, i.e. if $A \cap X_j \delta_j B \cap X_j$ for some j then $A \cap X_j$, $B \cap X_j \subset X_{ij}$ implies $A \cap X_j \delta_i B \cap X_j$ by the accordance, so $A\delta_i B$ again.

3° *$c(\delta^1)$ is coarser than c.* If $x \in c(B)$ then (1) is satisfied with $A = \{x\}$, so $\{x\}\delta^1 B$.

4° *$c(\delta^1)$ is finer than c.* Assume $x \notin c(B)$; we have to show that none of the conditions (1) to (3) holds with $A = \{x\}$. $\{x\} \cap c(B) = \emptyset$ is evident. For $y \in B$, $x \notin c(\{y\})$, thus we have $y \notin c(\{x\})$ from the symmetry of c, and so $c(\{x\}) \cap B = \emptyset$. Finally,

$$\{x\} \cap X_i \bar{\delta}_i B \cap X_i \quad (i \in I),$$

because the left hand side is empty if $x \notin X_i$, and, for $x \in X_i$, $x \notin c(B)$ implies $x \notin c(B \cap X_i) \cap X_i = c_i(B \cap X_i)$, thus $\{x\}\bar{\delta}_i B \cap X_i$.

5° *δ^1 is the finest extension.* Let δ be another extension; we have to show that $\delta^1 \subset \delta$. Assume $A\delta^1 B$. If (1) holds then $x \in c(B)$ for some $x \in A$, thus $\{x\}\delta B$ and $A\delta B$; similarly, (2) implies $A\delta B$. Finally, from (3) we have $A \cap X_i \delta B \cap X_i$ (since $\delta|X_i = \delta_i$), hence $A\delta B$ again. ◇

1.2 Our next aim is to construct the coarsest extension; its definition will be a little bit more complicated than that of the finest one.

Definition. For a family of proximities in a closure space, let β be a base for $\delta^0 \subset \exp X \times \exp X$, where $A\bar{\beta}B$ iff one of the following conditions holds:

$$|A| \leq 1 \text{ and } A \cap c(B) = \emptyset, \tag{1}$$

$$|B| \leq 1 \text{ and } c(A) \cap B = \emptyset, \tag{2}$$

$$A\bar{\delta}_i B \text{ for some } i \in I. \tag{3}$$

The notations $\delta^0(c, \delta_i)$, etc. will be used as in Definition 1.1 (and similar conventions will apply to all subsequent definitions). $\diamondsuit$

Theorem. *A family of proximities in a symmetric closure space always has a coarsest extension, namely* δ^0.

Proof. β clearly satisfies Axioms P1 to P4, so δ^0 is a proximity on X.

1° *$\delta^0|X_i$ is finer than δ_i.* If $A\bar{\delta}_i B$ then (3) holds, so $A\bar{\beta}B$ and $A\bar{\delta}^0 B$.

2° *$\delta^0|X_i$ is coarser than δ_i.* $\beta|X_i$ is a base for $\delta^0|X_i$, so it is enough to show that $\bar{\beta}|X_i \subset \bar{\delta}_i$. Assume that $A\bar{\beta}B$ and $A, B \subset X_i$. If (1) holds and $A \neq \emptyset$ then $A = \{x\}$ for some $x \in X_i$; now $x \notin c(B)$, so $x \notin c_i(B)$, implying $\{x\}\bar{\delta}_i B$, i.e. $A\bar{\delta}_i B$. The case of (2) is analogous. Finally, if $A\bar{\delta}_j B$ for some $j \in I$ then $A, B \subset X_{ij}$, so $A\bar{\delta}_i B$ follows from the accordance.

3° *$c(\delta^0)$ is finer than c.* If $x \notin c(B)$ then (1) holds with $A = \{x\}$, thus $\{x\}\bar{\beta}B$ and $\{x\}\bar{\delta}^0 B$.

4° *$c(\delta^0)$ is coarser than c.* Assume $\{x\}\bar{\delta}^0 B$. Then B can be written as a finite union $\bigcup_n B_n$ such that $\{x\}\bar{\beta}B_n$ for each n; it is now enough to show that

$$x \notin c(B_n), \tag{4}$$

because then $x \notin c(B)$ by Axiom C4. If (1) holds (with $A = \{x\}$ and $B = B_n$) then (4) is evident. If (2) holds and $B_n \neq \emptyset$ then $B_n = \{y\}$, and the symmetry of c implies $x \notin c(\{y\})$, which is the same as (4). Finally, if $\{x\}\bar{\delta}_i B_n$ for some i then $x \notin c_i(B_n)$, so $x \in X_i$ and $B_n \subset X_i$ imply (4) again.

5° δ^0 *is the coarsest extension.* Let δ be another extension; it is enough to show that $\bar{\beta} \subset \bar{\delta}$. Assume $A\bar{\beta}B$. If (1) holds and $A \neq \emptyset$ then $A = \{x\}$ for some $x \in X$, and $x \notin c(B)$, implying $A\bar{\delta}B$, which follows in the same way from (2), too. Finally, if (3) holds then $A\bar{\delta}B$ again, because $\delta_i = \delta|X_i$. ◇

Part 5° of the above proof uses only one half of the assumption that δ is an extension: δ^0 is the coarsest one among those proximities δ that induce a closure finer than c, and for which $\delta|X_i$ is finer than δ_i ($i \in I$). Similarly, δ^1 is the finest one among those proximities δ that induce a closure coarser than c, and for which $\delta|X_i$ is coarser than δ_i ($i \in I$), see 5° in the proof of Theorem 1.1. These observations are of some interest when compared with the results of §1C.

1.3 Recall that the proximities on a fixed set form a complete lattice with respect to the relation finer/coarser, and the infimum and the supremum of the proximities $\delta[i]$ on X ($i \in I \neq \emptyset$) can be described as follows: $\inf_i \delta[i] = \bigcup_i \delta[i]$, while $\bigcup_i \delta[i]$ is a base for $\sup_i \delta[i]$, (see e.g. [2] 38 A.1 and 38 A.5, where the infimum is called supremum, and vice versa). Infima and suprema of proximities commute with the restriction to a subset (evident) as well as with taking the induced closure ([2] 38 B.3); constructions of infima and suprema of closures are not needed here, see them e.g. in [2] 31 A.2 and 31 ex. 2.

For $i \in I$ fixed, let us write $\delta^0[i]$ for $\delta^0(c, \{\delta_i\})$, and denote by $\delta^{00}[i]$ the coarsest proximity δ on X (not necessarily compatible with c) for which $\delta|X_i = \delta_i$; this means that $A\bar{\delta}^{00}[i]B$ iff either $A\bar{\delta}_iB$ or $A = \emptyset$ or $B = \emptyset$. Now we have, for $I \neq \emptyset$,

$$(1) \qquad \delta^0 = \sup_i \delta_0[i] = \sup\{\delta^0(c), \sup_i \delta^{00}[i]\}.$$

This could be checked looking at the constructions, but in fact it is enough to know for the proof of (1) that proximities figuring in it do exist: Denote by δ' the proximity in the middle of (1), and by δ'' the one on the right hand side of it. $\delta^0 \subset \delta^0[i] \subset \delta^{00}[i]$ is evident, and so is $\delta^0[i] \subset \delta^0(c)$, therefore $\delta^0 \subset \delta' \subset \delta''$. Moreover, $c(\delta'')$ is finer than c, and $\delta''|X_i$ is finer than δ_i, because $\delta^{00}[i]|X_i = \delta_i$; hence $\delta'' \subset \delta^0$ by the remark at the end of 1.2 (and the construction of δ^0 is not really needed in that remark either: $\delta \cup \delta^0$ is an extension, so, δ^0 being the coarsest extension, we have $\delta \subset \delta \cup \delta^0 \subset \delta^0$).

Similarly, if $\delta^1[i] = \delta^1(c, \{\delta_i\})$, and $\delta^{11}[i]$ denotes the finest proximity δ on X (not necessarily compatible with c) for which $\delta|X_i = \delta_i$ ($A\delta^{11}[i]B$ iff $A \cap X_i \delta_i B \cap X_i$ or $A \cap B \neq \emptyset$) then, for $I \neq \emptyset$,

$$(2) \qquad \delta^1 = \inf_i \delta^1[i] = \inf \{\delta^1(c), \inf_i \delta^{11}[i]\}.$$

B. RIESZ PROXIMITIES IN A CLOSURE SPACE

1.4 If a family of proximities in a closure space has a Riesz extension then each proximity is Riesz, the closure is weakly separated, and the trace filters are compressed (because the neighbourhood filters have to be compressed with respect to the extension). We are going to show that these conditions are sufficient, too; there are again a finest and a coarsest extension.

Definition. For a family of Riesz proximities in a weakly separated closure space, let $\delta^1_R \subset \exp X \times \exp X$ be defined as follows: $A\delta^1_R B$ iff either

$$(1) \qquad c(A) \cap c(B) \neq \emptyset$$

or

$$(2) \qquad A \cap X_i \delta_i B \cap X_i \text{ for some } i. \quad \diamond$$

Lemma. *Given a family of Riesz proximities in a weakly separated closure space, δ^1_R is a compatible Riesz proximity on X; it is the finest one among those Riesz proximities δ that induce a closure coarser than c, and for which $\delta|X_i$ is coarser than δ_i ($i \in I$).*

Proof. δ^1_R is clearly a proximity on X.

1° *$\delta^1_R|X_i$ is coarser than δ_i.* If $A\delta_i B$ then (2) holds, implying $A\delta^1_R B$.

2° *$c(\delta^1_R)$ is coarser than c.* If $x \in c(B)$ then $c(\{x\}) \cap c(B) \neq \emptyset$, so $\{x\}\delta^1_R B$ by (1).

3° *$c(\delta^1_R)$ is finer than c.* Assume $x \notin c(B)$; we have to show that $\{x\}\bar{\delta}^1_R B$, i.e. that neither (1) nor (2) holds with $A = \{x\}$. $c(\{x\}) \cap$ $\cap c(B) = \emptyset$, since c is weakly separated. $\{x\} \cap X_i \bar{\delta}_i B \cap X_i$ follows as in 4° of the proof of Theorem 1.1.

4° δ_R^1 *is Riesz.* If $A\bar{\delta}_R^1 B$ then (1) does not hold, and we have already seen that $c = c(\delta_R^1)$.

5° δ_R^1 *is finest.* Let δ be another Riesz proximity with $\delta_i \subset \delta|X_i$ $(i \in I)$ and $c(\delta)$ coarser than c; we have to show that $\delta_R^1 \subset \delta$. Assume $A\delta_R^1 B$. If (1) holds then $c'(A) \cap c'(B) \neq \emptyset$ where $c' = c(\delta)$; now $A\delta B$, because δ is Riesz. If (2) holds then $A \cap X_i \delta B \cap X_i$, and so $A\delta B$ again. ◇

Theorem. *A family of Riesz proximities in a weakly separated closure space has a Riesz extension iff the trace filters are compressed; if so then δ_R^1 is the finest Riesz extension.*

Proof. In view of the lemma, it is enough to show that if the trace filters are compressed then $\delta_R^1|X_i$ is finer than δ_i $(i \in I)$. Assume $A\delta_R^1 B$, $A, B \subset X_i$. If (1) holds then, picking $x \in c(A) \cap c(B)$, we have $A, B \in \sec \mathsf{s}_i(x)$, hence $A\delta_i B$, because $\mathsf{s}_i(x)$ is δ_i-compressed. On the other hand, if $A \cap X_j \delta_j B \cap X_j$ for some j then $A\delta_i B$ by the accordance, just like in 2° of the proof of Theorem 1.1. ◇

If $\{\text{int } X_i : i \in I\}$ covers X then it is not necessary to assume that the trace filters are compressed. Indeed, if $A, B \subset X_i$, $A, B \in \sec \mathsf{v}(x)$, $x \in \text{int } X_j$ then $X_j \in \mathsf{v}(x)$, so $A \cap X_j$, $B \cap X_j \in \sec \mathsf{v}(x)$, implying $A \cap X_j \delta_j B \cap X_j$ (since δ_j is Riesz); hence $A\delta_i B$ by the accordance.

Corollary. *A family of Riesz proximities in a weakly separated closure space has a Riesz extension iff*

$$\delta_i \supset \delta_R^1(c)|X_i \qquad (i \in I). \tag{3}$$

Proof. The necessity is obvious. Conversely, if (3) holds then each $\mathsf{s}_i(x)$ is δ_i-compressed, because it is compressed with respect to the finer proximity $\delta_R^1(c)|X_i$; thus the theorem applies. ◇

1.5 Lemma. *If δ' and δ'' are proximities such that $c(\delta') = c(\delta'')$, δ' is Riesz, and δ'' is coarser than δ' then δ'' is Riesz, too.* ◇

Theorem. *Under the hypotheses of Theorem 1.4, δ^0 is the coarsest Riesz extension.*

Proof. Theorems 1.2 and 1.4, and the above lemma. ◇

1.6 Assume that the conditions of Theorem 1.4 are satisfied. Similarly to 1.3 (2),

$$\delta_R^1 = \inf_i \delta_R^1[i] = \inf \{\delta_R^1(c), \inf_i \delta^{11}[i]\}, \tag{1}$$

where $\delta_R^1[i] = \delta_R^1(c, \{\delta_i\})$. Just like the other proximities in (1), $\delta^{11}[i]$ is Riesz, since, with c_i' standing for $c(\delta^{11}[i])$, we have $c_i'(A) = A \cup c_i(A \cap \cap X_i)$. Concerning 1.3 (1), let us observe that $\delta^{00}[i]$ cannot be replaced by the "coarsest Riesz proximity δ on X for which $\delta|X_i = \delta_i$", because such a proximity may not exist: let $|X| = 3$, $|X_0| = 2$, and δ_0 be the discrete proximity on X_0.

1.7 Observe that $A\bar{\delta}^0(c)B$ iff either A is finite and $A \cap c(B) = \emptyset$ or B is finite and $c(A) \cap B = \emptyset$. The next lemma will be needed in §1C.

Lemma. *If c is and S_1-topology then $\delta_R^1(c)$ is Lodato; if c is a T_1-topology then $\delta^0(c)$ is Lodato as well.*

Proof. The first statement is evident. To prove the second one, assume that c is a T_1-topology, and $A\bar{\delta}^0(c)B$. Then, say, A is finite and $A\cap \cap c(B) = \emptyset$; hence $c(A) = A$ is finite, $c(c(B)) = c(B)$, so $c(A)\ \bar{\delta}^0(c)c(B)$. ◇

C. LODATO PROXIMITIES IN A CLOSURE SPACE

1.8 If a family of proximities in a closure space has a Lodato extension then each proximity is Lodato, the closure is an S_1-topology, and the trace filters are compressed (because a Lodato proximity is Riesz). Somewhat suprisingly, these conditions are not sufficient:

Example. Let $X = \mathrm{R}^2$, c be the Euclidean topology on X, $X_0 = \mathrm{R} \times \times \{0\}$, $X_1 = X \setminus X_0$, δ_0 the Euclidean proximity on X_0, and $\delta_1 = = \delta_R^1(c)|X_1$. Now c is an S_1-topology, δ_i is a Lodato proximity compatible with $c_i (i = 0, 1)$, for $i = 1$ by Lemma 1.7. Moreover, the trace filters are compressed, since the Euclidean proximity on X is a Lodato extension of δ_0, while $\delta_R^1(c)$ is a Lodato extension of δ_1.

Assume that the family $\{\delta_0, \delta_1\}$ has a Lodato extension δ. With $\mathrm{N}' = \{n + 2^{-n} : n \in \mathrm{N}\}$, consider $A = \mathbb{N} \times (\mathbb{R} \setminus \{0\})$ and $B = \mathrm{N}' \times (\mathbb{R}\setminus$

$\setminus\{0\})$. Now $c(A) = \mathbb{N} \times \mathbb{R}$, $c(B) = \mathbb{N}' \times \mathbb{R}$, hence $A\bar{\delta}_1 B$, and so $A\bar{\delta}B$. On the other hand, $c(A) \cap X_0 \delta_0 c(B) \cap X_0$ so that $c(A) \cap X_0 \delta c(B) \cap X_0$ and $c(A)\delta c(B)$, a contradiction. $\diamondsuit$

1.9 Definition. For a family of Lodato proximities in an S_1-space, let $\delta_L^1 \subset \exp X \times \exp X$ be defined as follows: $A\delta_L^1 B$ iff either

$$(1) \qquad c(A) \cap c(B) \neq \emptyset$$

or

$$(2) \qquad c(A) \cap X_i \delta_i c(B) \cap X_i \text{ for some } i. \ \diamondsuit$$

Lemma. *For a family of Lodato proximities in an S_1-space, δ_L^1 is a compatible Lodato proximity; it is the finest one among those Lodato proximities δ on X that induce a closure coarser than c, and for which $\delta|X_i$ is coarser than δ_i $(i \in I)$.*

Proof. It is easy to see that δ_L^1 is a proximity on X.

1^o *$\delta_L^1|X_i$ is coarser than δ_i.* If $A\delta_i B$ then (2) holds, and so $A\delta_L^1 B$.

2^o *$c(\delta_L^1)$ is coarser than c.* Just like in the proof of Lemma 1.4.

3^o *$c(\delta_L^1)$ is finer than c.* Assume $x \notin c(B)$; we have to show that neither (1) nor (2) holds with $A = \{x\}$. $c(\{x\}) \cap c(B) = \emptyset$ because c is S_1.

$$(3) \qquad c(\{x\}) \cap X_i \bar{\delta}_i c(B) \cap X_i$$

is evident if the left hand side is empty. Otherwise, one can take $y \in$ $\in c(\{x\}) \cap X_i$; now $c(\{x\}) = c(\{y\})$ (since c is S_1), thus (3) is equivalent to

$$(4) \qquad c_i(\{y\}) \bar{\delta}_i c(B) \cap X_i.$$

$x \notin c(B)$ implies $y \notin c(B)$ (again by S_1), therefore $y \notin c(B) \cap X_i =$ $= c_i(c(B) \cap X_i)$, i.e. $\{y\}\bar{\delta}_i c(B) \cap X_i$, and so (4) holds indeed (as δ_i is Lodato).

4^o *δ_L^1 is Lodato.* This is clear from $c = c(\delta_L^1)$, since (1) and (2) depend only on $c(A)$ and $c(B)$, and c is a topology.

5^o *δ_L^1 is finest.* Let δ be another Lodato proximity with $\delta_i \subset$ $\subset \delta|X_i$ $(i \in I)$ and $c(\delta)$ coarser than c; we have to show that $\delta_L^1 \subset \delta$. Assume $A\delta_L^1 B$. (1) implies $c'(A) \cap c'(B) \neq \emptyset$ where $c' = c(\delta)$, thus

(δ being Lodato) we have $A\delta B$. On the other hand, if (2) holds then $c(A) \cap X_i \delta c(B) \cap X_i$, so $c(A)\delta c(B)$, implying $A\delta B$ again. ◊

1.10 Definition. For a family of Lodato proximities in an S_1-space, let β be a base for $\delta_L^0 \subset \exp X \times \exp X$, where $A\bar{\beta}B$ iff one of the following conditions holds:

$$(1) \qquad A \subset c(\{x\}) \text{ for some } x \notin c(B), \text{ or } A = \emptyset,$$

$$(2) \qquad B \subset c(\{x\}) \text{ for some } x \notin c(A), \text{ or } B = \emptyset,$$

$$(3) \qquad \text{there are } i, A', B' \text{ with } A'\bar{\delta}_i B', A \subset c(A'), B \subset c(B'). \diamond$$

Lemma. *If a family of Lodato proximities is given in an S_1-space, and the trace filers are compressed then δ_L^0 is the coarsest one among those compatible Lodato proximities δ on X for which $\delta|X_i$ is finer than δ_i ($i \in I$).*

Proof. 1^o *δ_L^0 is a proximity.* β clearly satisfies Axioms P1, P2 and P4. To prove P3, assume $A\bar{\beta}B$. If (1) or (2) holds then $A \cap B = \emptyset$ follows from S_1. If (3) holds then $c(A') \cap c(B') = \emptyset$, because the trace filters are compressed; hence $A \cap B = \emptyset$ again, i.e. β fullfills P1 to P4. Consequently, δ_L^0 is a proximity indeed.

2^o *$\delta_L^0|X_i$ is finer than δ_i.* If $A\bar{\delta}_i B$ then (3) holds with $A' = A$ and $B' = B$, so $A\bar{\beta}B$ and $A\bar{\delta}_L^0 B$.

3^o *$c(\delta_L^0)$ is finer than c.* If $x \notin c(B)$ then (1) holds with $A = \{x\}$, thus $\{x\}\bar{\delta}_L^0 B$.

4^o *$c(\delta_L^0)$ is coarser than c.* Just as in 4^o of the proof of Theorem 1.2, it is enough to show that $\{y\}\bar{\beta}B$ implies

$$(4) \qquad y \notin c(B).$$

If (1) holds (with $A = \{y\}$) then $x \notin c(B)$ and S_1 imply (4). If (2) holds and $B \neq \emptyset$ then from $x \notin c(\{y\})$ and S_1 we have $y \notin c\{x\})$, which implies (4), since $c(\{x\}) = c(B)$ by S_1. Finally, if (3) holds then $y \in c(A')$, $B \subset c(B')$ and $A'\bar{\delta}_i B'$, thus $c(A') \cap c(B') = \emptyset$ (because the trace filters are compressed), and $y \notin c(B') = c(c(B')) \supset c(B)$.

5^o *δ_L^0 is Lodato.* If $A\bar{\beta}B$ then $c(A)\bar{\beta}c(B)$ follows directly from the definition (taking into account that c is a topology). Now δ_L^0 is Lodato, since we have already seen that $c = c(\delta_L^0)$.

6^o *δ_L^0 is coarsest.* Let δ be another compatible Lodato proximity with $\delta|X_i \subset \delta_i$ ($i \in I$); it is enough to show that $\bar{\beta} \subset \bar{\delta}$. If (1) holds then either $A = \emptyset$, in which case $A\bar{\delta}B$ is evident, or $\{x\}\bar{\delta}B$ (since δ is compatible), hence $c(\{x\})\bar{\delta}B$ (since δ is Lodato), and so $A\bar{\delta}B$. The case of (2) is analogous. Finally, if (3) holds then $A'\bar{\delta}B'$, therefore $c(A')\bar{\delta}c(B')$, and $A\bar{\delta}B$ again. ◇

It is not true that δ^0_L is the coarsest one among those Lodato proximities δ that induce a closure finer than c, and for which $\delta|X_i$ is finer than δ_i $(i \in I)$, not even when $I = \emptyset$:

Example. Let (X, c) be the topological sum of two infinite indiscrete spaces, and c' the discrete closure on X. Now c' is finer than c, but $\delta^0_L(c') = \delta^0(c')$ is not finer than $\delta^0_L(c)$, since there are infinite sets A and B with $A\bar{\delta}^0_L(c)B$, while $A\delta^0_L(c')B$ for any pair of inifinite sets. ◇

1.11 Lemma. *A family of Lodato proximities in an S_1-space has a Lodato extension iff $\delta^1_L \subset \delta^0_L$; if so then both δ^0_L and δ^1_L are Lodato extensions.*

Proof. 1° *Necessity.* If δ is a Lodato extension then $\delta^1_L \subset \delta \subset \delta^0_L$ by Lemmas 1.9 and 1.10 (the latter can be applied since the existence of δ implies that the trace filters are compressed).

2° *Sufficiency.* If $A\bar{\delta}_iB$ for some i then $c(A)\bar{\delta}^0_Lc(B)$ by 1.10 (3), so $c(A)\bar{\delta}^1_Lc(B)$, implying $c(A) \cap c(B) = \emptyset$ (because δ^1_L is a proximity by Lemma 1.9); this means that the trace filters are compressed and so Lemma 1.10 applies as well as Lemma 1.9. Consequently, δ^0_L and δ^1_L are compatible Lodato proximities, $\delta^0_L|X_i \subset \delta_i \subset \delta^1_L|X_i$, and from $\delta^1_L \subset \delta^0_L$ we have also $\delta^1_L|X_i \subset \delta^0_L|X_i$. Hence both δ^0_L and δ^1_L are extensions. ◇

Theorem. *A family of Lodato proximities in an S_1-space has a Lodato extension iff the trace filters are compressed, and, for any $i, j \in I$,*

$$(1) \qquad A\bar{\delta}_iB \Rightarrow c(A) \cap X_j\bar{\delta}_jc(B) \cap X_j;$$

if so then δ^0_L is the coarsest and δ^1_L is the finest Lodato extension.

Remark: Observe that (1) is a strengthening of the accordance.

Proof. 1° *Necessity.* If δ is a Lodato extension then $A\bar{\delta}_iB$ implies $A\bar{\delta}B$, hence $c(A)\bar{\delta}c(B)$ and $c(A) \cap X_j\bar{\delta}c(B) \cap X_j$, thus the right hand side of (1) holds.

2° *Sufficiency.* In consequence of Lemma 1.9, it is enough to prove that $\delta^1_L|X_i$ is finer than δ_i $(i \in I)$. Assume that $A\delta^1_LB$ and $A, B \subset X_i$.

If 1.9 (1) holds then $A\delta_i B$, because the trace filters are compressed. On the other hand, if 1.9 (2) holds, i.e. if $c(A)\cap X_j\delta_j c(B)\cap X_j$ for some j then we have $A\delta_i B$ from (1).

3° δ_L^0 and δ_L^1 are Lodato extensions by the foregoing lemma; they are coarsest, respectively finest by Lemmas 1.10 and 1.9. ◇

Corollary. *A family of proximities in an S_1-space has a Lodato extension iff $\{\delta_i,\delta_j\}$ has a Lodato extension for any $i,j\in I$.* ◇

1.12 Corollary. *A single Lodato proximity given in an S_1-space has a Lodato extension iff the trace filters are compressed.*

Proof. 1.11 (1) is always satisfied for $i=j$, because $c(S)\cap X_i=c_i(S)$ $(S\subset X_i)$, and δ_i is Lodato. ◇

1.13 Theorem. *Let a family of Lodato proximities be given in an S_1-space, assume that the trace filters are compressed, and*

$$(1)\qquad c(X_i\setminus X_j)\cap(X_j\setminus X_i)=\emptyset\qquad (i,j\in I).$$

Then there exists a Lodato extension.

Proof. We have to show that 1.11 (1) holds. Assume $A\bar{\delta}_i B$; it is enough to consider the following three cases because then Axioms C4 and P5 can be applied:

a) $A,B\subset X_i\setminus X_j$;
b) $A,B\subset X_{ij}$;
c) $A\subset X_i\setminus X_j,\ B\subset X_{ij}$.

Case a). From (1) we have $c(A)\cap X_j\subset X_{ij}$ and $c(B)\cap X_j\subset X_{ij}$, so, by the accordance, it is enough to prove that $c(A)\cap X_j\bar{\delta}_i c(B)\cap X_j$, which is true, because $c(A)\cap X_j\subset c(A)\cap X_i=c_i(A)$, similarly, $c(B)\cap$ $\cap X_j\subset c_i(B)$, and δ_i is Lodato.

Case b). The accordance implies $A\bar{\delta}_j B$, so the right hand side of 1.11 (1) holds again, now because δ_j is Lodato.

Case c). As in Case a), $c(A)\cap X_j\subset c_i(A)$, so $c(A)\cap X_j\bar{\delta}_i B$ (because δ_i is Lodato). The accordance implies $c(A)\cap X_j\bar{\delta}_j B$, therefore $c(A)\cap X_j\bar{\delta}_j c_j(B)$ (because δ_j is Lodato); $c_j(B)=c(B)\cap X_j$ completes the proof. ◇

Corollary. *Let a family of Lodato proximities be given in an S_1-space. Assume that either each X_i is open and the trace filters are compressed or each X_i is closed. Then there exists a Lodato extension.*

Proof. $c(X_i \backslash X_j) \cap (X_j \backslash X_i)$ does not change if c is replaced by $c|X_i \cup X_j$; $X_i \backslash X_j$ and $X_j \backslash X_i$ are disjoint closed (or open) sets in $(X_i \cup X_j, c|X_i \cup \cup X_j)$, thus (1) holds. ◇

If the sets X_i form an open cover of X then we do not have to assume that the trace filters are compressed, see after Theorem 1.4.

1.14 Assume that a non-empty family of Lodato proximities is given in an S_1-space. Similarly to 1.3 (2) and 1.6 (1), we have

(1) $$\delta_L^1 = \inf_i{}_L \delta_L^1[i] = \inf_L\{\delta_L^1(c), \inf_i{}_L \delta^{11}[i]\},$$

where $\delta_L^1[i] = \delta_L^1(c, \{\delta_i\})$, and $\inf_L$ denotes the infimum in the realm of the Lodato proximities (recall that the Lodato proximities on X form a complete lattice, see e.g. [7] (5.1); observe that $\delta^{11}[i]$ is Lodato). The proof is the same as that of 1.3 (1) and 1.3 (2). The proximity in the middle of (1) can be written as $\inf_i \delta_L^1[i]$, because the infimum of Lodato proximities inducing the same closure is Lodato, too. However, the right hand side of (1) cannot be replaced by $\inf\{\delta_L^1(c), \inf_i \delta^{11}[i]\}$:

Example. Let X, c, X_0, δ_0, A and B be as in Example 1.8, $I = \{0\}$. Then $A\bar{\delta}_L^1(c)B$, $A\bar{\delta}^{11}[0]B$, but $A\delta_L^1 B$. ◇

If, in addition, the trace filters are compressed then

(2) $$\delta_L^0 = \sup_i \delta_L^0[i].$$

(The supremum of Lodato proximities is always Lodato, see e.g. [7] (5.1).) An analogue of the right hand side of 1.3 (1) cannot be added to (2), because, in general, there is no coarsest Lodato proximity δ on X for which $\delta|X_i = \delta_i$ (see the example at the end of 1.6).

1.15 If the conditions of Theorem 1.11 are satisfied then we have the following five extensions:

(1) $$\delta^0 \supset \delta_L^0 \supset \delta_L^1 \supset \delta_R^1 \supset \delta^1.$$

If $I = \emptyset$ then $\delta^1_R = \delta^1_L$, and, assuming also that c is T_1, $\delta^0 = \delta^0_L$ (Lemma 1.7). If c is T_1, and each X_i is closed then $\delta^0 = \delta^0_L$ (look at the definitions); similarly, if each X_i is open then $\delta^1_L = \delta^1_R$. This observation yields an alternative proof of Corollary 1.13 (only in T_1-spaces if the subsets are closed, but then we can get rid of T_1 using a stock argument: switch over to the T_0-reflexion of (X, c), take an extension there, and carry it back to (X, c)).

All the proximities in (1) can be, however, different if $|I| = 1$, even when c is T_1:

Examples. a) In Example 1.14, $A\delta^1_L B$, but $A\bar{\delta}^1_R B$.

b) Let X, c, X_1, δ_1, A and B be as in Example 1.8, $I = \{1\}$. Then $c(A) \setminus X_1 \delta^0 c(B) \setminus X_1$, but $c(A) \setminus X_1 \bar{\delta}^0_L c(B) \setminus X_1$. $\diamond$

1.16 Concerning extensions of a single *Efremovich proximity*, see [22], [15] 3.25, [9] §4, [1], [10] §2, [12] 2.2., [14]. We know only the following about simultaneously extending Efremovich proximities:

a) If $\{\delta_1, \delta_2\}$ is a family of Efremovich proximities in a topological space, $X = X_1 \cup X_2$, either X_1 and X_2 are both open and the trace filters are round, or X_1 and X_2 are both closed then $\{\delta_1, \delta_2\}$ has an Efremovich extension; this follows from [13] Remark 1.13 c). (A filter $\mathfrak{s}$ in the proximity space (X, c) is *round* [22] if for any $S \in \mathfrak{s}$ there is an $S_0 \in \mathfrak{s}$ with $S_0 \bar{\delta} X \setminus S$.)

b) The above statement is false for three proximities, even if the subspaces are open-closed. (Essentially [13] Example 1.13b).)

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ON THE INVARIANT MEASURE FOR JACOBI-PERRON ALGORITHM

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Abstract: Since the days of Gauss it has been known that continued fraction algorithm admits an invariant measure. Its density may be written in the form $\rho(x) = \int (1 + x\xi)^{-2} d\xi$. The aim of this paper is to give an explicit expression for the density of 2-dimensional Jacobi-Perron algorithm. The result is given as $\rho(x, y) = \int\int (1 + f(\xi, \eta)x + g(\xi, \eta)y)^{-3} d\kappa(\xi, \eta)$ where the functions f and g are given by a limiting processs and κ is a singular measure.

0. Introduction

The Jacobi-Perron algorithm was introduced by C. G. Jacobi 1868 and later generalized by O. Perron 1907. The main point was the attempt to extend Lagrange's theorem to algebraic numbers of higher degree, namely to characterize algebraic numbers by the periodicity of the algorithm. In spite of several efforts the problem still waits for its solution (see L. Bernstein 1971, Bouhamza 1984).

The ergodic theory for the Jacobi-Perron algorithm was developped along the lines already known for continued fractions (Schweiger 1973). For continued fractions the density of the (up to a constant factor) unique invariant measure which is equivalent to Lebesgue measure has been known implicitly since the days of Gauss:

$$\rho(x) = \frac{1}{1+x}. \tag{0.1}$$

Let

$$Tx = \frac{1}{x} - [\frac{1}{x}] \tag{0.2}$$

be the map associated with continued fractions then for any measurable set

$$\int_{T^{-1}E} \rho(x)dx = \int_E \rho(x)dx.$$

The associated transfer operator is given by

$$(\mathcal{A}\psi)(x) = \sum_{k=1}^{\infty} \psi\left(\frac{1}{k+x}\right)\frac{1}{(k+x)^2}. \tag{0.3}$$

The invariant density ρ then is characterized by the property $\mathcal{A}\rho = \rho$.

For the Jacobi-Perron algorithm it can be shown that there exists a finite invariant measure μ which is equivalent to Lebesgue measure but no explicit expression comparable with (0.1) has been known. The paper aims to give an explicit expression which is however more complicated. Our approach will explain why this is to be expected. In order to illustrate the basic ideas more clearly we restrict the discussion to the case $n = 2$ but the arguments are valid in the general case.

1. A heuristic approach

We first consider continued fractions (see Khintchine 1963). Given a sequence $(k_1, k_2, \ldots, k_n)$ of digits we define p_n and q_n by

$$\begin{pmatrix} p_{n-1} & p_n \\ q_{n-1} & q_n \end{pmatrix} = \prod_{j=1}^{n} \begin{pmatrix} 0 & 1 \\ 1 & k_j \end{pmatrix}.$$

Then

$$(\mathcal{A}^n 1)(x) = \sum_{(n)} (q_n + q_{n-1}x)^{-2} =$$

$$= \sum_{(n)} q_n^{-2} \left(1 + \frac{q_{n-1}}{q_n} x\right)^{-2}.$$

Here the sum runs over all admissible sequences $(k_1, k_2, \ldots, k_n)$. Next we define a seqence of measures (ν_n), $n \geq 1$, by

$$\nu_n(\psi) := \sum_{(n)} q_n^{-2} \psi \left(\frac{q_{n-1}}{q_n}\right).$$

If $\nu = \lim_{n\to\infty} \nu_n$ exists, then

$$\rho(x) := \int_0^1 \frac{d\nu(z)}{(1+zx)^2}$$

should be the density of an invariant measure.
It is well known that

$$\frac{q_{n-1}}{q_n} = [k_n, k_{n-1}, \ldots, k_1].$$

We introduce $K_j := k_{n+1-j}$, $1 \leq j \leq n$, and we define sequences (P_n), (Q_n), $n = 1, 2, \ldots$ by

$$\begin{pmatrix} P_{n-1} & P_n \\ Q_{n-1} & Q_n \end{pmatrix} = \prod_{j=1}^{n} \begin{pmatrix} 0 & 1 \\ 1 & K_j \end{pmatrix}.$$

Then $q_{n-1} = P_n$ and $q_n = Q_n$. We define

$$f_n := \frac{P_n}{Q_n} = \frac{q_{n-1}}{q_n}$$

and for $\xi = [K_1, K_2, \ldots]$

$$f_n(\xi) := f_n$$

in an obvious manner. Then $\lim_{n\to\infty} f_n(\xi) = \xi$. The measure ν_n may be written as

$$\nu_n(\psi) = \sum_{(n)} Q_n^{-2} \psi(f_n).$$

Since $f_n \in \mathcal{B}(K_1, \ldots, K_n)$, the cylinder determined by the digits $K_1, \ldots, K_n$, it looks like a Riemann sum.
In fact one can prove

$$d\nu = \frac{d\lambda}{\log\ 2}$$

where λ denotes Lebesgue measure.
Now we consider the Jacobi-Perron algorithm for $n = 2$. The associated map is given by

$$T(x, y) = \left(\frac{y}{x} - a_1, \frac{1}{x} - b_1 \right),$$

$$a_1 = a_1(x, y) := \left[\frac{y}{x} \right], \qquad b_1 = b_1(x, y) := \left[\frac{1}{x} \right].$$

If $a_j(x, y) := a_1(T^{j-1}(x, y))$, $b_j(x, y) := b_1(T^{j-1}(x, y))$ then this sequence of digits is subject to the following conditions:

$$(1.1) \qquad 1 \leq b_j, \qquad 0 \leq a_j \leq b_j;$$

$$(1.2) \qquad \text{if} \quad a_j = b_j, \quad \text{then} \quad 1 \leq a_{j+1}.$$

Similarly one defines p_n, r_n, q_n by

$$(1.3) \qquad \begin{pmatrix} p_{n-2} & p_{n-1} & p_n \\ r_{n-2} & r_{n-1} & r_n \\ q_{n-2} & q_{n-1} & q_n \end{pmatrix} := \prod_{j=1}^{n} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & a_j \\ 0 & 1 & b_j \end{pmatrix}.$$

Then

$$\lim_{n\to\infty}\left(\frac{p_n}{q_n},\frac{r_n}{q_n}\right)=(x,y).$$

We put

$$\mathcal{B}_1 := \{(x,y) : 0 \le x < y \le 1\},$$
$$\mathcal{B}_2 := \{(x,y) : 0 \le y \le x \le 1\}.$$

Then the transfer operator $\mathcal{A}$ is given by

$$(1.4)\qquad \begin{aligned}(\mathcal{A}\psi)(x,y) &= \sum_{b=1}^{\infty}\sum_{a=0}^{b}\psi\left(\frac{1}{b+y},\frac{a+x}{b+y}\right)\frac{1}{(b+y)^3} \quad \text{on } \mathcal{B}_1\\ (\mathcal{A}\psi)(x,y) &= \sum_{b=1}^{\infty}\sum_{a=0}^{b-1}\psi\left(\frac{1}{b+y},\frac{a+x}{b+y}\right)\frac{1}{(b+y)^3} \quad \text{on } \mathcal{B}_2.\end{aligned}$$

Therefore

$$(1.5)\qquad \begin{aligned}(\mathcal{A}^n 1)(x,y) &= \sum_{(n)}(q_n+q_{n-1}y+q_{n-2}x)^{-3} = \\ &= \sum_{(n)} q_n^{-3}\left(1+\frac{q_{n-1}}{q_n}y+\frac{q_{n-2}}{q_n}x\right)^{-3}.\end{aligned}$$

Here the sum runs over all admissible sequences (a_j,b_j), $1 \le j \le n$ and depends on (x,y). Again $\rho(x,y)$ is the density of an invariant measure if and only if ρ satisfies Kuzmin's equation $\mathcal{A}\rho=\rho$. We define a sequence of measure (ν_n), $n \ge 1$ by

$$(1.6)\qquad \nu_n(\psi) := \sum_{(n)} q_n^{-3}\psi\left(\frac{q_{n-1}}{q_n},\frac{q_{n-2}}{q_n}\right).$$

If again $\nu := \lim_{n\to\infty}\nu_n$ exists then

$$\rho(x,y)=\int\int\frac{d\nu(z,w)}{(1+zy+wx)^3}$$

(where the domain of integration depends on (x,y)) should be the density of an invariant measure. To understand the following construction we note that

$$
(1.7)\qquad
\begin{aligned}
\frac{q_{n-1}}{q_n} &= \left(b_n + a_n\frac{q_{n-2}}{q_{n-1}} + \frac{q_{n-3}}{q_{n-1}}\right)^{-1},\\
\frac{q_{n-2}}{q_n} &= \frac{q_{n-2}}{q_{n-1}}\left(b_n + a_n\frac{q_{n-2}}{q_{n-1}} + \frac{q_{n-3}}{q_{n-1}}\right)^{-1}.
\end{aligned}
$$

2. The basic construction

We now introduce a modified Jacobi-Perron algorithm. Actually it is a change in notation only.

$$
S(\xi,\eta) := \left(\frac{\eta}{\xi} - B_1 + A_1,\ \frac{1}{\xi} - B_1\right),
$$

$$
B_1 = B_1(\xi,\eta) := \left[\frac{1}{\xi}\right],\quad A_1 = A_1(\xi,\eta) = B_1 - \left[\frac{\eta}{\xi}\right],
$$

$$
B_j(\xi,\eta) := B_1(S^{j-1}(\xi,\eta)),\quad A_j(\xi,\eta) := A_1(S^{j-1}(\xi,\eta)).
$$

Then this sequence of digits is subject to the following conditions:

$$
(2.1)\qquad 1 \le B_j,\qquad 0 \le A_j \le B_j;
$$

$$
(2.2)\qquad \text{if}\ \ A_j = 0,\qquad \text{then}\ \ A_{j+1} < B_{j+1}.
$$

The most important fact is the following observation:
Define $(A_j, B_j) := (a_{n+1-j}, b_{n+1-j})$, $1 \le j \le n$, then the sequence (a_j, b_j), $1 \le j \le n$, is admissible for T if and only if the sequence (A_j, B_j), $1 \le j \le n$, is admissible for S, in short: S is a dual or backward algorithm for T (see Schweiger 1979, Ito 1986). Similarly to (1.3) we define P_n, R_n, Q_n by

$$
(2.3)\qquad
\begin{pmatrix} P_{n-2} & P_{n-1} & P_n \\ R_{n-2} & R_{n-1} & R_n \\ Q_{n-2} & Q_{n-1} & Q_n \end{pmatrix}
= \prod_{j=1}^{n} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & B_j - A_j \\ 0 & 1 & B_j \end{pmatrix}.
$$

Again the property

$$\lim_{n\to\infty}\left(\frac{P_n}{Q_n},\frac{R_n}{Q_n}\right)=(\xi,\eta)$$

holds. Note that

$$S(\xi',\eta')=(\xi,\eta)$$

is equivalent to

$$(\xi',\eta')=\left(\frac{1}{B+\eta},\frac{B_1-A_1+\xi}{B+\eta}\right)$$

for appropriate A_1 and B_1.

Next we define sequences α_{jn}, β_{jn}, $0\le j\le 2$, $n\ge 1$ by the matrix relation

$$\begin{pmatrix}\alpha_{1,n-1} & \beta_{1n} & \alpha_{1n}\\ \alpha_{2,n-1} & \beta_{2n} & \alpha_{2n}\\ \alpha_{0,n-1} & \beta_{0n} & \alpha_{0n}\end{pmatrix}=\prod_{j=1}^{n}\begin{pmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 0 & A_j & B_j\end{pmatrix}. \tag{2.4}$$

If $(a_{n+1-j},b_{n+1-j})=(A_n,B_n)$, $1\le j\le n$, then

$$q_{n-2}=\alpha_{1n},\quad q_{n-1}=\alpha_{2n},\quad q_n=\alpha_{0n}. \tag{2.5}$$

Note that ν_n may be rewritten as

$$\nu_n(\psi)=\sum_{(n)}\alpha_{0n}^{-3}\psi\left(\frac{\alpha_{2n}}{\alpha_{0n}},\frac{\alpha_{1n}}{\alpha_{0n}}\right). \tag{2.6}$$

Here the sum runs over all admissible sequences (A_j,B_j), $1\le j\le n$, or over all admissiable sequences (A_j,B_j), $1\le j\le n$, with the initial condition $A_1<B_1$.

If $\widehat{\alpha}_{jn},\widehat{\beta}_{jn}$, $0\le j\le 2$, $n\ge 1$, are given by

$$\begin{pmatrix}\widehat{\alpha}_{1,n-1} & \widehat{\beta}_{1n} & \widehat{\alpha}_{1n}\\ \widehat{\alpha}_{2,n-1} & \widehat{\beta}_{2n} & \widehat{\alpha}_{2n}\\ \widehat{\alpha}_{0,n-1} & \widehat{\beta}_{0n} & \widehat{\alpha}_{0n}\end{pmatrix}=\prod_{j=2}^{n}\begin{pmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 0 & A_j & B_j\end{pmatrix},$$

then

$$\begin{aligned}\frac{\alpha_{2n}}{\alpha_{0n}} &= \left(B_1 + A_1\frac{\widehat{\alpha}_{2n}}{\widehat{\alpha}_{0n}} + \frac{\widehat{\alpha}_{1n}}{\widehat{\alpha}_{0n}}\right)^{-1}, \\ \frac{\alpha_{1n}}{\alpha_{0n}} &= \frac{\widehat{\alpha}_{2n}}{\widehat{\alpha}_{0n}}\left(B_1 + A_1\frac{\widehat{\alpha}_{2n}}{\widehat{\alpha}_{0n}} + \frac{\widehat{\alpha}_{1n}}{\widehat{\alpha}_{0n}}\right)^{-1}.\end{aligned} \tag{2.7}$$

The relation (2.7) reflects (1.7).
For any finite admissible sequence (A_j, B_j), $1 \leq j \leq n$, we define

$$f_n := \frac{\alpha_{2n}}{\alpha_{0n}}, \quad g_n := \frac{\alpha_{1n}}{\alpha_{0n}}.$$

If $(\xi, \eta) = ((A_j, B_j),\ j \geq 1)$ is the expansion of (ξ, η) into an infinite Jacobi-Perron algorithm we define

$$f_n(\xi, \eta) = f_n, \quad g_n(\xi, \eta) = g_n$$

in the obvious way. Then we can prove the following:

Lemma 1. *The following limit exists:*

$$\lim_{n\to\infty} (f_n(\xi,\eta), g_n(\xi,\eta)) =: (f(\xi,\eta), g(\xi,\eta)).$$

Proof. We follow an idea of R.Fischer 1972. Put $\pi_n := (f_n, g_n)$. Then

$$\pi_{n+1} = \kappa_n\pi_n + \lambda_n\pi_{n-1} + \mu_n\pi_{n-2}$$

where the weights are given by

$$\begin{aligned}\kappa_n &= \frac{B_{n+1}\alpha_{0n}}{B_{n+1}\alpha_{0n} + A_n\alpha_{0,n-1} + \alpha_{0,n-2}} \\ \lambda_n &= \frac{A_n\alpha_{0,n-1}}{B_{n+1}\alpha_{0n} + A_n\alpha_{0,n-1} + \alpha_{0,n-2}} \\ \mu_n &= \frac{\alpha_{0,n-2}}{B_{n+1}\alpha_{0n} + A_n\alpha_{0,n-1} + \alpha_{0,n-2}}.\end{aligned}$$

Therefore π_{n+1} lies in the triangle spanned by π_{n-2}, π_{n-1} and π_n. Since $\alpha_{0n} = B_n\alpha_{0,n-1} + \beta_{0,n-1}$ clearly $\kappa_n \geq \frac{1}{3}$. Similarly

$$\pi_{n+2} = \kappa_n'\pi_n + \lambda_n'\pi_{n-1} + \mu_n'\pi_{n-2}.$$

Here $\kappa'_n = \kappa_n \kappa_{n+1} \geq \frac{1}{9}$.
Put $\delta(n) := \max(||\pi_n - \pi_{n-1}||, ||\pi_n - \pi_{n-2}||, ||\pi_{n-1} - \pi_{n-2}||)$.
The function $\delta(n)$ is decreasing. We see

$$||\pi_n - \pi_{n+1}|| \leq (1 - \kappa_n)\delta(n),$$

$$||\pi_{n+1} - \pi_{n+2}|| \leq (1 - \kappa_{n+1})\delta(n+1),$$

$$||\pi_n - \pi_{n+2}|| \leq (1 - \kappa'_n)\delta(n).$$

Hence $\delta(n+2) \leq \frac{8}{9}\delta(n)$.

Lemma 2. *If*

$$(\xi', \eta') = \left(\frac{1}{B+\eta}, \frac{B - A + \xi}{B+\eta}\right)$$

then

$$\begin{aligned} f(\xi', \eta') &= (B + Af(\xi, \eta) + g(\xi, \eta))^{-1}, \\ g(\xi', \eta') &= f(\xi, \eta)(B + Af(\xi, \eta) + g(\xi, \eta))^{-1}. \end{aligned} \tag{2.8}$$

Proof. This follows immediately from (2.7).

Remark. *The functions f and g are not continuous and not injective. In the case of continued fractions the corresponding function reduces to the identity $f(\xi) = \xi$.*

With the help of the functions f and g we now define the sequence of measures (κ_n) on $\mathcal{B}^*$ by

$$\nu_n(\psi) =: \int_{\mathcal{B}^*} \psi(f_n(\xi, \eta), g_n(\xi, \eta)) d\kappa_n(\xi, \eta)$$

and the measure κ on $\mathcal{B}^*$ by

$$\nu(\psi) =: \int_{\mathcal{B}^*} \psi(f(\xi, \eta), g(\xi, \eta)) d\kappa(\xi, \eta)$$

(if the limit measure exists).

Lemma 3. *If $\nu = \lim_{n\to\infty} \nu_n$ exists then*

$$d\kappa(\xi', \eta') = \frac{d\kappa(\xi, \eta)}{(B + Af(\xi, \eta) + g(\xi, \eta))^3}. \tag{2.9}$$

Proof. This again follows from

$$\alpha_{0n} = B_1 \widehat{\alpha}_{0n} + A_1 \widehat{\alpha}_{2n} + \widehat{\alpha}_{1n}.$$

Remark. *If the map $(\xi, \eta) \mapsto (f(\xi, \eta), g(\xi, \eta))$ is absolutely continuous then* it is easily seen that

$$d\kappa(\xi, \eta) := \frac{\partial(f, g)}{\partial(\xi, \eta)} d\xi \, d\eta$$

has the transformation property (2.9).

3. The invariant measure

The heuristic considerations in section 1 now suggest:

Theorem 1. *The density of the invariant measure μ is given as follows:*

$$\rho(x, y) = \iint_{\mathcal{B}^*} \frac{d\kappa(\xi, \eta)}{(1 + f(\xi, \eta)y + g(\xi, \eta)x)^3} \quad \text{for} \quad (x, y) \in \mathcal{B}_1,$$

$$\rho(x, y) = \iint_{\mathcal{B}_1^*} \frac{d\kappa(\xi, \eta)}{(1 + f(\xi, \eta)y + g(\xi, \eta)x)^3} \quad \text{for} \quad (x, y) \in \mathcal{B}_2.$$

Here

$$\mathcal{B}_1^* = \{(\xi, \eta) : 0 \le \xi < \eta \le 1\},$$
$$\mathcal{B}_2^* = \{(\xi, \eta) : 0 \le \eta \le \xi \le 1\},$$
$$\mathcal{B}^* = \mathcal{B}_1^* \cup \mathcal{B}_2^*.$$

Proof. Remember that

$$\mathcal{B}^*(a,b) = \{(\xi,\eta) : A_1(\xi,\eta) = a,\ B_1(\xi,\eta) = b\}.$$

Then

$$\begin{aligned}
\mathcal{B}_1^* &= \bigcup_{b=1}^{\infty} \bigcup_{a=0}^{b-1} \mathcal{B}^*(a,b),\\
\mathcal{B}_2^* &= \bigcup_{b=1}^{\infty} \mathcal{B}^*(b,b),\\
S\mathcal{B}^*(a,b) &= \mathcal{B}^* \ \text{ if } \ a \geq 1,\\
S\mathcal{B}^*(0,b) &= \mathcal{B}_1^*.
\end{aligned}$$

The following identity is basic:

$$(1 + f(\xi,\eta)\frac{x+a}{y+b} + g(\xi,\eta)\frac{1}{y+b})^3\ (y+b)^3 =$$

$$= \left(1 + \frac{y}{b + af(\xi,\eta) + g(\xi,\eta)} + \frac{xf(\xi,\eta)}{b + af(\xi,\eta) + g(\xi,\eta)}\right)^3 .$$

$$\cdot (b + af(\xi,\eta) + g(\xi,\eta))^3.$$

Therefore we obtain (by use of relation (2.9))

$$\iint_{\mathcal{B}^*} \frac{d\kappa(\xi,\eta)}{(1 + f(\xi,\eta)\frac{x+a}{y+b} + g(\xi,\eta)\frac{1}{y+b})^3\ (y+b)^3} =$$

$$= \iint_{\mathcal{B}^*(a,b)} \frac{d\kappa(\xi',\eta')}{(1 + f(\xi',\eta')y + g(\xi',\eta')x)^3} \qquad \text{if} \quad a \geq 1$$

and

$$\iint_{\mathcal{B}_1^*} \frac{d\kappa(\xi,\eta)}{(1 + f(\xi,\eta)\frac{x}{y+b} + g(\xi,\eta)\frac{1}{y+b})^3 (y+b)^3} =$$

$$= \iint_{\mathcal{B}^*(0,b)} \frac{d\kappa(\xi',\eta')}{(1 + f(\xi',\eta')y + g(\xi',\eta')x)^3}.$$

Since

$$\mathcal{B}^* = \bigcup_{b=1}^{\infty} \bigcup_{a=1}^{b} \mathcal{B}^*(a,b) \cup \bigcup_{b=1}^{\infty} \mathcal{B}^*(0,b),$$

$$\mathcal{B}_1^* = \bigcup_{b=1}^{\infty} \bigcup_{a=1}^{b-1} \mathcal{B}^*(a,b) \cup \bigcup_{b=1}^{\infty} \mathcal{B}^*(0,b),$$

a comparison with (1.4) shows that ρ satisfies Kuzmin's equation $A\rho = \rho$.

Remark. *The map*

$$\varepsilon(x,y,\xi,\eta) = \left(\frac{y}{x} - a, \frac{1}{x} - b, \frac{1}{b+\eta}, \frac{b-a+\xi}{b+\xi}\right)$$

can be seen as the natural extension of T (see Ito 1986; Bosma-Jager--Wiedijk 1983). *The measure*

$$\frac{dx\ dy\ d\kappa(\xi,\eta)}{(1+f(\xi,\eta)y+g(\xi,\eta)x)^3}$$

is invariant with respect to ε.

4. κ is singular

As before we denote by μ the invariant absolutely continuous probability measure for the map T. We introduce a new set function τ as follows:

$$\tau(\mathcal{B}((a_1,b_1),\ldots,(a_n,b_n))) := \mu(\mathcal{B}((b_n-a_n,b_n),\ldots,(b_1-a_1,b_1))).$$

Then it is checked easily that τ is in fact an invariant measure for T. Furthermore τ also is an ergodic measure. Therefore $\tau = \mu$ or τ and μ are mutually singular. We will prove that $\tau = \mu$ is impossible.

Theorem 2. *The measure* τ *is singular with respect to* μ.

Proof. Suppose the contrary, namely $\tau = \mu$. This means

$$\mu(\mathcal{B}((a_1, b_1), \dots, (a_n, b_n)) = \mu(\mathcal{B}((b_n - a_n, b_n), \dots, (b_1 - a_1, b_1)))$$

for all admissible sequences.
It is well known that there is a constant $D \geq 1$ such that

$$D^{-1} q_n^{-3} \leq \mu(\mathcal{B}((a_1, b_1), \dots, (a_n, b_n))) \leq D q_n^{-3}$$

and hence

$$D^{-1} Q_n^{-3} \leq \mu(\mathcal{B}((b_n - a_n, b_n), \dots, (b_1 - a_1, b_1))) \leq D Q_n^{-3}.$$

If in fact $\mu = \tau$ we obtain

$$D^{-2} \leq \left(\frac{Q_n}{q_n}\right)^3 \leq D^2. \tag{4.1}$$

But this is impossible. It is sufficient to take $(a_j, b_j) = (0, 1)$, $1 \leq j \leq n$. Then

$$q_n \sim \alpha^n \quad \text{and} \quad Q_n \sim \beta^n$$

where α is the greatest root of $x^3 - x^2 - 1 = 0$ and β is the greatest root of $x^3 - x^2 - x - 1 = 0$. Then $1 < \alpha < \beta$ and (4.1) cannot be true.

Remark. *However the corresponding entropies coincide:*

$$h(\mu, T) = h(\tau, T).$$

Corollary.

$$\lim_{n \to \infty} \frac{Q_n}{\alpha_{0n}} = 0 \qquad \textit{for almost all} \quad (\xi, \eta) \in \mathcal{B}^*.$$

Proof. Since τ is singular with respect to μ the martingale convergence theorem shows that

$$\lim_{n \to \infty} \frac{\tau(B((a_1, b_1), \dots, (a_n, b_n))}{\mu(B((a_1, b_1), \dots, (a_n, b_n))} = \lim_{n \to \infty} \left(\frac{q_n}{Q_n}\right)^3 = 0$$

for (Lebesgue) almost all $(x, y) \in \mathcal{B}$. Therefore by symmetry we obtain

$$\lim_{n \to \infty} \left(\frac{Q_n}{q_n} \right)^3 = 0$$

for (Lebesgue) almost all $(\xi, \eta) \in \mathcal{B}^*$.
Finally note that $\alpha_{0n} = q_n$.

Remark. In the case of continued fractions it is well known that $Q_n = q_n$. Therefore $\mu(\mathcal{B}(k_1, \ldots, k_n))) = \mu(\mathcal{B}(k_n, \ldots, k_1)))$ *for all admissible sequences.*

Lemma. *The limits* $\nu = \lim_{n \to \infty} \nu_n$ *resp.* $\kappa = \lim_{n \to \infty} \kappa_n$ *exist.*

Proof. It is sufficient to show that

$$\lim_{n \to \infty} \nu_n(\psi) =: \nu(\psi)$$

exists for any function ψ which is Lipschitz continuous on $\mathcal{B}^*$.
The map $\Phi(\xi, \eta) := (f(\xi, \eta),\ g(\xi, \eta))$ does not satisfy a Lipschitz condition generally, but it satisfies an appropriate condition on cylinders. The proof of lemma 1 shows that if

$$(\xi, \eta),\ (\xi^*, \eta^*) \in \mathcal{B}^*((A_1, B_1), \ldots, (A_n, B_n)))$$

then

$$||\Phi(\xi, \eta) - \Phi(\xi^*, \eta^*)|| << \delta(n)$$

hence

$$|\psi(\Phi(\xi, \eta)) - \psi(\Phi(\xi^*, \eta^*))|| << \delta(n).$$

An examination of the proof of Kuzmin's theorem for Jacobi-Perron algorithm shows that the Lipschitz condition is used on cylinders only (Schweiger - Waterman 1973). Therefore

$$\sum_{(n)} \psi \left(\Phi \left(\frac{p_n + p_{n-1}y + p_{n-2}x}{q_n + q_{n-1}y + q_{n-2}x}, \frac{r_n + r_{n-1}y + r_{n-2}x}{q_n + q_{n-1}y + q_{n-2}y} \right) \right) \cdot$$

$$\cdot \frac{1}{(q_n + q_{n-1}y + q_{n-2}x)^3} \to \rho(x, y) c(\psi) \text{ as } n \to \infty.$$

Note that

$$\Phi\left(\frac{p_n}{q_n},\frac{r_n}{q_n}\right) = \left(\frac{\alpha_{2n}}{\alpha_{0n}},\frac{\alpha_{1n}}{\alpha_{0n}}\right).$$

Since the summation $\sum_{(n)}$ depends on $(x,y) \in \mathcal{B}_1$ resp. $(x,y) \in \mathcal{B}_2$ we take $\lim_{(x,y)\to(0,0)}$ with $(x,y) \in \mathcal{B}_1$ resp. $\lim_{(x,y)\to(0,0)}$ with $(x,y) \in \mathcal{B}_2$.

If we write L_1 for the first limit and L_2 for the second limit then we obtain as $n \to \infty$

$$\sum_{(n)} \psi\left(\frac{\alpha_{2n}}{\alpha_{0n}},\frac{\alpha_{1n}}{\alpha_{0n}}\right)\frac{1}{\alpha_{0n}^3} \to L_1\, c(\psi)$$

if the sum runs over all admissible sequences $((A_1,B_1),\ldots,(A_n,B_n))$ and

$$\sum_{(n)} \psi\left(\frac{\alpha_{2n}}{\alpha_{0n}},\frac{\alpha_{1n}}{\alpha_{0n}}\right)\frac{1}{\alpha_{0n}^3} \to L_2\, c(\psi)$$

if the sum runs over all admissible sequences $((A_1,B_1),\ldots,(A_n,B_n))$ with the additional condition $A_1 < B_1$.

Theorem 3. *The limit measure κ is singular with respect to Lebesgue measure on $\mathcal{B}^*$.*

Proof. Since

$$\lambda(\mathcal{B}^*((A_1,B_1),\ldots,(A_n,B_n))) \sim Q_n^{-3},$$

this follows at once from Theorem 2 (applied to the measure τ transposed to S) or its Corollary.

Remark. Since ε maps the set $\mathcal{B}((a_1,b_1),\ldots,(a_n,b_n)) \times \mathcal{B}^*$ onto $\mathcal{B}\times\mathcal{B}^*((b_n-a_n,b_n),\ldots,(b_1-a_1,b_1)))$ (if $a_n < b_n$), well known theorems from ergodic theory (see Friedman 1970) show that ε *does not admit a finite invariant measure equivalent to Lebesgue measure on $\mathcal{B} \times \mathcal{B}^*$.*

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FACTOR - UNION REPRESENTATION OF PHENOTYPE SYSTEMS

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Abstract: An algorithm is provided in order to decide whether a given phenotype system is a factor-union system and to construct a corresponding factor-union representation (if such a representation exists).

Factor-union phenotype systems were introduced by Cotterman (cf. [1]). A characterization of such systems was given in [2]. In [4] Markowsky provided an algorithm in order to decide whether a given phenotype system is a factor-union system and to construct a corresponding factor-union representation (if such a representation exists). The aim of this paper is to present another such algorithm which is very simple.

In the following let G denote a fixed non-empty finite set (of genes) and put $M := \{A \in 2^G | 1 \leq |A| \leq 2\}$. The following two definitions are essentailly a "more algebraic" reformulation of the corresponding definitions originally given by Cotterman (cf. [1]).

Definition 1. By a *phenotype system* (with respect to G) one means an equivalence relation on M.

In the following let α denote a fixed phenotype system.

Definition 2. By a *factor-union representation* (FU-*representation*) of α on means an ordered pair (F, f) where F is some set (of so-called "factors") and f is a mapping from G to 2^F such that $\{(A, B) \in \in M^2 | \bigcup_{x \in A} f(x) = \bigcup_{x \in B} f(x)\} = \alpha\}$. α is called a *factor-union system* (FU-*system*) if there exists an FU-representation of α.

In the following, for every $\beta \subseteq (2^G \backslash \{\emptyset\})^2$ let $<\beta>$ denote the congruence on $(2^G \backslash \{\emptyset\}, \cup)$ generated by β.

Remark 1. $<\alpha>$ is the transitive hull of $\{(A \cup C, B \cup C) | (A, B) \in \in \alpha; C \subseteq G\}$.

Remark 2. Let (F, f) be an FU-representation of α. Since $(2^G \backslash \{\emptyset\}, \cup)$ is the free semilattice with free generating set G (every $g \in G$ is here identified with the one-element set $\{g\}$), f can be extended to a homomorhism g from $(2^G \backslash \{\emptyset\}, \cup)$ to $(2^F, \cup)$. Now $\ker g$ is a congruence on $(2^G \backslash \{\emptyset\}, \cup)$ and $< (\ker g) \cap M^2 > \subseteq \ker g$. Here, in general, equality does not hold as can be seen from the following example: Put $G := \{a, b, c, d\}$ (a, b, c, d mutually distinct), $F := \{1, 2, 3, 4, 5\}$ and $f := \{(a, \{1, 2\}), (b, \{2, 3\}), (c, \{3, 4\}), (d, \{1, 2, 3, 5\})\}$. Then (F, f) is an FU-representation of $\{\{a\}\}^2 \cup \{\{b\}\}^2 \cup \{\{c\}\}^2 \cup \{\{d\}, \{a, d\}, \{b, d\}\}^2 \cup \{\{a, b\}\}^2 \cup \{\{a, c\}\}^2 \cup \{\{b, c\}\}^2 \cup \{\{c, d\}\}^2$, $< (\ker g) \cap M^2 > = = \{\{a\}\}^2 \cup \{\{b\}\}^2 \cup \{\{c\}\}^2 \cup \{\{d\}, \{a, d\}, \{b, d\}, \{a, b, d\}\}^2 \cup \{\{a, b\}\}^2 \cup \cup \{\{a, c\}\}^2 \cup \{\{b, c\}\}^2 \cup \{\{c, d\}, \{a, c, d\}, \{b, c, d\}, \{a, b, c, d\}\}^2 \cup \{\{a, b, c\}\}^2$ and $\ker g = < (\ker g) \cap M^2 > \cup \{\{a, c\}, \{a, b, c\}\}^2$.

Definition 3. By a *(join-)semilattice* on means a commutative idempotent semigroup, i.e. an algebra $(S, \vee)$ of type 2 in which the laws $x \vee y = y \vee x$, $x \vee x = x$ and $(x \vee y) \vee z = x \vee (y \vee z)$ hold. The corresponding partial order $\leq$ is defined by $x \leq y$ iff $x \vee y = y$ $(x, y \in S)$. $a \in S$ is called *meet-irreducible* if for all $x, y \in S$ with $x \wedge y = a$ it holds $a \in \{x, y\}$ ($\wedge$ denotes the infimum with respect to $\leq$).

The aim of this paper is to prove the following

Theorem.

(i) *α is an* FU-*system iff* $<\alpha> \cap M^2 = \alpha$.

(ii) *If* $<\alpha> \cap M^2 = \alpha$ *and if L denotes the set of all meet-irreducible elements of the (join-)semilattice* $(2^G \backslash \{\emptyset\}, \cup)/ <\alpha>$ *then* $(L, \{(x, \{y \in L | y \not\geq [\{x\}] <\alpha>\}) | x \in G\})$ *is an* FU-*representation of* α.

The proof of the theorem makes use of the following

Proposition (cf. [3]). *Let* $(S, \vee)$ *be a finite join-semilattice and let L denote the set of all meet-irreducible elements of* $(S, \vee)$. *Then* $\{(x, \{y \in \in L | y \not\geq x\}) | x \in S\}$ *is an injective homomorphism from* $(S, \vee)$ *to* $(2^L, \cup)$.

Proof. Put $f := \{(x, \{y \in L | y \not\geq x\}) | x \in S\}$. Since S is finite, every element a of S is the meet of elements of L and hence the meet of all elements of L which are $\geq a$. This shows injectivity of f. A straightforward calculation yields $f(x \vee y) = f(x) \cup f(y)$ for all $x, y \in S$.

Proof of the Theorem. First assume α to be an FU-system. Then there exists an FU-representation (F, f) of α. Let g denote the homomorphism from $(2^G \backslash \{\emptyset\}, \cup)$ to $(2^F, \cup)$ extending f. (This homomorphism exists according to Remark 2 following Definition 2.) Then $\ker g$ is a congruence on $(2^G \backslash \{\emptyset\}, \cup)$ and since (F, f) is an FU-representation of α, $\alpha = (\ker g) \cap M^2$. Because of $\alpha \subseteq \ker g$ we have $<\alpha> \subseteq \ker g$ and hence $\alpha \subseteq <\alpha> \cap M^2 \subseteq (\ker g) \cap M^2 = \alpha$ which shows $<\alpha> \cap M^2 = \alpha$. Conversely, assume $<\alpha> \cap M^2 = \alpha$. Let L denote the set of all meet-irreducible elements of $(2^G \backslash \{\emptyset\}, \cup)/ <\alpha>$ and put $h := \{(x, \{y \in L | y \not\geq x\}) | x \in (2^G \backslash \{\emptyset\})/ <\alpha>\}$. Then, according to the above proposition, h is an injective homomorphism from $((2^G \backslash \{\emptyset\})/ <\alpha>, \cup)$ to $(2^L, \cup)$. Now for all $A, B \in M$ the following are equivalent: $\bigcup_{x \in A} h([\{x\}] <\alpha>) = \bigcup_{x \in B} h([\{x\}] <\alpha>)$, $h([A] <\alpha>) = h([B] <\alpha>)$, $[A] <\alpha> = [B] <\alpha>$, $(A, B) \in <\alpha>$, $(A, B) \in \alpha$. This completes the proof of the theorem.

Remark. The FU-systems (with respect to G) are exactly the restrictions of the congruences of $(2^G \backslash \{\emptyset\}, \cup)$ (or of $(2^G, \cup)$) to M; for let β be some congruence of $(2^G \backslash \{\emptyset\}, \cup)$ and suppose $\alpha = \beta \cap M^2$. Then $\alpha \subseteq \beta$ and hence $<\alpha> \subseteq \beta$. From this we conclude $\alpha \subseteq <\alpha> \cap M^2 \subseteq \subseteq \beta \cap M^2 = \alpha$ whence $\alpha = < \alpha > \cap M^2$, i.e. α is an FU-system.

The above theorem gives rise to the following

Algorithm. Construct the undirected graph (with vertex-set $2^G\setminus\{\emptyset\}$) corresponding to α. Next construct in an obvious graph-theoretical manner $<\alpha>$ as the transitive hull of $\{(A\cup C, B\cup C)|(A,B)\in\alpha; C\subseteq \subseteq G\}$. Then check if $<\alpha>\cap M^2=\alpha$. If this is the case then construct the Hasse-diagram of $(2^G\setminus\{\emptyset\},\cup)/<\alpha>$ in order to obtain the FU-representation of α described within the above theorem.

Example (human ABO blood group system). Put $G:=\{A,B,O\}$ and $\alpha:=\{\{A\},\{A,O\}\}^2\cup\{\{B\},\ \{B,O\}\}^2\cup\{\{O\}\}^2\cup\{\{A,B\}\}^2$. Then $<\alpha>=\alpha\cup\{\{A,B\},\{A,B,O\}\}^2$ and hence $<\alpha>\cap M^2=\alpha$. Therefore α is an FU-system. The above theorem yields the FU-representation $(\{a,b,c\},\{(A,\{b\}),(B,\{a\}),(O,\emptyset)\})$ of α where $a:=[\{A\}]<\alpha>$, $b:= := [\{B\}]<\alpha>$ and $c:=[\{A,B\}]<\alpha>$.

Remark. From the above theorem it follows that if α is an FU-system (with respect to G) then there exists an FU-representation (F,f) of α with $|F|\leq 2^{|G|}-1$. Knowing this, the problem whether α is an FU-system or not can be decided in a finite number of steps. (Take an arbitrary fixed set of cardinality $2^{|G|}-1$ as the set of possible factors.)

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A NON-COMPLETELY REGULAR QUIET QUASI-METRIC*

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Abstract: We improve on an example of Fletcher, Hejcman and Hunsaker [4] for a non-completely regular quiet-uniformity: our example is quasi-metrizable.

1. Introduction

Doitchinov [2,3] introduced a class of quasi-metrics, respectively quasi-uniformities, admitting a satisfactory theory of completeness and completion. We shall only deal here with these classes, and not with completions. See [5] for basic definitions concerning quasi-metrics and quasi-uniformities.

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Definitions. A *sequence pair* (a *filter pair*) is an ordered pair of sequences (of filters).

The sequence pair $(<x_k>, <y_n>)$ in the quasi-metric space (X,d) is *Chauchy* if for any $\varepsilon > 0$, there is an $m \in \mathbb{N}$ such that $d(x_k, y_n) < \varepsilon \quad (k,n > m)$. A filter pair (f,g) in the quasi-uniform space $(X,\mathcal{U})$ is *Chauchy* if for any $U \in \mathcal{U}$, there are $F \in \mathsf{f}$ and $G \in \mathsf{g}$ with $F \times G \subset U$.

The quasi-metric d is *balanced* [2] if for any Chauchy sequence pair $(<x_k>, <y_n>)$, and for any $x, y \in X$, we have

$$(1) \qquad d(x,y) \le \sup_n d(x,y_n) + \sup_k d(x_k,y).$$

(Equivalently: one can write lim sup instead of sup.) The T_1-quasi-uniformity $\mathcal{U}$ is *quiet* [3] provided that for any $U \in \mathcal{U}$ there is a $V \in \mathcal{U}$ such that if $x, y \in X$, (f,g) is a Cauchy filter pair, $Vx \in \mathsf{g}, V^{-1}y \in \mathsf{f}$, then xUy. (See [1] §§7 – 8 for related notions.) $\diamondsuit$

The notions of a balanced quasi-metric and of a quiet quasi-uniformity are in close connexion: if d is balanced then $\mathcal{U}(d)$ is quiet ([3] p. 6); it is also pointed out in [3] that the quietness of $\mathcal{U}(d)$ can be reformulated in terms of d, namely: $\mathcal{U}(d)$ is quiet iff for any $\varepsilon > 0$ there is a $\delta > 0$ such that $d(x,y) < \varepsilon$ whenever there is a Cauchy sequence pair $(<x_k>, <y_n>)$ with $d(x,y_n) < \delta \quad (n \in \mathbb{N})$ and $d(x_k,y) < \delta$ $(k \in \mathbb{N})$.

Conversely, if $\mathcal{U}$ is quasi-metrizable and quiet then it can be induced by a quasi-metric d satisfying a condition strictly stronger than the one in the preceding paragraph, but strictly weaker than the one in the definition of balanced quasi-metrics: there is a constant C such that for any Cauchy sequence pair $(<x_k>, <y_n>)$, and for any $x, y \in X$,

$$(2) \qquad d(x,y) \le C(\sup_n d(x,y_n) + \sup_k d(x_k,y)).$$

A routine application of the Metrization Lemma ([6], 6.12) gives this with $C = 8$; taking then $d' = \sqrt[j]{d}$ with some $j \in \mathbb{N}$, (2) will be satisfied for d' with $C = \sqrt[j]{8}$; i.e. for any $C > 1$, there is a quasi-metric compatible with $\mathcal{U}$ such that (2) holds with this C.

The topology induced by a balanced quasi-metric is completely regular [2], while a quiet quasi-uniformity induces a regular topology (Doitchinov, cited in [4]). Considering the similarity of the two notions, it is somewhat surprising that, as shown by an example of Fletcher, Hejcman and Hunsaker [4], a quiet quasi-uniformity is not necessarily completely regular.

The aim of this note is to give a similar example, which, in addition, is quasi-metrizable. Observe that the two notions are now even closer to each other: compare (1) and (2), where, as we have seen, one can take $C = 1 + \varepsilon$.

2. The example

Let $\mathbb{Q}$ denote the set of the rationals, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $Q_i =]i, i+1[\cap \mathbb{Q}$ $(i \in \mathbb{N}_0)$, $\mathcal{E}$ the Euclidean topology on $\mathbb{Q}$. For a convergent sequence s in $\mathbb{Q}$, denote its $\mathcal{E}$-limit by $\lambda(s)$. Let an injection $\nu : \mathbb{Q} \to \mathbb{N}$ be fixed. Take a maximal almost disjoint collection A_0 of strictly decreasing sequences in Q_0 that $\mathcal{E}$-converge to some point in Q_0. For a sequence $s = < x_j >$ in Q_0, and for each $i \in \mathbb{N}_0$, let $s + i = < x_j + i >$. Define $A_i = \{s + i : s \in A_0\}$ $(i \in \mathbb{N})$, $A = \bigcup_1^\infty A_i$, $Q = \bigcup_0^\infty Q_i$, $X = \{\omega\} \cup Q \cup A$. For $i \in \mathbb{N}$ and $s = < x_j > \in A_i$, let $s^* = < 2i - x_j >$. Now $A_i^* = \{s^* : s \in A_i\}$ is a maximal almost disjoint collection of increasing sequences in Q_{i-1} that $\mathcal{E}$-converge to some point in Q_{i-1}, while A_{i-1} is a similar collection of decreasing sequences, and $A_{i-1} \cup A_i^*$ is clearly almost disjoint, too. Define a function d on $X \times X$ as follows:

$$d(x,y) = \begin{cases} 1/i & \text{if } x = \omega,\ y \in A_i \cup Q_{i-1},\ i \in \mathbb{N}, \\ y - \lambda(x) & \text{if } y \in x \in A,\ \nu(y) > \nu(\lambda(x)), \\ \lambda(x^*) - y & \text{if } x \in A,\ y \in x^*,\ \nu(y) > \nu(\lambda(x^*)), \\ 0 & \text{if } x = y, \\ 1 & \text{otherwise.} \end{cases}$$

Claim 1. *d is a non-Archimedian quasi-metric.*

Proof. In the second line of the definiton, y is in a decreasing sequence tending to $\lambda(x)$, hence the value of d is positive in this case, and similarly in the third line. So we have only to check that

$$d(x,z) \leq \max\{d(x,y), d(y,z)\}.$$

This is evident if the right hand side is 1 or $x = y$ or $y = z$; otherwise $x = \omega$, $y \in A_i$ for some $i \in \mathbb{N}$, and $z \in Q_{i-1} \cup Q_i$, thus the right hand side is $\geq 1/i$, and the left hand side is $1/i$ or $1/(i+1)$. $\Diamond$

Claim 2. *The topology $\mathcal{T}(d)$ is regular.*

Proof. All the open balls round points different form ω are closed (because if x_1, x_2 are distinct points in A then $x_1 \cup x_1^*$ and $x_2 \cup x_2^*$ are almost disjoint), while

$$\left\{ \{\omega\} \cup \bigcup_{i=j}^{\infty} (A_i \cup Q_i) : j \in \mathbb{N} \right\}$$

is a neighbourhood base of ω consisting of closed sets. $\Diamond$

Claim 3. *Any $\mathcal{T}(d)$-neighbourhood of the $\mathcal{T}(d)$-closure of an interval in Q_i contains an interval in Q_{i-1}.*

Proof. Let $\emptyset \neq H =]a, b[\cap Q_i$, $a^* = 2i - a$, $b^* = 2i - b$, $H^* =]b^*, a^*[\cap Q_{i-1}$, F the $\mathcal{T}(d)$-closure of H, G a $\mathcal{T}(d)$-neighbourhood of F. Now $s \in F$ whenever $s \in A_i$ with $\lambda(s) \in H$, thus each $t \in A_i^*$ with $\lambda(t) \in H^*$ is almost contained by G.

Assume indirectly that G does not contain a subinterval of H^*. Then we can pick a strictly increasing sequence in $H^* \setminus G$ that $\mathcal{E}$-converges to some point of H^*, contradicting the maximality of the almost disjoint collection A_i^*. $\Diamond$

Claim 4. *$\mathcal{T}(d)$ is not completely regular.*

Proof. If f is a continuous real function on X, and $f(\omega) > 0$ then, according to Claim 3, there is a $q \in Q_0$ with $f(q) > 0$; thus ω and Q_0 cannot be separated by a continuous function. $\Diamond$

Claim 5. *The topology $\mathcal{T}(d^{-1})$ is discrete.*

Proof. For $y \in Q_i$, choose $\varepsilon > 0$ such that $z \in]y-\varepsilon, y+\varepsilon[\cap (Q_i \setminus \{y\}) \Rightarrow \nu(z) > \nu(y)$, and assume also that $\varepsilon < 1/(i+1)$. Then the d^{-1}-ball of radius ε round y is equal to $\{y\}$ (see the condition on $\nu(y)$ in the definiton of d). The points outside Q are evidently isolated. $\Diamond$

Claim 6. *If* (f,g) *is a* $\mathcal{U}(d)$*-Cauchy filter pair then* $\{z\} \in \mathsf{f}$ *for some* $z \in X$*, and* g $\mathcal{T}(d)$*-converges to* z.

Proof. It is enough to show that f contains a singleton, because the second assertion is then clear from the definition of the Cauchy property.

Choose $F \in \mathsf{f}$ and $G \in \mathsf{g}$ such that

$$d(x,y) < 1 \qquad (x \in F,\ y \in G). \tag{3}$$

Now if F is finite then f contains a smallest element F_0, and, according to the Cauchy property, g $\mathcal{T}(d)$-converges to each element of F_0, i.e. F_0 is a one-point set, since $\mathcal{T}(d)$ is T_2. On the other hand, if F is infinite then there are different points $x_1, x_2 \in F \cap A$, because (3) implies that $|F \cap Q| \leq 1$. Thus from (3) we have

$$G \subset (\{x_1\} \cup x_1 \cup x_1^*) \cap (\{x_2\} \cup x_2 \cup x_2^*) \in \mathsf{g},$$

and this intersection is finite by the almost disjointness, i.e. there is a point $z \in \bigcap \mathsf{g}$. According to the Cauchy property, f $\mathcal{T}(d^{-1})$-converges to z, and then Claim 5 implies that $\{z\} \in \mathsf{f}$. ◇

Claim 7. *The quasi-uniformity* $\mathcal{U}(d)$ *is quiet.*

Proof. Let $U_j = \{(x,y) : d(x,y) \leq 1/j\}$. We are going to show that the condition in the defintion of quietness holds for $U = U_j$ and $V = U_{j+1}$. Take a filter pair (f,g) with $\{z\} \in \mathsf{f}$ and g $\mathcal{T}(d)$-converging to z (by Claim 6, all the Cauchy filter pairs are of this form). We have to show that if $U_{j+1}x \in \mathsf{g}$ and $U_{j+1}^{-1}y \in \mathsf{f}$ then xU_jy; this is a consequence of the following statement: if

$$d(x,y_n) \leq 1/(j+1) \qquad (n \in \mathrm{N}), \tag{4}$$

$$d(z,y_n) < 1/n \qquad (n \in \mathrm{N}), \tag{5}$$

$$d(z,y) \leq 1/(j+1) \tag{6}$$

then

$$d(x,y) \leq 1/j. \tag{7}$$

It is indeed enough to prove that (4), (5) and (6) imply (7): if $U_{j+1}x \in \mathsf{g}$ then points $y_n \in U_{j+1}x$ satisfying (5) can be chosen because g converges to z, and then (4) holds evidently; moreover, $U_{j+1}^{-1}y \in \mathsf{f}$ implies (6), and xU_jy is equivalent to (7).

If $z = \omega$ then (4) and (5) imply $x = \omega$, thus (7) follows from (6). If $z \in Q$ then $y = z = y_1$ by (6) and (5), thus (7) follows from (4). Finally, assume that $z \in A_i$ for some $i \in \mathbb{N}$. From (4), (5) and the almost disjointness we have $x = z$ or $x = \omega$. (6) implies (7) in the first case; on the other hand, if $x = \omega$ then, by (5), $y_1 \in Q_{i-1} \cup Q_i \cup A_i$, thus $d(x, y_1)$ is either $1/i$ or $1/(i+1)$, hence (4) implies $i \geq j$; according to (6), $y \in Q_{i-1} \cup Q_i \cup A_i$, thus $d(x,y) \leq 1/i \leq i/j$. $\diamondsuit$

Remarks. a) Similarly to the example in [4], our example is complete in the sense of Doitchinov [3]. (Clear from Claim 6, since, by definition, the completeness of $\mathcal{U}$ means that the second element of any Cauchy filter pair is $\mathcal{T}(\mathcal{U})$-convergent.)

b) The topology $\mathcal{T}(d)$ can be regarded as a special case of a general construction from [7], and it was very likely described long before. (*Added in proof.* See the addition in proof in [7]).

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DISTRIBUTION AND MOMENT CONVERGENCE OF SUMS OF ASSOCIATED RANDOM VARIABLES

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Abstract: Let $\{X_n, n\geq 1\}$ be a sequence of associated random variables with zero mean, and let $S_{n,k} = \sum_{1\leq i_1<\ldots<i_k\leq n} X_{i_1}\ldots X_{i_k}$, $k\geq 1$. We present sufficient conditions for the distribution and moment convergence of $S_{n,k}/\mathrm{Var}(S_{n,1})$, to the distribution and moments of $H_k(\mathcal{N})/k!$, where H_k is the Hermite polynomial of degree k and $\mathcal{N}$ is a standard normal variable.

1. Introduction

Let $\{X_n, n \geq 1\}$ be a sequence of random variables, defined on some probability space $(\Omega, \Im, P)$, such that $EX_n = 0$, $EX_n^2 < \infty$, $n \geq 1$.

Let us put:

$$S_0 = 0,\ S_n = \sum_{k=1}^{n} X_k,\ \sigma_n^2 = ES_n^2,\ n \geq 1,$$

$$S_{n,k} = \sum_{1 \leq i_1 < \ldots < i_k \leq n} X_{i_1} X_{i_2} \ldots X_{i_k},\quad k \geq 1$$

$$Y_{n,k} = S_{n,k}/\sigma_n,\ U_n^2 = \sum_{i=1}^{n} X_i^2/\sigma_n^2.$$

Let us observe that $S_{n,1} = S_n$.

Let $H_k(z)$ denote the Hermite polynomial of degree k, defined by

$$\left(\frac{d}{dz}\right) \exp(-z^2/2) = (-1)^k H_k(z) \exp(-z^2/2).$$

One can note that setting $H_0(z) \equiv 1$ we have

$$H_{k+1} = zH_k(z) - kH_{k-1}(z),\quad k \geq 1.$$

Let $\mathcal{N}$ denote a standard normal variable.

In this paper we study convergence of distributions and moments of the sequence $\{Y_{n,k}, n \geq 1\}$ to the distribution and moments of $H_k(\mathcal{N})/k!$. This problem has been studied by Teicher [5] in the case when $\{X_n, n \geq 1\}$ is a sequence of square-integrable martingale differences. We investigate sequences $\{X_n, n \geq 1\}$ that satisfy a condition of positive dependence called association.

We recall that a collection $\{X_1, \ldots, X_n\}$ of random variables is *associated* if for any two coordinatewise nondecreasing functions f_1, f_2 on $\mathbb{R}^n$ such that $\widehat{f_i} = f_i(X_1, \ldots, X_n)$ has finite variance for $i = 1, 2$, thus $\mathrm{Cov}(\widehat{f_1}, \widehat{f_2}) \geq 0$ holds. An infinite collection is associated if every finite subcollection is associated (cf. [4]).

Recently many papers have been published concerning weak convergence of the sequence $\{Y_{n,1}, n \geq 1\}$ to the distribution of $H_1(\mathcal{N})$ (cf. [2] and the references given there). But, as we know, there are no results concerning weak convergence of the sequence $(Y_{n,k}, n \geq 1)$ for associated sequences. The convergence of moments of $\{Y_{n,k}, n \geq 1\}$ has not been studied yet, even in the case $k = 1$. The results presented in this paper fill in this gap.

2. Results

Studying limit properties of associated sequences we need the following coefficient

$$u(n) = \sup_{n \in \mathbb{N}} \sum_{j:|j-k| \geq n} \operatorname{Cov}(X_j, X_k), \quad n \in \mathbb{N} \cup \{0\}.$$

In what follows we will use the following conditions

(1) $\lim_{n \to \infty} E(U_n^2)^p = 1$,
(2) $\sup_{n \in \mathbb{N}} E|X_n|^{2p+\delta} = M < \infty$,
(3) $u(n) = \mathrm{O}(n^{-(2p+\delta/2-2)(2p+\delta)/\delta})$, and $u(1) < \infty$,
(4) $\inf_{n \in \mathbb{N}} n^{-1}\sigma_n^2 > 0$.

Theorem 1. *Let $\{X_n, n \geq 1\}$ be a sequence of associated random variables such that $EX_n = 0$ and $EX_n^2 < \infty$, $n \geq 1$. If for some $p > 1$ and $\delta \in (0,1)$ the conditions* (2), (3) *and* (4) *hold, then*

$$S_n/\sigma_n \xrightarrow{D} \mathcal{N} \quad as \quad n \to \infty, \tag{5}$$

and

$$E|S_n/\sigma_n|^{2p} \longrightarrow E|\mathcal{N}|^{2p} \quad as \quad n \to \infty. \tag{6}$$

Theorem 2. *Let $\{X_n, n \geq 1\}$ be a sequence of associated random variables such that $EX_n = 0$ and $EX_n^2 < \infty$, $n \geq 1$. If for some $p > 1$ and $\delta \in (0,1)$ the conditions* (1), (2), (3) *and* (4) *hold then for every $k \in \mathbb{N}$*

$$Y_{n,k} \xrightarrow{D} H_k(\mathcal{N})/k! \quad as \quad n \to \infty, \tag{7}$$

and

$$E|Y_{n,k}|^{2p/k} \longrightarrow E|H_k(\mathcal{N})/k!|^{2p/k} \quad as \quad n \to \infty. \tag{8}$$

3. Proofs

Proof to Theorem 1. By (4), $\sigma_n^2 \to \infty$ as $n \to \infty$. Thus, by (2) and (4), we get the Lyapunov condition of order $2p + \delta$

$$\sigma_n^{-2p-\delta} \sum_{i=1}^{n} E|X_i|^{2p+\delta} \leq nM\sigma^{-2p-\delta} =$$

$$= M(n/\sigma_n^2)\sigma^{-2(p-1)-\delta} \to 0 \quad as \quad n \to \infty.$$

Hence the Lindeberg condition of order $2p + \delta$ holds and therefore the classical Lindeberg condition is satisfied:

$$\sigma_n^{-2} \sum_{i=1}^{n} E|X_i|^2 I[|X_i| \geq \sigma_n \varepsilon] \to 0 \text{ as } n \to \infty \text{ for every } \varepsilon > 0. \tag{9}$$

Taking into account (3), (4) and (9), by Theorem 3 of Birkel [2] we have (5).
By (4) we have $n/\sigma^2 < C < \infty$ for some constant C and every $n \in \mathrm{N}$. Thus applying Theorem 1 of Birkel [3] we get

$$E|S_n/\sigma_n|^{2p+\delta/2} \leq \sup_{n \in \mathbb{N} \cup \{0\}} E|S_{n+m} - S_m|^{2p+\delta/2}/\sigma_n^{2p+\delta/2} \leq$$

$$\leq Bn^{(2p+\delta/2)/2}/(\sigma_n^2)^{(2p+\delta/2)/2} \leq K,$$

where K is a constant not depending on n.
Thus the sequence $\{|S_n/\sigma_n|^{2p}, n \geq 1\}$ is uniformly integrable so, by Theorem 5.4 of Billingsley [1] we get (6).

Proof of Theorem 2. In order to obtain (7) it suffices, by Remark 1 of Teicher [5], to prove the following three assertions:

(i) $S_n/\sigma_n \xrightarrow{D} \mathcal{N}$,

(ii) $U_n^2 \xrightarrow{P} 1$,

(iii) $\max_{1\le i\le n} |X_i|/\sigma_n \xrightarrow{P} 0$ as $n \to \infty$.

(i) follows from Theorem 1.
To prove (ii), we first show that

$$EU_n^2 \to 1 \quad \text{as} \quad n \to \infty. \tag{10}$$

Clearly:

$$1 \ge EU_n^2 = \sigma_n^{-2} E(X_1^2 + \ldots + X_n^2) = 1 - \sigma_n^{-2} \sum\sum_{1\le i\ne j\le n} \operatorname{Cov}(X_i, X_j) \ge$$

$$\ge 1 - 2\sigma_n^{-2} \sum_{j=1}^{n} u(j) \ge 1 - 2\sigma_n^{-2} \sum_{j=1}^{\infty} u(j).$$

But by assumption (3), for sufficiently large j we have

$$u(j) \le C/j^{(1+\delta/4)},$$

thus $\sum_{j=1}^{\infty} u(j) < \infty$, and therefore

$$1 \ge EU_n^2 \ge 1 - 2\sigma_n^{-2} \textstyle\sum_{j=1}^{\infty} u(j) \to 1 \quad \text{as} \quad n \to \infty.$$

By (1) and (10), applying Lemma 1 of Teicher [5] we get

$$E|U_n^2 - 1|^p \to 0 \text{ as } n \to \infty, \text{ thus } U_n^2 \xrightarrow{P} 1 \text{ as } n \to \infty.$$

To prove (iii) let us observe that under our assumptions the Lyapunov conditon holds and therefore

$$P[\max_{1\le i\le n} |X_i|/\sigma_n \ge \varepsilon] \le \sum_{i=1}^{n} P[|X_i|/\sigma_n \ge \varepsilon] \le$$

$$\le \varepsilon^{-2p-\delta} \sigma_n^{-2p-\delta} \sum_{i=1}^{n} E|X_i|^{2p+\delta} \to 0 \text{ as } n \to \infty.$$

Thus (i), (ii) and (iii) hold hence (7) is satisfied.

By Theorem 1 we have (8) in the case $k = 1$. Taking into account condition (1) and applying Lemma 1 of Teicher [5] we get the uniform integrability of the sequence $\{U_n^{2p}, n \geq 1\}$. From the proof of Theorem 1 the sequence $\{|Y_n|^{2p}, n \geq 1\}$ is uniformly integrable, where $Y_n = Y_{n,1}$. Let us observe that

$$|2Y_{n,2}|^p = |Y_n^2 - U_n^2|^p \leq 2^p(|Y_n|^{2p} + U_n^{2p}),$$

thus the sequence $\{|Y_{n,2}|^p,\ n \geq 1\}$ is uniformly integrable. This fact and (7) yields (8) in the case $k = 2$.

Now we proceed by induction on k.

Assume that (8) holds for $m = 1, 2, \ldots, k$ then by (7) and Theorem 5.4 of Billingsley [1], the sequence $\{|Y_{n,m}|^{2p/m}, n \geq 1\}$ is uniformly integrable for $m = 1, 2, \ldots, k$. For $0 \leq j < k$ we have

$$|Y_{n,k-j} \sum_{i=1}^{n} (X_i/\sigma_n)^{j+1}| \leq (k-j)/(k+1)|Y_{n,k-j}|^{(k+1)/(k-j)} +$$

$$+(j+1)/(k+1)| \sum_{i=1}^{n} (X_i/\sigma_n)^{j+1}|^{(k+1)/(j+1)},$$

and

$$|Y_{n,k-j} \sum_{i=1}^{n} (X_i/\sigma_n)^{j+1}|^{2p/(k+1)} \leq C \left(\frac{k-j}{k+1} \right)^{2p/(k+1)} |Y_{n,k-j}|^{2p/(k-j)} +$$

$$+ C \left(\frac{j+1}{k+1} \right)^{2p/(k+1)} | \sum_{i=1}^{n} (X_i/\sigma_n)^{j+1}|^{2p/(j+1)} \leq$$

$$\leq C \left(|Y_{n,k-j}|^{2p/(k-j)} + | \sum_{i=1}^{n} (X_i/\sigma_n)^{j+1}|^{2p/(j+1)} \right),$$

where $C = 2^{2p/(k+1)}$.

Let us observe that for $1 \leq j < k$ we have

$$| \sum_{i=1}^{n} X_i^{j+1}|^{2p/(j+1)} \leq \left(\sum_{i=1}^{n} X_i^2 \right)^p .$$

Thus taking into account the uniform integrability of $\{U_n^{2p}, n \geq 1\}$ and $\{|Y_{n,m}|^{2p/m}, n \geq 1\}$, for $m = 1, 2, \ldots, k$, and the above inequalities we see that the sequence

$$\{|Y_{n,k-j} \sum_{i=1}^{n} (X_i/\sigma_n)^{j+1}|^{2p/(k+1)}, n \geq 1\}$$

is uniformly integrable for $0 \leq j \leq k$.
Now using the equality

$$(k+1)Y_{n,k+1} = \sum_{j=0}^{k} (-1)^j Y_{n,k-j} \left(\sum_{i=1}^{n} (X_i/\sigma_n)^{j+1} \right)$$

we get the uniform integrability of the sequence

$$\left\{ |Y_{n,k+1}|^{2p/(k+1)}, n \geq 1 \right\},$$

this together with (7) implies (8) in the case $m = k+1$, which completes the proof of Theorem 2.

References

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